The Algebra of Taylor Series and the Roots of a General Polynomial

John Herrera

October 11, 2002

Superconducting Magnet Division

DISCLAIMER

The Algebra of Taylor Series and the Roots of a General Polynomial

John Herrera

October 11, 2002

The Algebra of Taylor Series and the Roots of a General Polynomial

J. C. Herrera

Brookhaven National Laboratory, Upton, NY 11973-5000 (Retired)

Abstract

We develop the basic elements of the algebra of Taylor series. This knowledge allows us to derive a series expression for an exact root of a general polynomial of arbitrary degree.

1. Introduction:

Taylor series are commonly used in the mathematical analysis of physical systems. It is our purpose in this paper to present systematically the algebra of Taylor series, and then to use this knowledge to derive an expression for an exact root of a polynomial of arbitrary degree.

Drawing on the discussion of Taylor series as it appears in Whittaker and Watson [1], we write the expansion of a function about an offset z_0 as

$$f(z) = \sum_{n=0}^{\infty} a_n(z_0)(z - z_0)^n \tag{1}$$

All the quantities in Eq. (1) are complex and the coefficients are defined by the contour integral centered at z_0

$$a_n(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)\,dz}{(z - z_0)^{n+1}} \ , \quad n = 0,1,2,\ldots \tag{2}$$

The radius of convergence for the Taylor series is the lower bound

$$R_a = \underline{\lim} \left| a_n(z_0) \right|^{-1/n} \ , \quad n \to \infty \ . \tag{3}$$

2. Algebraic structure for addition and multiplication:

Our approach to the algebraic properties of the set of Taylor series of the form of Eq. (1) is entirely based on Eq. (2) for the coefficients of each element of the set. Thus with respect to the binary operation of addition, it is clear that such a set constitutes an additive abelian group. The identity element for additon is the element with coefficients $a_n(z_0) = 0$. Similarly, as we shall verify in the following two sections, the set of elements characterized by having $a_0(z_0) \neq 0$, form with respect to multiplication, a multiplicative abelian group with the identity: $a_n(z_0) = (1,0,0,\ldots)$. Since the distributive law over addition also holds, we can say that the set of Taylor series expanded about a given point z_0 constitutes an infinite field over the field of complex numbers [2,3].

3. Product of two Taylor series:

As a simple illustration of the method of basing the analysis on Eq. (2), we consider the product of the series $f(z)$ with another one $g(z)$. Thus,

$$h(z) = f(z)g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \sum_{\ell=0}^{\infty} b_\ell (z - z_0)^\ell , \qquad (4)$$

or

$$h(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} a_k b_\ell (z - z_0)^{k+\ell} . \qquad (5)$$

Dividing Eq. (5) by the quantity $(z - z_0)$ and performing the contour integration yields

$$\frac{1}{2\pi i} \oint \frac{f(z)g(z)\, dz}{z - z_0} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} a_k b_\ell \, \delta_{k+\ell,0} . \qquad (6)$$

Here $\delta_{k+\ell,0}$ is the Kronecker delta function equal to one when $(k+\ell) = 0$ and zero otherwise. Consequently, the zero degree coefficient in the expansion of $h(z)$ is, as expected,

$$c_0 = a_0 b_0 . \qquad (7)$$

Moving on to the nth degree coefficient, we derive in a similar way

$$\frac{1}{2\pi i} \oint \frac{f(z)g(z)\, dz}{(z - z_0)^{n+1}} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} a_k b_\ell \, \delta_{k+\ell,n} = \sum_{k=0}^{n} a_k b_{n-k} , \qquad (8)$$

Or, more explicitly,

$$c_n(z_0) = \sum_{k=0}^{n} a_k(z_0)\, b_{n-k}(z_0) , \quad n = 0,1,2,\dots \qquad (9)$$

It is important to point out that in the derivation of the recursion relation for the coefficients, Eq. (9), we have dealt with three series that are all expanded about the same non-singular offset point z_0. The product expansion converges within its own radius of convergence, R_c, specified by Eq. (3) and centered at z_0.

4. Reciprocal of a Taylor series:

We next consider the reciprocal of $f(z)$

$$f^{-1}(z) = \sum_{\ell=0}^{\infty} b_\ell (z - z_0)^\ell . \qquad (10)$$

Since the identity

$$\frac{1}{2\pi i} \oint \frac{f(z)f^{-1}(z)\, dz}{(z - z_0)^n} = \frac{1}{2\pi i} \oint \frac{dz}{(z - z_0)^n} = \delta_{n,1}, \ n \ge 1, \qquad (11)$$

2

is valid, it follows that

$$\frac{1}{2\pi i}\oint \frac{dz\, f(z)}{(z-z_0)^n}\sum_{\ell=0}^{\infty}b_\ell(z-z_0)^\ell = \delta_{n,1}.$$ (12)

Therefore,

$$\sum_{\ell=0}^{n-1}b_\ell\, a_{n-\ell-1} = \delta_{n,1}.$$ (13)

This recursion relation specifies that the successive values of the coefficients for the reciprocal of Eq. (1) are

$$b_0 = 1/a_0, \quad a_0 \neq 0,$$ (14a)

and

$$b_n = -\frac{1}{a_0}\sum_{\ell=1}^{n}b_{n-\ell}\, a_\ell, \quad n \geq 1.$$ (14b)

5. A Taylor series taken to a power:

Let us examine the non-binary operation of exponentiation,

$$g(z) = \left[f(z)\right]^\alpha,$$ (15)

or

$$\sum_{k=0}^{\infty}b_k(z-z_0)^k = \left[\sum_{n=0}^{\infty}a_n(z-z_0)^n\right]^\alpha, \quad a_0 \neq 0.$$ (16)

The zero degree coefficient for $g(z)$ is evidently

$$b_0 = g(z_0) = \left[a_0\right]^\alpha.$$ (17)

Differentiating Eq. (15) with respect to z gives

$$f(z)g'(z) = \alpha\, g(z)f'(z),$$ (18)

and, as a result,

$$f(z)\sum_{k=1}^{\infty}k\, b_k(z-z_0)^{k-1} = \alpha g(z)\sum_{k=1}^{\infty}k\, a_k(z-z_0)^{k-1}.$$ (19)

A division by the quantity $(z-z_0)^\ell$ followed by a contour integration leads to the equality

$$\sum_{k=1}^{\infty}k b_k \oint \frac{f(z)dz}{(z-z_0)^{\ell-k+1}} = \alpha\sum_{k=1}^{\infty}k a_k \oint \frac{g(z)dz}{(z-z_0)^{\ell-k+1}}.$$ (20)

Whence, the recursion relationship assumes the form

$$\sum_{k=1}^{\ell}k b_k\, a_{\ell-k} = \alpha\sum_{k=1}^{\ell}k a_k\, b_{\ell-k},$$ (21)

3

with the result that

$$b_1 a_0 = \alpha a_1 b_0 , \quad \ell = 1 , \tag{22a}$$

and, for $\ell \geq 2$

$$b_\ell a_0 = \alpha a_\ell b_0 + \sum_{k=1}^{\ell-1} \frac{(\ell-k)\alpha - k}{\ell} a_{\ell-k} b_k . \tag{22b}$$

We observe that since the exponent α has not been restricted to integers, the coefficient b_0, Eq. (17), may be multivalued and corresponds to different elements of the set of Taylor series, each equally valid.

6. Reversion of a Taylor series:

The series in Eq. (1) converges over a region in the complex z plane centered at $z = z_0$. Instead, the reversion of $f(z)$ will be a series in the complex f plane centered at the offset $a_0 = f(z_0)$. Accordingly, the reversion series is expressed as

$$z(f) = \sum_{n=0}^{\infty} b_n(a_0)(f - a_0)^n , \tag{23}$$

with

$$z(a_0) = b_0(a_0) = z_0 . \tag{24}$$

We note that this series is expanded about the point a_0, and therefore it does not belong to our original set expanded about z_0. Eq. (23) is rewritten in the convenient expanded form

$$z(f) = z_0 + \sum_{n=1}^{\infty} b_n(a_0)(f - a_0)^n \tag{25}$$

Our task at this point is that of expressing the $b_n(a_0)$ coefficients in terms of the $a_n(z_0)$ coefficients of Eq. (1). To accomplish this end, we write

$$f - a_0 = a_1(z - z_0)g(z) , \tag{26}$$

where the function

$$g(z) = \sum_{k=0}^{\infty} A_k (z - z_0)^k , \tag{27}$$

has the coefficients

$$A_0 = 1; \quad A_k = a_{k+1}/a_1, \quad k \geq 1 . \tag{28}$$

Since the differential relation

$$z'(f) df = dz \tag{29}$$

holds, it is evident that

$$\frac{z'(f)\,df}{(f-a_0)^\ell} = \frac{dz}{(f-a_0)^\ell}. \tag{30}$$

Integration of both sides of Eq. (30) about the corresponding points a_0 and z_0 results directly in

$$\oint \frac{z'(f)\,df}{(f-a_0)^\ell} = \oint \frac{dz}{a_1^{\ell}(z-z_0)^\ell} \left[g(z) \right]^{-\ell}. \tag{31}$$

When we develop that last factor in Eq. (31) into a Taylor series

$$\left[g(z) \right]^{-\ell} = \sum_{k=0}^{\infty} B(\ell,k)(z-z_0)^k, \tag{32}$$

and realize that the left-hand side of Eq. (31) is the Taylor coefficient for the derivative of $z(f)$, we arrive at the equation

$$\ell\, b_\ell = B(\ell, \ell-1)\big/ a_1^{\ell}. \tag{33}$$

Because Eq. (32) is itself the $(-\ell)$-th power of a Taylor series, we can explicitly, per Eq. (22), write $B(\ell,k)$ as the sequence:

$$B(\ell,0) = 1, \tag{34a}$$

$$B(\ell,1) = -\ell \left(\frac{a_2}{a_1} \right), \tag{34b}$$

$$\vdots$$

$$B(\ell,k) = -\ell \left(\frac{a_{k+1}}{a_1} \right) - \sum_{j=1}^{k-1} \frac{(k-j)\ell + j}{k} \left(\frac{a_{k-j+1}}{a_1} \right) B(\ell,j), \tag{34c}$$

$$\vdots$$

$$B(\ell,\ell-1) = -\ell \left(\frac{a_\ell}{a_1} \right) - \sum_{j=1}^{\ell-2} (\ell-j) \left(\frac{a_{\ell-j}}{a_1} \right) B(\ell,j). \tag{34d}$$

In summary, the reversion series is Eq. (25) with the $b_n(a_0)$ coefficients dependent upon the original [see Eq. (1)] $a_n(z_0)$ coefficients through Eq. (34d). Briefly then

$$z(f) = z_0 + \sum_{\ell=1}^{\infty} \frac{B(\ell,\ell-1)}{\ell \left[a_1(z_0) \right]^\ell} \left[f - a_0(z_0) \right]^\ell. \tag{35}$$

We have not found in the literature the recursion relations for the reversion of a Taylor series. This observation is consistent with the statements made by K. Knopp in his two texts [4,5] that the recursion relations are unknown.

7. Root of a polynomial of arbitrary degree:

A polynomial is a Taylor series characterized by $a_n(z_0) = 0$ above some term of degree N, that is

$$f(z) = \sum_{n=0}^{N} a_n(z_0)(z - z_0)^n, \quad a_0(z_0) \neq 0. \tag{36}$$

In agreement with the previous development, such a series has a corresponding reversion series of the form of Eq. (35). Under the condition that the value of the polynomial vanishes $f = 0$, Eq. (35) becomes an explicit expression for the root, $z(0)$,

$$z(0) = z_0 + \sum_{\ell=1}^{\infty} \frac{B(\ell, \ell-1)}{\ell} \left[-\frac{a_0(z_0)}{a_1(z_0)} \right]^{\ell} \tag{37}$$

It is important to mention that the right-hand side of this formula depends only on the value of the offset z_0. It will be the root $z(0)$ when the value, $f = 0$, is within the circle of convergence of the reversion series, Eq. (35), centered at $a_0(z_0)$ in the complex f plane.

It behooves us, in view of this situation, to locate an appropriate offset z_0 in the complex z plane. We therefore give, in the following, one way of doing this. To start, let us assume that we are initially given a polynomial having N distinct roots distributed arbitrarily in the complex z plane about the origin, $z_0 = 0$, of the coordinate system. Therefore

$$f(z) = \sum_{n=0}^{N} a_n(0) z^n, \quad a_0(0) \neq 0. \tag{38}$$

We now carry out the following three numerical manipulations:

1. With the aid of Eqs. (14a,b), we calculate the coefficients of the reciprocal of the given polynomial, Eq. (38).

2. Applying Eq. (3) up to some high-degree coefficient, we derive an estimate of the radius of convergence, R_b, of this reciprocal series.

3. We now numerically sweep at small angular intervals along this approximate circle of convergence searching for a complex value of z for which the initial polynomial, Eq. (38), takes on its smallest absolute value. This value of z is the complex offset z_0 that we seek.

At this juncture in our discussion we must emphasize that the successful outcome of our 3-step algorithm is that it give a value of z_0 such that the circle of convergence of Eq. (35) contains the value $f = 0$. Subject to this condition, the particular complex value of z_0 is not critical. When the initial polynomial, Eq. (38), is displaced by this value of the offset z_0, it becomes

$$f(z) = \sum_{n=0}^{N} a_n(z_0)(z - z_0)^n. \tag{39}$$

The coefficients $a_n(z_0)$ that appear in this displaced form of the polynomial are the ones we insert in Eq. (37), and thereby calculate $z(0)$, a root of Eq. (38).

8. Summary and Conclusions:

We have presented the basic elements of the algebra of Taylor series and verified that for the binary operations of addition and multiplication the set of Taylor series expanded about a point constitutes a field over the field of complex numbers. For the non-binary operations of exponentiation and reversion we have derived the appropriate recursion relations for the expansion coefficients. Section 7 has dealt with the problem of solving for a root of a general polynomial. The remaining roots are found by following, for each of them, the same procedure of locating the pertinent offset with the help of the 3-step algorithm, and then calculating the nearby root employing Eq. (37). As a check on the validity and generality of these ideas, we have written a computer program and successfully solved for the numerical roots of a large variety of polynomials of many degrees (up to $N = 17$). The accuracy of these calculated roots has been equal to the expected machine precision of about one part in 10^{15}. In these exercises, polynomials with roots of multiplicity greater than one have been reduced to equivalent ones that have the same roots but with a multiplicity of one. The procedure for carrying out this operation is clearly explained in reference [6].

In conclusion, we can state that Eq. (37), in conjunction with a method of locating an appropriate offset z_0, is an answer to the long-standing problem of finding an expression for the exact roots of a general polynomial of arbitrary degree. This statement does not contradict the "Impossibility Theorem" of Abel [7], since his landmark theorem addresses itself only to the specific question of which polynomials are solvable algebraically by radicals [2,8,9]. As a final remark, we would like to add that by "an exact root" we mean one that is obtainable directly from a formula, like Eq. (37), without employing successive approximations or iterations.

Acknowledgements:

References:

[1] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, (Cambridge University Press, American Edition, 1948.)

[2] John E. Maxfield and Margaret W. Maxfield, *Abstract Algebra and Solution by Radicals*, (Dover Publications, Inc., N.Y., 1971.)

[3] P. Henrici, *Applied and Computational Analysis*, Vol. 1, (John Wiley and Sons, Inc. 1974.)

[4] K. Knopp, *Infinite Sequences and Series*, p. 122, footnote 2, (Dover Publications, N.Y., 1956.)

[5] K. Knopp, *Theory and Application of Infinite Series*, p. 187, (Dover Publications, N.Y., 1990.)

[6] A. P. Mishina and I. V. Proskuryakov, *Higher Algebra*, p. 162. (Pergamon Press, First English Edition, 1965.)

[7] N. H. Abel, *Oeuvres Complètes*, 1881 Tome Second, Memoire VII, pp. 66-94. (Johnson Reprint Corp, N.Y., 1995).

[8] A. Cayley, '*Equation*', in *Encyclopedia Britannica*, 11 Edition, Vol. 9, p. 709 (1911)

[9] R. G. Ayoub, *American Mathematical Monthly* 89, 397 (1982).