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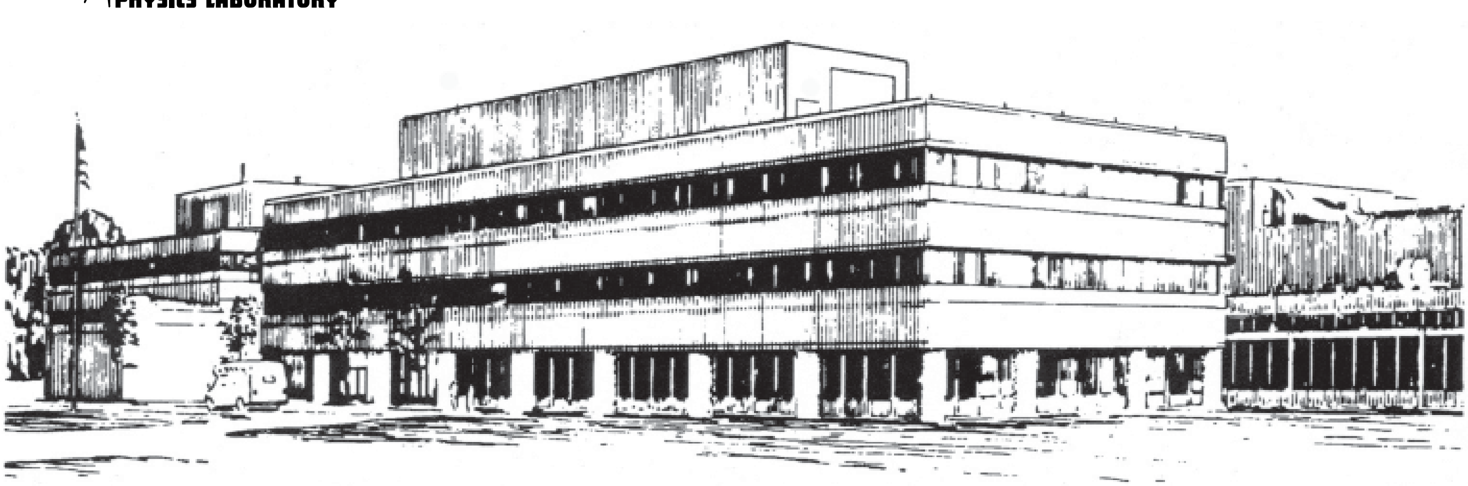

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Lagrangian Description of Nonadiabatic Particle Motion in Spherical Tori

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1. Introduction. The ability of a device to provide adiabatic motion of charged particles is crucial for magnetic confinement. As the magnetic field in the present-day spherical tori, e.g., MAST and NSTX, is much lower than in the conventional tokamaks, effects of the finite Larmor radius (FLR) on the motion of fast ions are of importance in these devices, affecting the stochasticity threshold for the interaction of the ions with electromagnetic perturbations. In addition, FLR by itself may result in non-conservation (jumps) of the magnetic moment of particles [4]. In this work we propose a Lagrangian approach to description of the resonant collisionless motion of charged particles under a perturbation, allowing for FLR. The work generalizes results of Ref. [1], where only time-independent perturbations were considered. The approach is used to find the stochasticity thresholds for the Goldston–White–Boozer (GWB) diffusion [2] and the cyclotron-resonance-induced (CRI) diffusion (for the case of the first cyclotron resonance, the latter was discovered in Ref. [3]). In addition, a new expression for the magnetic moment variation caused by FLR is found.

2. Perturbative approach. We present the exact Lagrangian of the particle motion, $\mathcal{L}$, as the sum of a Lagrangian of an integrable system and the residual considered as a perturbation. As the integrable system, we take the lowest-order Lagrangian of the guiding-center motion [5] in the axisymmetric magnetic field $\mathbf{B}$,

$$\mathcal{L}_{gc}(\mathbf{X}, \mathcal{E}, \mu, \dot{\mathbf{X}}, \dot{\Phi}) = \frac{e}{c}\mathbf{A}(\mathbf{X}) \cdot \dot{\mathbf{X}} + \frac{Mv_\parallel}{B}\mathbf{B}(\mathbf{X}) \cdot \dot{\mathbf{X}} + \frac{Mc}{e}\mu\dot{\Phi} - \mathcal{E}, \tag{1}$$

where $\mathbf{X}$ is the guiding center position; e, M, $\mu = Mv_\perp^2/[2\mathbf{B}(\mathbf{X})]$, and $\mathcal{E}$ are the charge, mass, magnetic moment, and energy of the particle; Φ is the gyrophase; $\mathbf{A}$ is the vector potential; $v_\parallel = \{2[\mathcal{E} - \mu B(\mathbf{X})]/M\}^{1/2}$. The guiding center position, $\mathbf{X}$, and the particle position, $\mathbf{x}$, are connected to the considered order by the relation $\mathbf{x} = \mathbf{X} + \rho_\perp \hat{\mathbf{a}}$, where $\rho_\perp = v_\perp/\omega_B$, $\omega_B = eB/(Mc)$, $\hat{\mathbf{a}} = \cos(\Phi)\hat{\mathbf{e}}_1(\mathbf{X}) - \sin(\Phi)\hat{\mathbf{e}}_2(\mathbf{X})$, $-\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are the binormal and the normal to the magnetic field line, respectively. The Lagrangian given by Eq. (1) indeed describes an integrable system, possessing three constants of motion: $\mathcal{E}$, the canonical angular momemtum (J_3), and μ. Then we present $\mathcal{L}$ as follows:

$$\mathcal{L} = \mathcal{L}_{gc} + \underbrace{(\mathcal{L}_{as} - \mathcal{L}_{gc})}_{\delta\mathcal{L}_1} + \underbrace{(\mathcal{L} - \mathcal{L}_{as})}_{\delta\mathcal{L}_2} = \mathcal{L}_{gc} + \delta\mathcal{L}_1 + \delta\mathcal{L}_2, \tag{2}$$

where

$$\mathcal{L}(\mathbf{x},\mathbf{v},\dot{\mathbf{x}}) = \frac{e}{c}\left[\mathbf{A}(\mathbf{x}) + \tilde{\mathbf{A}}(\mathbf{x})\right]\cdot\dot{\mathbf{x}} + M\mathbf{v}\cdot\dot{\mathbf{x}} - \frac{M}{2}v^2 - e\tilde{\phi}, \tag{3}$$

$$\mathcal{L}_{as}(\mathbf{x},\mathbf{v}) = \frac{e}{c}\mathbf{A}(\mathbf{x})\cdot\dot{\mathbf{x}} + M\mathbf{v}\cdot\dot{\mathbf{x}} - \frac{M}{2}v^2, \tag{4}$$

is the Lagrangian of the particle motion in the axisymmetric field; $\mathbf{v}$ is the particle velocity; ϕ is the scalar potential of the electromagnetic field; tildes refer to the electromagnetic perturbation. We will consider both $\delta\mathcal{L}_1$ and $\delta\mathcal{L}_2$ as perturbations, the former being responsible for all nonadiabatic effects in the axisymmetric field, and the latter describing the effect of the electromagnetic perturbation.

3. Nonconservation of the magnetic moment. To derive an equation for $\dot{\mu}$ in the axisymmetric field, we set $\delta\mathcal{L}_2 = 0$ and obtain from the Euler-Lagrange equations [1]:

$$\dot{\mu} = \frac{e}{Mc}\left[\frac{\partial\delta\mathcal{L}_1}{\partial\Phi} - \frac{d}{dt}\left(\frac{\partial\delta\mathcal{L}_1}{\partial\dot{\Phi}}\right)\right] = M\frac{v_\parallel^2 v_\perp}{BR_c}\cos\Phi, \tag{5}$$

where R_c is the curvature radius of the magnetic field line, and we have omitted terms $\propto \exp(2i\Phi)$. To compare Eq. (5) with previous results, one should have in mind that here μ is defined in terms of the magnetic field at the guiding center. Taking the magnetic field at the particle position, as in most previous works, we find:

$$\frac{d}{dt}\left(\frac{Mv_\perp^2}{2B(\mathbf{x})}\right) = \dot{\mu} + M\frac{v_\perp^3}{2BR_c}\cos\Phi - M\frac{2\pi v_\perp^3}{B^3}\left[(\hat{\mathbf{e}}_2\cdot\nabla p)\cos\Phi + (\hat{\mathbf{e}}_1\cdot\nabla p)\sin\Phi\right]. \tag{6}$$

The first two terms of Eq. (6) yield the equation for $\delta\mu$ of Refs. [6,7]. Thus, Eq. (5) differs from the mentioned previous formulae by allowing for effects of plasma pressure. Introducing a curvature-like parameter $8\pi B^{-2}\nabla p$, we could obtain a formula for $\dot{\mu}$ similar to that derived in Ref. [8]. However, the equation for $\dot{\mu}$ given by Eq. (5) is much simpler, while correctly describing the mentioned effects.

4. Lagrangian description of resonant motion. In the action-angle variables the Lagrangian $\mathcal{L}_{gc}$ takes the form

$$\mathcal{L}_{gc} = J_1\dot{\varsigma}_1 + J_2\dot{\varsigma}_2 + J_3\dot{\varsigma}_3 - \mathcal{E}, \tag{7}$$

where $J_1 = \mu Mc/e$; J_3 is the canonical angular momentum;

$$J_2(J_1, J_3, \mathcal{E}) = \frac{1}{2\pi}\oint d\mathbf{X}\cdot\left[\frac{e}{c}\mathbf{A}(\mathbf{X}) + \frac{Mv_\parallel}{B}\mathbf{B}(\mathbf{X})\right]; \tag{8}$$

ς_i, $i = 1,2,3$, are the angle variables canonically conjugate to J_i and characterizing the gyrophase, the bounce phase, and the toroidal angle, respectively; the integral in Eq. (8) is taken along a poloidal contour that lies on an invariant torus of the system.

Considering the motion near the resonance $l\omega_1 - s\omega_2 - N\omega_3 + \Omega = 0$, where $\omega_i = \dot{\varsigma}_i$, Ω is the perturbation frequency, we divide the phase-space variables in the vicinity of the resonance into "slow" variables (J_1, J_2, J_3, and $l\varsigma_1 - s\varsigma_2 - N\varsigma_3 + \Omega t$) and "fast" ones (two other independent angular coordinates) and average the Lagrangian over the fast variables. Hence, we introduce the new action-angle variables $(I_1, I_2, I_3; \alpha_1, \alpha_2, \alpha_3)$ with $\alpha_1 = l\varsigma_1 - s\varsigma_2 - N\varsigma_3 + \Omega t$, $\alpha_2 = \varsigma_2$, $\alpha_3 = \varsigma_1$. Applying the averaging procedure similar to that described in Ref. [1] to the exact Lagrangian, we obtain

$$\mathcal{L} = \bar{I}_1\dot{\bar{\alpha}}_1 + I(\bar{\alpha}_1, \bar{I}_1, \bar{I}_3, \mathcal{E})\dot{\bar{\alpha}}_2 + \bar{I}_3\dot{\bar{\alpha}}_3 - H, \tag{9}$$

where bars refer to the new variables modified by the averaging;

$$I(\bar{\alpha}_1, \bar{I}_1, \bar{I}_3, \mathcal{E}) = I_2(\bar{I}_1, \bar{I}_3, \mathcal{E}) + \tilde{I}(\bar{\alpha}_1, \bar{I}_1, \bar{I}_3, \mathcal{E}), \tag{10}$$

is the adiabatic invariant of the resonant motion; $I_2 = J_2 - (s/N)J_3$; $H = \mathcal{E} - J_3/N$;

$$\tilde{I} = \frac{\Omega}{8\pi^3 N^3}\int_{-\pi n_3}^{\pi n_3} d\varsigma_{10}\int_{-\pi N/\Omega}^{\pi N/\Omega} dt_0 \int_{L(\varsigma_{10}, t_0, \alpha_1, P, \mu, \mathcal{E})}\left[d\mathbf{x}\cdot\frac{e}{c}\tilde{\mathbf{A}}(\mathbf{x}) - dt\, e\tilde{\phi}\right]; \tag{11}$$

$L(\varsigma_{10}, t_0\alpha_1, J_3, \mu, \mathcal{E})$ is the curve in the phase space determined by the equations $-\pi N \leq \varsigma_2 \leq \pi N$, $t = t_0 + \varsigma_2/\omega_2$, $\varsigma_1 = \varsigma_{10} + (\omega_1/\omega_2)\varsigma_2$, $\varsigma_3 = (l\varsigma_1 - s\varsigma_2 + \Omega t - \alpha_1)/N$, $J_3 = \text{const}$, $\mu = \text{const}$, $\mathcal{E} = \text{const}$. The adiabatic invariant I can be used to determine the width of resonance islands and evaluate the threshold amplitude of the perturbation at which the transition to global stochasticity takes place due to the overlap of the islands.

5. Stochasticity threshold for the motion in a rippled magnetic field. When the perturbation in Eq. (11) is the magnetic field ripple, the integrand is typically a rapidly oscillating function. It is known that the main contribution to such integrals comes from the vicinities of the points of stationary phase, which at the exact global resonance coincide with the points of "local" resonance, $l\omega_B = N\dot{\varphi}$, where N is the number of the toroidal field coils, φ is the toroidal angle (one should distinguish the local resonance from the "global" one, $l\omega_1 - s\omega_2 - N\omega_3 + \Omega = 0$). Typically, there are two points of the local resonance for a given l on a particle orbit. Then, in the simplest case of up-down symmetry, we obtain [1]:

$$I = \frac{N}{2\pi}\,\varphi_p'\big|_{I_1 = I_{1r}}(I_1 - I_{1r})^2 + \frac{1}{\pi}\delta I_1(I_1, I_3, \mathcal{E})\cos\left[\frac{N}{2}\varphi_b(I_1, I_3, \mathcal{E})\right]\cos\alpha_1, \tag{12}$$

where "r" refers to the value at the global resonance, prime denotes $\partial/\partial I_1 = -N\partial/\partial J_3 + (le/Mc)\partial/\partial\mu$, δI_1 characterizes the jump of I_1 for one local resonance event, φ_p and φ_b, the unidirectional and alternate parts of the perturbation phase variation between the events. We observe that the mathematical description of the resonant motion for different l is rather similar. The transition to chaos for the GWB diffusion (l=0) was studied in Ref. [9]. The transition occurs in two different ways, depending on the parameter $Q = \phi_b'(N\delta I_1/\phi_p')^{1/2}$. When Q is small, the level contours of I describe a conventional Chirikov island with the width $\propto \delta I_1^{1/2}$. When Q exceeds ~ 2, the contours show chains of islands near the resonance, the width of the chain being $\propto \delta I_1$.

Using the mentioned similarity, we generalize the known results for $l = 0$ to arbitrary l and obtain the following stochasticity criterion for the field ripple of the form $\delta_{rip} = \delta_0 \exp(R/L_{rip})$, where R is the distance to the axis of symmetry, δ_0 is a constant:

$$\bar{\delta}_{rip} > \delta_{crit} \equiv \frac{(2|\Xi|)^{1/2} c}{(\pi N)^{1/2} e \bar{B} v_\perp L_{rip} a_l(\rho_\perp / L_{rip}) (|\varphi_p'| + |\varphi_b'|)}, \tag{13}$$

where all quantities are taken at the local resonance point, bars refer to the guiding center, $a_l(x) = [\mathsf{I}_{|l-1|})(x) + \mathsf{I}_{|l+1|}(x)]/2$, $\mathsf{I}_j(x)$ are modified Bessel functions,

$$\Xi = \frac{v^2}{2qR^2} \frac{\partial \ln B}{\partial \theta} \left(1 + \frac{l^2 R^2}{N^2 \rho^2}\right) + \frac{l \omega_B}{N} \frac{d \ln R}{dt}. \tag{14}$$

θ is the poloidal angle. This criterion agrees with previously obtained criteria for the GWB diffusion ($l = 0$) for $\rho \ll L_{rip}$ but allows for the effect of FLR. For instance, this effect results in a decrease of δ_{crit} by a factor of 1.5 for $\rho_\perp / L_{rip} = 2$, as is the case for 80-keV deuterons in NSTX. Equation (13) may not agree with previously obtained stochasticity thresholds for the CRI diffusion, especially for large magnitudes of $N\rho/R$, because the previous results were obtained without taking account of jumps in J_3.

6. Summary. A Lagrangian formalism for the description of the nonadiabatic effects on the particle motion is developed. Based on this formalism, a general expression for the critical ripple amplitude is obtained, which for the GWB diffusion of particles with $\rho \ll L_{rip}$ agrees with previous results. In addition, a formula for the variation of the magnetic moment due to FLR is derived. The formula agrees with that obtained by a different procedure in Refs. [6,7] but takes into account effects of the plasma pressure and has a simpler form.

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