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Single Axioms for Boolean Algebra

by

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Contents

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Abstract

Explicit single axioms are presented for Boolean algebra in terms of (1) the Sheffer stroke; (2) disjunction and negation; (3) disjunction, conjunction, and negation; and (4) disjunction, conjunction, negation, 0, and 1. It was previously known that single axioms exist for these systems, but the procedures to generate them are exponential, producing huge equations. Automated deduction techniques were applied to find axioms of lengths 105, 131, 111, and 127, respectively, each with six variables.

1 Introduction

This article is about finding short single equational axioms for Boolean algebra under a variety of treatments. In 1973, Padmanabhan and Quackenbush presented a method for constructing a single axiom for any finitely based theory that has certain distributive and permutable congruences [9]. Boolean algebra has these properties, so it has been known since then that single axioms exist. However, straightforward application of the method usually yields single axioms of enormous length, sometimes with tens of millions of symbols. Our goal was to find single axioms of reasonable length. We used a variety of automated deduction techniques to search for shorter single axioms.

Boolean algebra can be axiomatized with various sets of operations, including several subsets of the standard operations of disjunction, conjunction, negation, 0, and 1 $\{+, \cdot, ', 0, 1\}$. Boolean algebra can also be presented with just one binary operation, the Sheffer stroke, $x|y = x' \cdot y'$, and this was our primary interest. We also considered $\{+, '\}$ (the operations of the Robbins problem [3]), $\{+, \cdot, '\}$, and the full set of standard operations, $\{+, \cdot, ', 0, 1\}$.

Otter [2, 1] is an automated deduction system for first-order logic with equality. We used the equational capabilities of Otter extensively and in several ways. First, we used Otter to search for proofs of equational conjectures. These conjectures were either set up by hand or run automatically by a driver program Otter-loop (see the next paragraph). Second, we used Otter as a simple rewrite system to transform equations; for example, to expand with definitions or to simplify. Third, we used it to generate large numbers of consequences from a theory to serve as candidates when looking for equations with particular properties. Finally, we used it to generate variants of equations to serve as candidates for various operations, for example, to insert terms at all possible positions of a given equation.

Otter-loop is a program that automatically runs a sequence of closely related Otter jobs. The user supplies a fixed search strategy and a fixed set of clauses (axioms and/or goals) along with a set of candidate clauses; a separate Otter job is run with each candidate clause by appending it to the fixed set of clauses. Otter-loop was used, for example, to search for single axiom schemata and to search for an equation that, when added to a given set of equations, forms a basis for Boolean algebra.

This article has a companion page on the World Wide Web, `http://www.mcs.anl.gov/~mccune/ba-axioms`. That web page contains links to Otter input files and other data files related to the work presented here. In this article we refer to those files with bold-faced underlined pseudolinks **like this**.

2 Single Axioms via Pixley Terms

DEFINITION 1 A *Pixley term* is a term $p(x, y, z)$ with the properties

$$p(y, y, x) = x, \quad p(x, y, x) = x, \quad p(x, y, y) = x. \qquad \text{(Pixley)}$$

Boolean algebra admits Pixley terms, for example,

$$p(x, y, z) = (x \cdot y') + ((x \cdot z) + (y' \cdot z)). \tag{Pixley-1}$$

Pixley terms are sometimes called *two-thirds minority, one-third majority terms*.

DEFINITION 2 An *absorption equation* has the form *term* $=$ *variable*; that is, it has a variable on one side.

In [9], Padmanabhan and Quackenbush show that one can construct a single axiom for any finitely based equational theory that admits Pixley terms. The general idea of the procedure is to replace nonabsorption equations with absorption equations and to replace pairs of absorption equations with single nonabsorption equations. Iterating in this way, we can capture all of the theory, except possibly the Pixley properties, in a single absorption equation. Finally, we can apply a schema to obtain a single axiom for the theory.

The Reduction Procedure. The following two steps are applied (nondeterministically) until neither one applies.

1. Let $\alpha = \beta$ be a nonabsorption equation that does not contain the variable x. Then, given the Pixley properties, $p(x, \alpha, \beta) = x$ is equivalent to $\alpha = \beta$. So we can replace $\alpha = \beta$ with $p(x, \alpha, \beta) = x$.

2. Let $g(x) = x$ and $h(y) = y$ be a pair of absorption equations that do not contain the variable z. Then, given the Pixley properties, $p(z, g(x), x) = p(z, h(y), y)$ is equivalent to the pair, so we can replace the pair with this new equation.

If we keep applying these two steps, the procedure terminates with exactly one absorption equation, say $f(x) = x$, such that the set of four equations

$$p(y, y, x) = x, \quad p(x, y, x) = x, \quad p(x, y, y) = x, \quad f(x) = x \tag{Basis-1}$$

is equivalent to the original set of equations.

The following theorem shows that we can replace that set of four equations with a single axiom for the theory.

THEOREM 1 (Padmanabhan-Quackenbush [9])

$$p(p(x, x, y), p(f(z), u, z), z) = y \Leftrightarrow \left\{ \begin{array}{l} p(y, y, x) = x \\ p(x, y, x) = x \\ p(x, y, y) = x \\ f(x) = x \end{array} \right\}. \tag{Schema-1}$$

There is no need to expand the Pixley terms to prove this, and the proof is trivial for Otter. The Otter input files for the two directions of the equivalence are **schema-1a.in** and **schema-1b.in**.

If we take the schema, expand the $f(z)$ term with the constructed absorption equation, and expand all of the p terms (with any Pixley term), we obtain a single axiom for the theory.

3 A Single Axiom for the Sheffer Stroke

This section contains a straightforward application of the reduction procedure. We start with the following well-known Sheffer stroke 2-basis of Meredith [7].

$$(x|x)|(y|x) = x \tag{Meredith-1}$$

$$(x|(y|(x|z))) = (((z|y)|y)|x) \tag{Meredith-2}$$

If we rewrite the Boolean algebra Pixley term

$$p(x, y, z) = (x \cdot y') + (x \cdot z) + (y' \cdot z) \tag{Pixley-1}$$

in terms of the Sheffer stroke (using rewrite rules $x' = (x|x)$, $x \cdot y = (x|x)|(y|y)$, and $x + y = (x|y)|(x|y)$; see Otter input file **pixley-2.in**), we obtain

$$\begin{aligned} p(x, y, z) = &(((x|x)|((y|y)|(y|y)))|((((x|x)|(z|z))|(((y|y)|(y|y))| \\ &(z|z)))|(((x|x)|(z|z))|(((y|y)|(y|y))|(z|z)))))| \\ &(((x|x)|((y|y)|(y|y)))|((((x|x)|(z|z))|(((y|y)|(y|y))| \\ &(z|z)))|(((x|x)|(z|z))|(((y|y)|(y|y))|(z|z))))). \end{aligned} \tag{Pixley-2}$$

Using this Pixley term, if we apply Step 1 of the reduction procedure to (Meredith-2), we obtain an absorption equation of length 345. Then if we apply Step 2 to the pair of absorption equations, we get a nonabsorption equation of length 8559. Applying Step 1 gives an absorption equation of length 138,969. Finally, we can apply (Schema-1) to obtain a single axiom of length 40,025,985. These lengths were calculated with Lisp programs which are available in file **lengths.lisp**.

4 Shorter Single Axioms for the Sheffer Stroke

Nothing in the reduction procedure depends on the syntactic structure of the Pixley term, so we are free to simplify it. All that matters is that it satisfies the three Pixley properties:

$$p(y, y, x) = x, \quad p(x, y, x) = x, \quad p(x, y, y) = x. \tag{Pixley}$$

We do not know whether a good (e.g., canonical) term rewriting system exists for simplifying Boolean algebra expressions in terms of the Sheffer stroke, so we set up an Otter job (input file **pixley-sheffer-shorter.in**) to search for short terms that are equal to Pixley-2. The search produced 30 Pixley terms of length 15 (available in file **pixley-sheffer-30**). Here are 6 of those 30 that we found useful.

$$p(x, y, z) = (x|(y|(z|z)))|((y|(y|x))|z) \tag{Pixley-3}$$
$$p(x, y, z) = (x|(y|(y|z)))|((y|(y|x))|z) \tag{Pixley-4}$$
$$p(x, y, z) = (x|(y|(z|z)))|(((x|x)|y)|z) \tag{Pixley-5}$$
$$p(x, y, z) = (x|(y|(y|z)))|(((x|x)|y)|z) \tag{Pixley-6}$$
$$p(x, y, z) = (x|(y|(x|z)))|(((x|x)|y)|z) \tag{Pixley-7}$$
$$p(x, y, z) = ((y|(x|z))|z)|(x|(y|(z|z))) \tag{Pixley-8}$$

Because our goal is shorter single axioms, we must consider the number of occurrences of individual variables in the Pixley terms. These six are representative of all the variable-occurrence combinations in the 30 short Pixley terms that we found.

Applying the reduction procedure (in the same way as in the preceding section) with (Pixley-3) through (Pixley-8) gives single axioms of lengths 3699, 4853, 2319, 3719, 2069, and 2161, respectively.

By symmetry in the first and third arguments of the Pixley properties, we are free to use $p(\alpha, \beta, x)$ instead of $p(x, \beta, \alpha)$ in Step 1 of the reduction procedure. Also, by symmetry of equality, we can swap α and β. By using these observations, we found a single axiom of length 1057 from Meredith's basis and (Pixley-8). This is much better than 40 million, but still not very satisfying.

4.1 Using Different Bases

We do not have to use Meredith's basis. Any Sheffer stroke basis for Boolean algebra is acceptable. A simpler one is the following 2-basis, conjectured by Stephen Wolfram [12] and proved to be a basis by Robert Veroff [11].

$$(x|y) = (y|x) \tag{Commutativity}$$
$$(x|z)|(x|(y|z)) = x. \tag{27a}$$

This gives a single axiom of length 833, using (Pixley-8).

Starting with a basis that has fewer nonabsorption equations should give us a head start in the reduction procedure, because we can skip some costly steps of replacing nonabsorption equations with absorption equations. We found that with the $\{$(27a),(Commutativity)$\}$ basis we can simply replace commutativity with a commuted variant of (Meredith-1).

THEOREM 2 *The equations*

$$(y|x)|(x|x) = x \tag{Meredith-1c}$$
$$(x|z)|(x|(y|z)) = x. \tag{27a}$$

are a basis for Boolean algebra in terms of the Sheffer stroke.

Proof. A simple calculation shows that the equations hold. For the other side of the proof, Otter easily derives commutativity (with input file **mccune.in**), giving us the $\{$(27a),(Commutativity)$\}$ basis.

This basis yields a single axiom of length 577, using (Pixley-8).

4.2 A Schema for Two Absorption Equations

Theorem 1 gives us a schema that applies to one absorption equation. If we can find an analogous schema that applies to two or more absorption equations, we can eliminate explosive steps in the reduction procedure. For two absorption equations, this amounts to finding an equation equivalent to the following set.

$$\begin{array}{ll} p(y,y,x) = x & f_1(x) = x \\ p(x,y,x) = x & f_2(x) = x \\ p(x,y,y) = x & \end{array} \tag{Basis-2}$$

To search for such an equation, we used Otter (with input file **schema-2-candidates.in**) to derive a large number of equational consequences of this set, then extracted the equations (1985 of them, in file **schema-2-candidates**) that contain exactly one occurrence of f_1 and exactly one occurrence of f_2. Then, we used the program Otter-loop (with input files **schema-2-head** and **schema-2-candidates**) to run a sequence of Otter jobs, one for each candidate, trying to prove the five equations listed above. Otter succeeded with one of the candidates, giving us the following theorem.

THEOREM 3

$$p(p(f_1(x), x, f_2(y)), y, p(z, u, p(z, v, v))) = z \Leftrightarrow \left\{ \begin{array}{l} p(y,y,x) = x \\ p(x,y,x) = x \\ p(x,y,y) = x \\ f_1(x) = x \\ f_2(x) = x \end{array} \right\}. \tag{Schema-2}$$

The proof is easy for Otter if the Pixley terms are not expanded. The Otter input files for the two directions of the equivalence are **schema-2a.in** and **schema-2b.in**.

With this new schema and the absorption 2-basis of Theorem 2, we do not need to apply Steps 1 or 2 of the reduction procedure; to construct a single axiom we simply plug in the two absorption equations for f_1 and f_2, then expand the schema with one of the Pixley terms. The shortest single axiom we could find with this method, length 185, comes from using (Pixley-4). The Otter input file to generate this axiom is **sheffer-185.in**.

4.3 Taking Advantage of the Pixley Properties

Consider the Pixley term

$$p(x,y,z) = ((y|(x|z))|z)|(x|(y|(z|z))) \tag{Pixley-8}$$

and the corresponding Pixley properties (which we obtain by taking the appropriate instances):

$$((x|(x|z))|z)|(x|(x|(z|z))) = z \tag{P1}$$
$$((y|(x|y))|y)|(x|(y|(y|y))) = x \tag{P2}$$
$$((y|(x|x))|x)|(x|(y|(x|x))) = x. \tag{P3}$$

If we start with a basis that includes the Pixley properties, we do not have to apply Steps 1 or 2 of the reduction procedure to the Pixley properties, because the reduction schema (e.g., Theorem 1 or Theorem 3) tells us we can always derive the Pixley properties. In particular, if we can find a basis consisting of {(P1),(P2),(P3)} and one additional absorption equation, we can immediately apply the schema of Theorem 1 to obtain a single axiom. Any such additional absorption equation, say $f(x) = x$, must contain at least three variables, because every basis for Boolean algebra must contain an equation with at least three variables, and each of the Pixley properties contains exactly two variables.

An obvious candidate is (27a), because it is short, has three variables, and needs only commutativity to form a basis. A second candidate is the following, also conjectured by Wolfram [12] to form a basis with commutativity which was later proved by Veroff [11]:

$$(x|y)|(x|(y|z)) = x \tag{26a}$$

We considered each of (27a) and (26a) with each of the 30 Pixley terms. [1] The 60 jobs were run automatically by otter-loop, with input files **sheffer-head** and **60-sheffer-base-candidates**

Success occurred with 41 of the candidates (in file **41-pixley-sheffer-bases**), including one that gives us the following theorem (proved with Otter input file **pixley-sheffer-base.in**).

THEOREM 4 *The four equations*

$$((x|(x|z))|z)|(x|(x|(z|z))) = z \tag{P1}$$
$$((y|(x|y))|y)|(x|(y|(y|y))) = x \tag{P2}$$
$$((y|(x|x))|x)|(x|(y|(x|x))) = x \tag{P3}$$
$$(x|z)|(x|(y|z)) = x. \tag{27a}$$

are a basis for Boolean algebra in terms of the Sheffer stroke.

The first three equations in this basis are the Pixley properties corresponding to the Pixley term (Pixley-8), so we can simply plug the fourth equation into (Schema-1) of Theorem 1 and expand with (Pixley-8) to obtain a single axiom. The Otter input file to generate the axiom is **sheffer-generate.in**. This gives us the following theorem.

THEOREM 5 *The equation*

$$\begin{aligned}&(((((x|(((y|z)|(y|(u|z)))|y))|y)|(((y|z)|(y|(u|z)))|(x|(y|\\&\quad y))))|((((v|(v|w))|w)|(v|(v|(w|w))))|y))|y)|((((v|(v|w))|\\&\quad w)|(v|(v|(w|w))))|((((x|(((y|z)|(y|(u|z)))|y))|y)|(((y|\\&\quad z)|(y|(u|z)))|(x|(y|y))))|(y|y))) = w\end{aligned} \tag{Sh}$$

is a single axiom for Boolean algebra in terms of the Sheffer stroke.

[1] We used all 30 of the Pixley terms instead of the 6 representing the variable-occurrence combinations, because we were searching for new bases rather than simply applying a reduction schema to look for short axioms.

This equation has length 105 and is the shortest single axiom we have found. To double check our reasoning in deriving this axiom, we have two Otter proofs: one showing that (Sh) holds for Boolean algebra in terms of the Sheffer stroke, and a second showing that Meredith's 2-basis (Meredith-1, Meredith-2) can be derived from (Sh). The input files that generate these proofs are **sheffer-105-sound.in** and **sheffer-105-complete.in**.

5 Single Axioms with Standard Operators

Now we consider the problem of finding short single axioms for Boolean algebra in terms of the ordinary operations of disjunction, conjunction, negation, 0, and 1. To search for axioms, we used methods similar to those that led to the shortest axioms for the Sheffer stroke; in particular, we took advantage of the Pixley properties by finding bases consisting of three Pixley properties and an additional absorption equation. For each of the sets of operations $\{+,'\}$, $\{+,\cdot,'\}$, and $\{+,\cdot,',0,1\}$, we give the (1) Pixley term, (2) the absorption equation that goes together with the corresponding Pixley properties to give a basis, and (3) the single axiom.

5.1 Disjunction and Negation

The Pixley term was obtained by rewriting the dual of (Pixley-1) with $x * y = (x' + y')'$, then simplifying with $x'' = x$:

$$p(x,y,z) = ((x+y')' + ((x+z)' + (y'+z)'))'. \qquad \text{(Pixley-9)}$$

The absorption equation was found by considering identities with variable pattern structure similar to the Sheffer identity (27a).

$$((((x+z)'+y)' + (y+x)')') = y. \qquad \text{(DN-absorb)}$$

The single axiom was obtained from (Schema-1) by using (Pixley-9) and (DN-absorb). The Otter input file is **dn-generate.in**.

THEOREM 6 *The equation*

$$\begin{aligned}&((((x+x')' + ((x+y)' + (x'+y)'))' + (((((z+u)'+v)' +\\ &\quad (v+z)')' + w')' + (((((z+u)'+v)' + (v+z)')' + v)' +\\ &\quad (w'+v)'))'')' + ((((x+x')' + ((x+y)' + (x'+y)'))' +\\ &\quad v)' + ((((((z+u)'+v)' + (v+z)')' + w')' + (((((z+u)' +\\ &\quad v)' + (v+z)')' + v)' + (w'+v)'))'' + v)'))' = y\end{aligned} \qquad \text{(DN)}$$

is a single axiom for Boolean algebra in terms of disjunction and negation.

Axiom (DN) has length 131. Theorem 6 was double checked by using Otter input files **dn-131-sound.in** and **dn-131-complete.in**.

5.2 Disjunction, Conjunction, and Negation

We use the standard Pixley term:

$$p(x,y,z) = (x \cdot y') + ((x \cdot z) + (y' \cdot z)). \qquad \text{(Pixley-1)}$$

The absorption equation is a commuted variant of (DN-absorb):

$$(((y + (z+x)')' + (y+x)')') = y. \qquad \text{(CDN-absorb)}$$

The single axiom was obtained from (Schema-1) by using (Pixley-1) and (CDN-absorb). The Otter input file is **cdn-generate.in**.

THEOREM 7 *The equation*

$$
\begin{aligned}
&(((x \cdot x') + ((x \cdot y) + (x' \cdot y))) \cdot ((((z + (u + v)')' + (z + v)')' \cdot w') + \\
&\quad ((((z + (u + v)')' + (z + v)')' \cdot z) + (w' \cdot z)))') + ((((x \cdot x') + \\
&\quad ((x \cdot y) + (x' \cdot y))) \cdot z) + (((((z + (u + v)')' + (z + v)')' \cdot w') + \\
&\quad ((((z + (u + v)')' + (z + v)')' \cdot z) + (w' \cdot z)))' \cdot z)) = y
\end{aligned}
\tag{CDN}
$$

is a single axiom for Boolean algebra in terms of conjunction, disjunction, and negation.

Axiom (CDN) has length 111. Theorem 7 was double checked by using Otter input files **cdn-111-sound.in** and **cdn-111-complete.in**.

5.3 Disjunction, Conjunction, Negation, 0, and 1

Again, we use the standard Pixley term:

$$p(x, y, z) = (x \cdot y') + ((x \cdot z) + (y' \cdot z)). \tag{Pixley-1}$$

The candidate absorption equations were obtained from (CDN-absorb) by using the identities $x + 0 = x$ and $x * 1 = x$ to insert one 0 and one 1 at all possible positions. An Otter-loop job found three promising candidates, and a separate Otter job showed the following one to be sufficient.

$$((((y + (z + x)')' + 0) \cdot 1) + (y + x)')' = y. \tag{CDN01-absorb}$$

The single axiom was obtained from (Schema-1) by using (Pixley-1) and (CDN01-absorb). The Otter input file is **cdn01-generate.in**.

THEOREM 8 *The equation*

$$
\begin{aligned}
&(((x \cdot x') + ((x \cdot y) + (x' \cdot y))) \cdot (((((((z + (u + v)')' + 0) \cdot 1) + \\
&\quad (z + v)')' \cdot w') + (((((((z + (u + v)')' + 0) \cdot 1) + (z + v)')' \cdot z) + \\
&\quad (w' \cdot z)))') + ((((x \cdot x') + ((x \cdot y) + (x' \cdot y))) \cdot z) + (((((((z + \\
&\quad (u + v)')' + 0) \cdot 1) + (z + v)')' \cdot w') + (((((((z + (u + v)')' + 0) \cdot \\
&\quad 1) + (z + v)')' \cdot z) + (w' \cdot z)))' \cdot z)) = y
\end{aligned}
\tag{CDN01}
$$

is a single axiom for Boolean algebra in terms of conjunction, disjunction, negation, 0, and 1.

Axiom (CDN01) has length 127. Theorem 8 was double checked by using Otter input files **cdn01-127-sound.in** and **cdn01-127-complete.in**.

6 Related Work

In [8], Padmanabhan and the author presented single axioms for Boolean algebra in terms of negation and a ternary operation f,

$$f(x, y, x) = (x \cdot y) + ((y \cdot z) + (z \cdot x)). \tag{TBA}$$

The system is often called ternary Boolean algebra (TBA). The single axioms were found by the same kind of Padmanabhan/Quackenbush reduction method with Pixley terms; it was relatively simple to find ones of reasonable length, because $f(x, y', z)$ is a Pixley term, and the TBA basis from which we were working includes two of the three Pixley properties (the third is dependent). The single axiom from straightforward reduction has length 34 and 9 variables:[8]

$$\begin{aligned} f(f(x,x',y), f(f(z,f(f(u,v,w),v_6,f(u,v,v_7))', \\ f(v,f(v_7,v_6,w),u)),v_8',z)',z) = y. \end{aligned} \tag{TBA-1}$$

Starting with this axiom and using automated reasoning techniques, we found simpler axioms; among the shortest is the following of length 26 with 7 variables.[8]

$$f(f(x,x',y), f(f(z,u,v),w,f(z,u,v_6))', f(u,f(v_6,w,v),z)) = y. \tag{TBA-2}$$

This is still the shortest single equational axiom known to us for Boolean algebra in any set of operations, although it has more variables (seven) than the ones presented in Sections 4 and 5 (six).

In [4, p. 169], the author and Padmanabhan refer to a single axiom for Boolean algebra in terms of $\{+,\cdot,',1\}$, of length 3183 with 15 variables, that arose during work on self-dual bases.

For Hilbert-style sentential logic (which uses modus ponens instead of equational inference), C. A. Meredith presents single axioms for three types of two-valued propositional calculus [6]. The axiom for implication/negation has length 22, for implication/0 has length 20, and for disjunction/negation has length 25. In [10], T. W. Scharle considers Hilbert-style propositional logic in terms of the Sheffer stroke. He proves that previously claimed axioms (all length 23) of Nicod, Łukasiewicz, and Wajsberg are in fact single axioms.

In [5] the author and Padmanabhan present a single equational axiom for lattice theory (in terms of meet and join) of length 79 with 7 variables. The method used to find the axiom is related to the one presented in this paper, but with majority polynomials, $m(x,x,y) = m(x,z,x) = m(w,x,x) = x$, rather than Pixley polynomials.

7 Further Work

The single axioms we have presented, in terms of $\{|\}$, $\{+,'\}$, $\{+,\cdot,'\}$, and $\{+,\cdot,',0,1\}$, are the shortest we know of for these sets of operations. Each axiom has six variables. Are there shorter (equational) axioms for these systems? Are there (equational) axioms with fewer variables for any system of Boolean algebra? Given a short Hilbert-style single axioms and its completeness proof, is there a method to help build a short single equational axiom for the corresponding equational system?

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