Chord Distributions of a Spherical Shell

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CHORD DISTRIBUTIONS OF A SPHERICAL SHELL

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1. Introduction. Many years ago, Dirac [1], [2] derived an approximation for α, the eigenvalue of the Neutron Transport Equation which is physically the criticality of a chunk of nuclear material, in terms of the distribution of the chord lengths in the chunk. The criticality of an assemblage of nuclear material is linked to the chord lengths of the assembly, because the net number of secondary neutrons which is produced in a collision by a primary neutron with an atom of the fissionable medium is proportional to the number of mean free paths that the primary neutron can travel in the assembly. Dirac derived the chord length distributions for three convex solids; an oblate ellipsoid, a hemi-sphere, and a hemi-ellipsoid.

In this paper, we derive the distributions for two kinds of chords for a spherical shell. The first kind are truncated chords which are drawn in the left figure of Fiq. 1. The second kind are complete chords which are drawn in the right figure of Fig. 1. Truncated chords are chords which originate at a point r and terminate at the nearest boundary of the shell. We can think of a truncated chord as the path in which a particle is born at point r and travels in the direction μ to the nearest boundary of the shell. For the case of the truncated chords, we can think of the region bounded by the inner surface of the shell as being filled with a perfect absorber. If the particle leaves the shell through the inner surface, it is absorbed by the perfect absorber. On the other hand, complete chords are straight lines that spans the sphere as if inner void region of the shell was not there. If a complete chord leaves the shell through the inner surface, it re-enters the shell through the other side of the void. The length of a complete chord is the length of the portion of the straight line that is within the shell.

2. The Distribution of Truncated Chord Lengths of a Spherical Shell. If the inner and outer radii of a spherical shell are a and b respectively, then the volume of the spherical shell is $V = \frac{4\pi}{3}(b^3 - a^3)$. If we assume that the interior points of the spherical shell are equally probable, then the interior points are distributed according to the distribution law

$$(2.1) \qquad \rho_r = \frac{4\pi r^2}{V} = \frac{3r^2}{b^3 - a^3} , \qquad a \leq r \leq b .$$

It is easy to verify that $\int_a^b dr \rho_r = 1$. Furthermore let μ be the cosine of the angle that is between the radius vector and the direction of the path of a neutron. If we assume that at each interior point r, the paths through this point is isotropically distributed, then the 'angular' distribution law of the paths is

$$(2.2) \qquad \rho_\mu = \frac{1}{2} , \qquad -1 \leq \mu \leq 1 .$$

Since the joint probability density is the product of ρ_r and ρ_μ, then the conditional probability density of the angle, given r, is $\rho(\mu|r) = \rho_\mu = 1/2$.
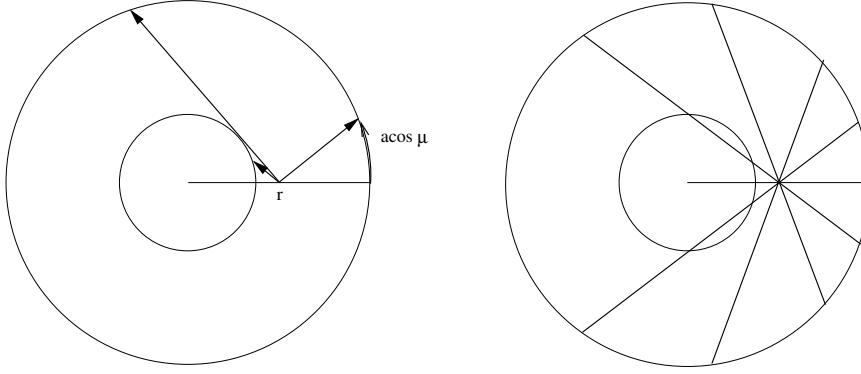
FIG. 2.1. *The left figure illustrates truncated chords in a spherical shell, and the right figure illustrates complete chords in a spherical shell.*

The derivation of the distribution of path lengths ρ_l from the joint probability density $\rho_r \rho(\mu|r)$ is an excercise in the change of integration variables from r and μ to r and l, where l is the length of the path from the point r in the direction μ to the nearest boundary of the shell. The left figure in Fig. 1 shows that there are two types of truncated chords which emanate from the point r in a spherical shell. The first type of path is from the point r to the inner wall of the shell, while the second type of path is from the point r the outer wall of the shell. It is clear from the left figure of Fig. 1 that the path length l is discontinous with respect to the angle μ. At the angle of discontinuity $\mu = -\sqrt{1 - a^2/r^2}$, the path grazes the inner boundary. This discontinuity adds a small degree of difficulty to the change of variables, but this small difficulty is easy to overcome.

In order to keep the presentation simple, let us define the integral which is the focus of our attention as

$$Q \equiv \int_a^b dr\ \rho_r \int_{-1}^1 d\mu\ \rho(\mu|r)$$

(2.3)
$$= \int_a^b dr\ \rho_r \int_{-1}^{-\frac{\sqrt{r^2-a^2}}{r}} d\mu\ \rho(\mu|r)\ +\ \int_a^b dr\ \rho_r \int_{-\frac{\sqrt{r^2-a^2}}{r}}^1 d\mu\ \rho(\mu|r)\ ,$$

where the μ integral is divided into two pieces in order to account for the two types of paths mentioned above. If we define the μ integrals of the above equation as

(2.4)
$$P_a(r) \equiv \int_{-1}^{-\frac{\sqrt{r^2-a^2}}{r}} d\mu\ \rho(\mu|r)\ ,$$

(2.5)
$$P_b(r) \equiv \int_{-\frac{\sqrt{r^2-a^2}}{r}}^1 d\mu\ \rho(\mu|r)\ ,$$

then Q of (2.3) can be written as $Q = Q_a + Q_b$, where

(2.6)
$$Q_a \equiv \int_a^b dr\ \rho_r P_a(r)\ ,$$

(2.7)
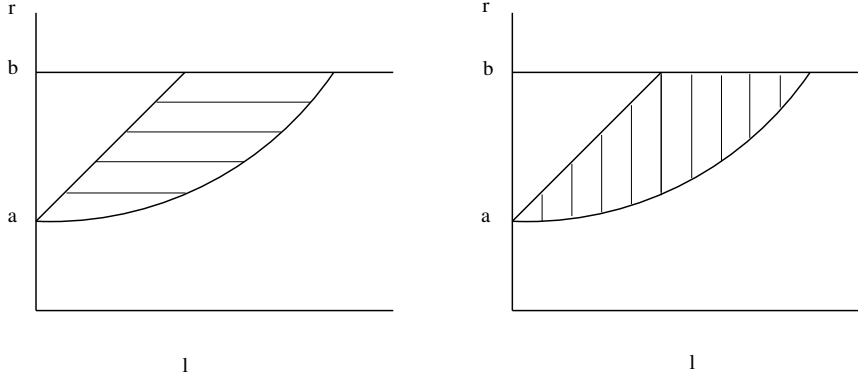$$Q_b \equiv \int_a^b dr\ \rho_r P_b(r)\ .$$

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Fig. 2.2. *The domain of integration of Q_a can be decomposed in two ways. The figure on the left describes the double integral as holding r fixed while integrating over l. The figure on the right describes the double integral as holding l fixed while integrating over r.*

The paths of (2.4) are those that intersect inner wall of the shell. The lengths of these paths can be determined simply by geometry, and are

$$(2.8) \qquad l_a = -\sqrt{a^2 - (1 - \mu^2)r^2} - \mu r \ .$$

The integration limit $\mu = -1$ corresponds to the path length $l_a = r - a$, and the integration limit $\mu = -\sqrt{1 - a^2/r^2}$ corresponds to the path length $l_a = \sqrt{r^2 - a^2}$. Since we wish to express the integral in (2.4) in terms of l_a, we solve (2.8) for μ, i.e.

$$(2.9) \qquad \mu = \frac{a^2 - r^2 - l_a^2}{2rl_a} \ ,$$

from which we get

$$(2.10) \qquad \frac{d\mu}{dl_a} = \frac{r^2 - a^2 - l_a^2}{2rl_a^2} \ .$$

The substitution of (2.10) into (2.4) gives

$$(2.11) \qquad P_a(r) = \int_{r-a}^{\sqrt{r^2-a^2}} \rho(\mu|r) \frac{d\mu}{dl_a} \, dl_a = \int_{r-a}^{\sqrt{r^2-a^2}} \frac{r^2 - a^2 - l^2}{4rl^2} dl$$

where the subscript a on l_a is dropped in the last integral since l_a is a dummy integration variable.

Now the substitution of (2.11) into (2.6) gives

$$(2.12) \qquad Q_a = \int_a^b dr \, \rho_r \int_{r-a}^{\sqrt{r^2-a^2}} \frac{r^2 - a^2 - l^2}{4rl^2} dl \ .$$

We would like to change the order of integration in (2.12). Figure 2 shows the domain of integration and the limits of the double integral of (2.12). When the order of integration is reversed, Fig. 2 shows that the integral of (2.12) can be writtten as

$$Q_a = \int_0^{b-a} dl \int_{\sqrt{a^2+l^2}}^{a+l} dr \, \rho_r \frac{r^2 - a^2 - l^2}{4rl^2}$$

$$(2.13) \qquad + \int_{b-a}^{\sqrt{b^2-a^2}} dl \int_{\sqrt{a^2+l^2}}^{b} dr \, \rho_r \frac{r^2 - a^2 - l^2}{4rl^2} \ .$$

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Since the r integrals of (2.13) are analytical, then Q_a can be written as

$$(2.14) \qquad\qquad Q_a = \int_0^{\sqrt{b^2-a^2}} dl \; \rho_a(l) \;,$$

where

$$(2.15) \qquad\qquad \rho_a(l) = \begin{cases} \frac{3a^2}{4(b^3-a^3)} \;, & 0 \le l \le (b-a) \\[2mm] \frac{3(b^2-a^2-l^2)^2}{16(b^3-a^3)l^2} \;, & (b-a) \le l \le \sqrt{b^2-a^2}, \end{cases}$$

Turning to the integral P_b of (2.5). The paths in (2.5) are those that intersect the outer wall of the shell. The lengths of these paths can also be determined simply by geometry, and are

$$(2.16) \qquad\qquad l_b = \sqrt{b^2 - (1-\mu^2)r^2} - \mu r.$$

We can solve (2.16) for μ to give

$$(2.17) \qquad\qquad \mu = \frac{b^2 - r^2 + l_b^2}{2rl_b} \;,$$

from which we get

$$(2.18) \qquad\qquad \frac{d\mu}{dl_b} = -\frac{b^2 - r^2 - l_b^2}{2rl_b^2} \;.$$

The substitution of (2.18) into (2.5) gives

$$(2.19) \quad P_b(r) = \int_{b-r}^{\sqrt{b^2-a^2}+\sqrt{r^2-a^2}} \rho(\mu|r) \frac{d\mu}{dl_b} \, dl_b = \int_{b-r}^{\sqrt{b^2-a^2}+\sqrt{r^2-a^2}} \frac{b^2 - r^2 + l^2}{4rl^2} \, dl$$

where the subscript b on l_b is dropped in the last integral since l_b is a dummy integration variable.

The substitution of (2.19) into (2.7) gives

$$(2.20) \qquad\qquad Q_b = \int_a^b dr \; \rho_r \int_{b-r}^{\sqrt{b^2-a^2}+\sqrt{r^2-a^2}} \frac{b^2 - r^2 - l^2}{4rl^2} dl \;.$$

Figure 3 shows the domain of integration and the limits of the double integral of (2.20). When the order of integration is reversed, we can write the integral of (2.20) as

$$Q_b = \int_0^{b-a} dl \int_{b-l}^b dr \; \rho_r \frac{b^2 - r^2 - l^2}{4rl^2}$$

$$+ \int_{b-a}^{\sqrt{b^2-a^2}} dl \int_a^b dr \; \rho_r \frac{b^2 - r^2 - l^2}{4rl^2}$$

$$(2.21) \qquad\qquad + \int_{\sqrt{b^2-a^2}}^{2\sqrt{b^2-a^2}} dl \int_{\sqrt{(l-\sqrt{b^2-a^2})^2+a^2}}^b dr \; \rho_r \frac{b^2 - r^2 - l^2}{4rl^2} \;.$$
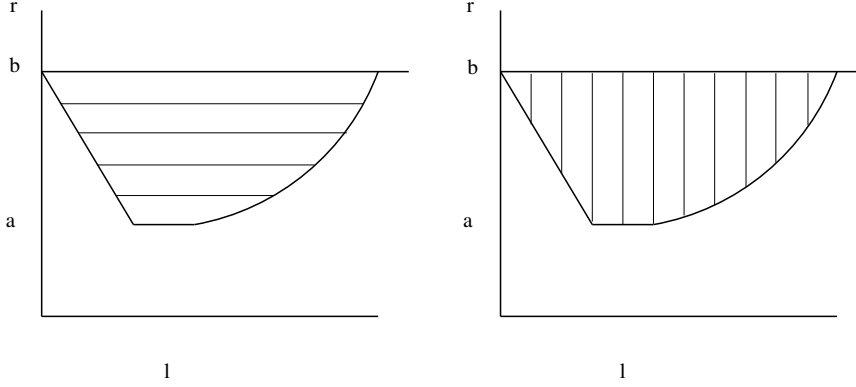
FIG. 2.3. *The domain of integration of Q_b can be decomposed in the same two ways that the domain of integration of Q_a is decomposed.*

Since the r integrals of (2.21) are analytical, then Q_b can be written as

$$(2.22) \qquad Q_b = \int_0^{2\sqrt{b^2 - a^2}} dl \, \rho_b(l) \, ,$$

where

$$(2.23) \qquad \rho_b(l) = \begin{cases} \frac{3(4b^2 - l^2)}{16(b^3 - a^3)} \, , & 0 \le l \le (b - a) \\[2ex] \frac{3(b^2 - a^2 + 2l^2)(b^2 - a^2)}{16(b^3 - a^3)l^2} \, , & (b - a) \le l \le \sqrt{b^2 - a^2} \\[2ex] \frac{12(b^2 - a^2) - 3l^2}{16(b^3 - a^3)} \, , & \sqrt{b^2 - a^2} \le l \le 2\sqrt{b^2 - a^2} \, . \end{cases}$$

Since $Q = Q_a + Q_b$, and Q_a and Q_b are given by (2.14) and (2.22) respectively, then Q can be written as

$$(2.24) \qquad Q = \int_0^{2\sqrt{b^2 - a^2}} \rho_l \, dl \, ,$$

where

$$(2.25) \quad \rho_l = \begin{cases} \frac{3(4b^2 - l^2)}{16(b^3 - a^3)} + \frac{3a^2}{4(b^3 - a^3)} \, , & 0 \le l \le (b - a) \\[2ex] \frac{3(b^2 - a^2 + 2l^2)(b^2 - a^2)}{16(b^3 - a^3)l^2} + \frac{3(b^2 - a^2 - l^2)^2}{16(b^3 - a^3)l^2} \, , & (b - a) \le l \le \sqrt{b^2 - a^2} \\[2ex] \frac{12(b^2 - a^2) - 3l^2}{16(b^3 - a^3)} \, , & \sqrt{b^2 - a^2} \le l \le 2\sqrt{b^2 - a^2} \, , \end{cases}$$

is the distribution of truncated chord lengths of the shell.

2.1. The Average Path Length of a Spherical Shell. The average path length $< l >$ in a spherical shell is

$$(2.26) \qquad < l > \equiv \int_0^\infty \rho_l \, l \, dl \, .$$

The substitution of (2.25) into (2.26) with a bit of algebra gives

$$(2.27) \qquad < l > = \frac{3(b - a)^2}{16(b^3 - a^3)} \left(4b^2 + 6ab + 4a^2 + (b + a)^2 \log_e \frac{b + a}{b - a} \right) .$$

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3. The Distribution of Complete Chord Lengths of a Spherical Shell.
We can derive the distribution of complete chord lengths in the same way that we derived the distribution of truncated chord lengths in the previous section. The only change that we need to make in the derivation is in the definition of the length of a chord. In order to be complete, we assume that the interior point of a spherical shell are equally probable and that the chords are isotropically distributed. So ρ_r and ρ_μ are given by (2.1) and (2.2) respectively. We see that there two kinds of complete chords in the right figure of Fig. 1. The first type is entirely within the shell, while the second type crosses the hollow region of the shell. If μ, the 'angle of inclination' of the chord with respect the vector $\mathbf{r}$, is between $-\sqrt{1-a^2/r^2}$ and $\sqrt{1-a^2/r^2}$, then the chord is the first type. If $|\mu| > \sqrt{1-a^2/r^2}$, then the chord is the second type. The chord length of the first type and the chord length of the second type are respectively

(3.1)
$$l_a = 2\sqrt{b^2 - (1-\mu^2)r^2} \ , \qquad\qquad 0 \le |\mu| \le \sqrt{1-a^2/r^2}$$
$$l_b = 2\sqrt{b^2 - (1-\mu^2)r^2} - 2\sqrt{a^2 - (1-\mu^2)r^2} \ , \quad \sqrt{1-a^2/r^2} < |\mu| \le 1 \ .$$

In the last section we broke the μ integral of Q into 2 regions. In this section, we break the μ integral of Q into 4 regions according to the breakpoints of (3.1), i.e.

$$Q = \int_a^b dr\ \rho_r \int_{-1}^1 d\mu\ \rho(\mu|r)$$

$$= \int_a^b dr \left\{ \int_{-1}^{-\sqrt{1-a^2/r^2}} + \int_{-\sqrt{1-a^2/r^2}}^0 + \int_0^{\sqrt{1-a^2/r^2}} + \int_{\sqrt{1-a^2/r^2}}^1 \right\} d\mu\ \rho(\mu|r)$$

(3.2) $$= 2\int_a^b dr\, \rho_r \int_0^{\sqrt{1-a^2/r^2}} d\mu\ \rho(\mu|r) \ + \ 2\int_a^b dr\, \rho_r \int_{\sqrt{1-a^2/r^2}}^1 d\mu\ \rho(\mu|r) \ ,$$

where we took into account that the μ integrals are symmetric with respect to $-\mu$. Since Q is broken into two pieces, we can write it as $Q = Q_a + Q_b$, where

(3.3) $$Q_a \equiv 2\int_a^b dr\ \rho_r \int_0^{\sqrt{1-a^2/r^2}} d\mu\ \rho(\mu|r) \ ,$$

(3.4) $$Q_b \equiv 2\int_a^b dr\ \rho_r \int_{\sqrt{1-a^2/r^2}}^1 d\mu\ \rho(\mu|r) \ .$$

The chord lengths for Q_a are given by the first equation of (3.1), and the chord lengths for Q_b are given by the second equation of (3.1).

For Q_a, we change the variable of integration from μ to l_a. Solving the first equation of (3.1) for μ,

(3.5) $$\mu = -\frac{\sqrt{\left(\frac{l_a}{2}\right)^2 + r^2 - b^2}}{r} \ ,$$

and its derivative is

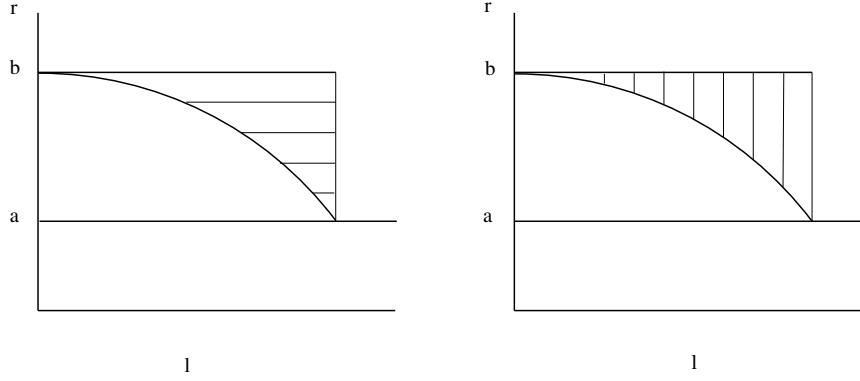(3.6) $$\frac{d\mu}{dl_a} = \frac{\frac{l_a}{2}}{2r\sqrt{\left(\frac{l_a}{2}\right)^2 + r^2 - b^2}} \ .$$

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FIG. 3.1. *The domain of integration of Q_a for complete chords.*

The substitution of (3.6) into (3.3) gives

$$(3.7) \qquad Q_a = 2 \int_a^b dr \; \rho_r \int_{2\sqrt{b^2-r^2}}^{2\sqrt{b^2-a^2}} dl_a \; \rho(\mu|r) \frac{\frac{l_a}{2}}{2r\sqrt{\left(\frac{l_a}{2}\right)^2 + r^2 - b^2}}$$

Figure 4 shows the the domain of integration and the limits of the double integral of (3.7). When the order of integration is reversed, Fig. 4 shows that the integral of (3.7) can be written as

$$Q_a = \int_0^{2\sqrt{b^2-a^2}} dl_a \int_{\sqrt{b^2-(l_a/2)^2}}^b dr \; \rho_r \rho(\mu|r) \frac{\frac{l_a}{2}}{r\sqrt{\left(\frac{l_a}{2}\right)^2 + r^2 - b^2}}$$

$$(3.8) \qquad = \int_0^{2\sqrt{b^2-a^2}} dl \; \frac{3l^2}{8(b^3 - a^3)}$$

For Q_b, we change the variable of integration from μ to l_b. Solving the second equation of (3.1) for μ,

$$(3.9) \qquad \mu = \frac{\sqrt{((l_b/2)^2 + 2r^2 - b^2 - a^2)^2 + 4(b^2 - r^2)(r^2 - a^2)}}{rl_b} \;,$$

we get

$$(3.10) \qquad \frac{d\mu}{dl_b} = \frac{(l_b/2)^4 - \left(b^2 - a^2\right)^2}{rl_b^2\sqrt{((l_b/2)^2 + 2r^2 - b^2 - a^2)^2 + 4(b^2 - r^2)(r^2 - a^2)}} \;.$$

The substitution of (3.10) into (3.4) gives

$$(3.11) \, Q_b = \int_a^b dr \int_{2(b-a)}^{2\sqrt{b^2-a^2}} dl_b \frac{2\rho_r \rho(\mu|r)\left(\left(b^2 - a^2\right)^2 - (l_b/2)^4\right)}{rl_b^2\sqrt{((l_b/2)^2 + 2r^2 - b^2 - a^2)^2 + 4(b^2 - r^2)(r^2 - a^2)}} \;.$$

Since the integration limits of the inner integral in the above equation are independent of r, we can change the order of integration without changing the limits of integration.

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Furthermore the r integral is analytical, so we get

$$(3.12) \qquad Q_b = \int_{2(b-a)}^{2\sqrt{b^2-a^2}} dl \frac{3}{2(b^3-a^3)} \left(\frac{\left(b^2-a^2\right)^2}{l^2} - \frac{l^2}{16} \right) .$$

Combining (3.8) and (3.12), we get

$$(3.13) \quad Q = \int_{0}^{2\sqrt{b^2-a^2}} dl \frac{3l^2}{8(b^3-a^3)} + \int_{2(b-a)}^{2\sqrt{b^2-a^2}} dl \frac{3}{2(b^3-a^3)} \left(\frac{\left(b^2-a^2\right)^2}{l^2} - \frac{l^2}{16} \right) .$$

Therefore the distribution of the complete chord lengths in a spherical shell is

$$(3.14) \qquad \rho(l) = \begin{cases} \frac{3l^2}{8(b^3-a^3)} , & 0 \le l \le 2(b-a) \\[2ex] \frac{9l^2}{32(b^3-a^3)} + \frac{12(b^2-a^2)^2}{8l^2(b^3-a^3)} , & 2(b-a) \le l \le 2\sqrt{b^2-a^2} . \end{cases}$$

3.1. Average Complete Chord Length in a Spherical Shell. The average length $< l >$ of a complete chord in a spherical shell can be determined easily from the density (3.14) and the formula

$$(3.15) \qquad\qquad\qquad < l > \equiv \int_{0}^{\infty} \rho_l \, l \, dl .$$

The substitution of (3.14) into (3.15) gives

$$(3.16) \qquad < l > = \frac{3(b-a)^4 + 9(b^2-a^2)^2}{8(b^3-a^3)} + \frac{3\left(b^2-a^2\right)^2}{4(b^3-a^3)} \log_e \frac{b+a}{b-a} .$$

4. Acknowledgments. I would like to thank Bill Moran for introducing me to this subject, and I would like to thank David Miller for his insights into this problem.

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