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A Circuit Model for Gun Driven Spheromaks

K. I. Thomassen

In this note we derive circuit equations for sustained spheromaks, in the phase after a spheromak is detached from the gun and sustained in a flux conserver. The impedance of the spheromak during the formation and "bubble burst" phase has been discussed by Barnes et. al.[1]. We assume here that the spheromak is formed and helicity is being delivered to it from the gun, currents are above the threshold current, and the λ-gradients are outward (λ decreasing inward). We follow an open field line that begins and ends at the gun electrodes, encircling the closed flux surfaces of the spheromak, and apply power and helicity balance equations for this gun-driven system. In addition to these equations one will need to know the initial conditions (currents, stored energies) after the "bubble burst" in order to project forward in time.

<u>Fields Equations</u>

We begin by applying the generalized Ohms law $\mathbf{E_{tot}} + \mathbf{v} \times \mathbf{B} = \eta\mathbf{j}$ to the gun circuit, consisting of an applied field $\mathbf{E_1}$ and response fields (back emf's, potential drops). The total field is[*] $\mathbf{E_{tot}} = \mathbf{E_1} - \nabla\phi - \dfrac{\partial\mathbf{A}}{\partial t}$ and there are 5 distinct types of electric field encountered in the volume of open field lines (gun circuit) between the gun electrodes; ohmic field, sheath field, Hall field, Faraday electric field, and dynamo electric field. Associating the sheath drops with the potential gradient,

$$\mathbf{E_1} = \mathbf{E_{sh}} + \eta\mathbf{j_1} + \frac{\partial\mathbf{A}}{\partial t} - \mathbf{v} \times \mathbf{B} + \mathbf{E_{dyn}}$$

where $\mathbf{E_{dyn}} = - \langle\delta\mathbf{v} \times \delta\mathbf{B}\rangle$. Our notation will be that region 1 is the volume of open field lines while region 2 is the volume of closed field lines. We take the line integral of the electric field from the positive electrode around the closed loop and find the applied gun voltage,

$$V_g = \int\mathbf{E_1}\ d\mathbf{l_1} = V_{sh} + \int(\eta\mathbf{j_1} - \mathbf{v} \times \mathbf{B} + \mathbf{E_{dyn}} + \frac{\partial\mathbf{A}}{\partial t})\ d\mathbf{l_1}$$

The Hall currents and fields are important during formation[1] but here we will assume that any fluid flow is along the open lines, as appropriate for an established Taylor equilibrium, so that the Hall term $\mathbf{v} \times \mathbf{B}$ is zero. Using $\mathbf{B} = \nabla \times \mathbf{A}$ we have

$$V_g = V_{sh} + \int(\eta\mathbf{j_1} + \mathbf{E_{dyn}})\ d\mathbf{l_1} + \frac{\partial}{\partial t}\int\mathbf{B_2}\ d\mathbf{S_2}$$

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[*] See for example, Ramo and Whinnery, "Fields and Waves in Modern Radio" for a discussion of making circuit equations from field equations.

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Here, $\mathbf{dl_1}$ is a path along any open line around the spheromak, from the positive gun electrode to the negative one, and $\mathbf{dS_2}$ is the unit area vector perpendicular to the poloidal plane encircled by $\mathbf{dl_1}$. The last term can be written $\frac{\partial \Phi_2}{\partial t}$ with

$$\Phi_2 = \int \mathbf{B_2} \ \mathbf{dS_2} \quad = \mu_o \int \frac{1}{\lambda_2} \mathbf{j_2 dS_2} \quad = \frac{\mu_o}{<\lambda_2>} \int \ \mathbf{j_2} \ \mathbf{dS_2} \quad = LI_2$$

with $<\lambda_2>L = \mu_o$

The open line paths around the spheromak are all the flux surfaces between the wall and the separatrix, and each enclose a slightly different flux. But we assume here and elsewhere that these open line surfaces constitute a relatively thin layer, and that the flux conserver volume is dominated by closed flux surfaces. In that approximation the toroidal flux defined by any particular path is nearly the same as that defined by all other paths.

The current I_1 along open lines flows through a cross sectional area $A_o(\iota)$ so the current density is $j_1(\iota) = \frac{A_o}{I_1}$ and there is a resistance $R_1 = \int \frac{<\eta_1>dl_1}{A_o(\iota)}$. The proper value of $<\eta_1>$ to use is the one that gives the same value for $I_1^2R_1$ as the volume integral of $\eta_1 j_1^2$.

Power and Helicity Equations

We define the gun power $P_g = V_g I_1$ as the product of I_1 with the individual voltages. The gun supplies the ohmic losses, dynamo fields, sheath power, and any increases in stored energy of the plasma. Here we assume all the gun current flows around the spheromak, but we can later allow for diverting a fraction directly across the gun channel, in which case the remaining current is what we are now calling I_1.

For the dynamo power term we use the volume integral of the power density $\mathbf{j \cdot E_{dyn}}$, along with a model to be described later. For now we simply call it P_{dyn}. Then,

$$P_g = V_{sh}I_1 + I_1^2R_1 + I_1 \frac{\partial}{\partial t}(LI_2) \quad + P_{dyn}$$

with

$$P_{dyn} = \int \mathbf{j_1} \ \mathbf{E_{dyn}} \ dV_1 = \frac{1}{\mu_o} \int \lambda_1 \mathbf{E_{dyn}} \ \mathbf{B_1} \ dV_1$$

Conservation of helicity gives an additional equation between the dynamo power and gun power involving the λ-ratios of the two regions. The gun power (less the sheath power) produces helicity at a rate

$$\frac{<\lambda_1>}{2\mu_o} \frac{dK_{gun}}{dt} \quad = P_g - P_{sh}$$

Helicity is lost at the rate

$$\frac{dK}{dt} = 2\int \mathbf{E}_\eta \cdot \mathbf{B} \; dV = 2\mu_o \int \frac{\eta \mathbf{j^2}}{\lambda} dV = \frac{2\mu_o}{<\lambda>} \int \eta \mathbf{j^2} \; dV = \frac{2\mu_o}{<\lambda>} P_{oh}$$

Equating helicity production to helicity loss

$$\frac{(P_g - P_{sh})}{<\lambda_1>} = \{\frac{P_{oh1}}{<\lambda_1>} + \frac{P_{oh2}}{<\lambda_2>}\} \qquad \Rightarrow \qquad P_{oh2} = \frac{<\lambda_2>}{<\lambda_1>} \{P_g - P_{sh} - P_{oh1})$$

This latter form displays the usual "efficiency" ratio that determines how much power goes to ohmic loss in the closed region.
From the power equation the power available to the dynamo is

$$P_{dyn} = P_g - P_{sh} - P_{oh1} - I_1 \frac{\partial}{\partial t}(LI_2)$$

To interpret the dynamo term, two different heuristic models are used. The first is quite simple and states how the dynamo power generated in region 1 must divide between regions 1 and 2. Clearly some of that power can be dissipated in region 1, but some must flow across flux surfaces and provide power to region 2 where there is no power source (the gun driven current in region 1, working against the dynamo electric field, is the power source there). We call these powers P_1 and P_2 respectively.
Next, P_2 must at least supply the ohmic losses in region 2, but could be larger depending on the strength of the driving power in region 1. We therefore have $P_2 = P_{oh2} + P_3$. While we can only calculate P_{oh2} and not the other two terms in the dynamo power, there is nonetheless a useful relationship that is simply derived from the power equations and helicity constraint. Note that the character of the powers P_1 & P_3 is different, in that these are the result of turbulent processes that invest power in waves that are dissipated via different channels. The power can be dissipated by Landau damping, ion cyclotron damping, or other processes, and at wave numbers and frequencies dependent on details of the turbulence. In contrast, the ohmic powers represents the heat from electron-ion collisions, presumably as calculated using Spitzer resistivity.
With these definitions, and the helicity constraint, the dynamo power is

$$P_{dyn} = P_1 + P_{oh2} + P_3 = \frac{<\lambda_1>}{<\lambda_2>} P_{oh2} - I_1 \frac{\partial}{\partial t}(LI_2)$$

Then, with $\Delta\lambda = <\lambda_1> - <\lambda_2>$ and $P_L = I_1 \frac{\partial}{\partial t}(LI_2)$, another interesting relationship arising from the helicity constraint is revealed;

$$\frac{\Delta\lambda}{<\lambda_2>} P_{oh2} = P_1 + P_3 + P_L = \varepsilon P_{oh2}$$

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Any difference in λ between that in the open and closed regions requires this power equality, with εP_{oh2} the amount available for dynamo dissipation and inductive power buildup. This relationship is not dependent on any particular dynamo model. We will argue below that the dissipation terms $P_{1,3}$ likely require gradients in λ to exist, thus a relationship between the field buildup rate and $\Delta\lambda$ also exists.

So we proceed to a second model, using a hyper resistivity κ, to describe the dynamo power. If there are λ-gradients we expect power flow across flux surfaces to be related to these gradients. A model developed by Hooper[2] (suggested from the work of Boozer[3] and Strauss[4]) gives $\mathbf{E}_{dyn} \bullet \mathbf{B} = -\nabla \bullet \{\frac{\kappa B^2}{\mu_o} \nabla\lambda\}$. A calculation of the dynamo power loss in a volume V then gives

$$P_{dyn} = \int \frac{\lambda}{\mu_o} \mathbf{E}_{dyn} \ \mathbf{B} dV \quad = -\int \frac{\lambda}{\mu_o}\nabla \quad \{\frac{\kappa B^2}{\mu_o}\nabla\lambda\}dV$$

$$= \int \{\frac{\kappa B^2}{\mu_o}\nabla\lambda\} \ \frac{\nabla\lambda}{\mu_o}dV \quad - \oint \frac{\lambda}{\mu_o} \ \{\frac{\kappa B^2}{\mu_o} \ \nabla\lambda\}\bullet \mathbf{dS}_1))$$

Applied to the open line region 1, the above two terms are simply our $P_1 + P_2$.

$$P_1 + P_2 = \int \kappa j_1^2 \frac{[\nabla\lambda]^2}{\lambda^2}dV_1 \quad - \oint \frac{\lambda}{\mu_o} \ \{\frac{\kappa B^2}{\mu_o} \ \nabla\lambda\}\bullet\mathbf{dS}_1$$

The power P_1 is dissipated in region 1 if $\nabla\lambda_1 \neq 0$, and P_2 is positive since $\nabla\lambda$ and $\mathbf{dS}_1$ are in opposite directions. The surface S_1 surrounding volume 1 consists of 3 kinds of surface: the portion where the gun flux and helicity enter volume 1; the flux conserver on the outside of volume 1 where $\nabla\lambda\bullet\mathbf{dS}_1 = 0$, and the separatrix surface between regions 1 and 2 where $\nabla\lambda\bullet\mathbf{dS}_1 < 0$. On the last of these surfaces the power P_2 flows into volume 2.

Applied to the closed line region 2 the model gives,

$$\int \kappa j_2^2 \frac{[\nabla\lambda]^2}{\lambda^2}dV_2 \quad - \oint \frac{\lambda}{\mu_o} \ \{\frac{\kappa B^2}{\mu_o} \ \nabla\lambda\}\bullet\mathbf{dS}_2 = 0$$

(Note that the surface vector $\mathbf{dS}_2 = -\mathbf{dS}_1$ is now pointed outward from region 2 to region 1, so and $\nabla\lambda\bullet\mathbf{dS}_2 > 0$). Therefore,

$$P_2 \equiv \oint \frac{\lambda}{\mu_o} \ \{\frac{\kappa B^2}{\mu_o} \ \nabla\lambda\}\bullet\mathbf{dS}_2 = \int \kappa j_2^2 \frac{[\nabla\lambda]^2}{\lambda^2}dV_2 \quad = P_{oh2} + P_3$$

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Here, one uses the value of λ and its gradient on the separatrix surface for the surface term, and the λ-profile in region 2 for the volume term. And, to repeat, while there are fluctuations at work in region 2 to rearrange field and current profiles to a near force-free state, there is no "internal" dynamo power source. The power dissipation and flow in region 2 is fed from the separatrix by P_2. Also, if there is a gradient in λ in region 2 some of this power will be dissipated and some will flow further inward until all of the power is dissipated.

While the above model does not allow the powers $P_{1,3}$ to be calculated without a model for hyper resistivity, it does gives a plausible relationship between λ-gradients and power dissipation and flow. In a sharp boundary model, where the λ-gradients are near zero everywhere except across a narrow layer of separatrix, the excess power is zero everywhere except in that layer. There is both power dissipation in that layer and flow into region 2. But if $\nabla\lambda_2 = 0$ there would be no flow of power towards the magnetic axis to supply the internal ohmic loss, so there must be an internal gradient and, concomitantly, $P_3 \neq 0$. Since dynamo power can be generated on each of the open lines between the flux conserver and the separatrix, a gradient in λ is not necessary in region 1, except near the separatrix to provide flow for $P_{oh2} + P_3$. Details of the dynamo process are obviously needed to make more definitive statements involving the dissipation and flow of power in the two regions.

Summarizing the equations,

$$V_g = V_{sh} + I_1 R + \int \mathbf{E_{dyn}}\ \mathbf{dl_1} \quad + \frac{\partial}{\partial t}(LI_2) \qquad\qquad \underline{\text{Voltage}}$$

$$P_g = V_{sh}I_1 + I_1{}^2R_1 + I_1\frac{\partial}{\partial t}(LI_2) \quad + P_{dyn} \qquad\qquad \underline{\text{Power}}$$

with

$$P_{dyn} - P_{oh2} = P_1 + P_3 = \frac{\Delta\lambda}{\langle\lambda_2\rangle}\ P_{oh2} - I_1\frac{\partial}{\partial t}(LI_2) \qquad \underline{\text{Helicity constraint}}$$

or

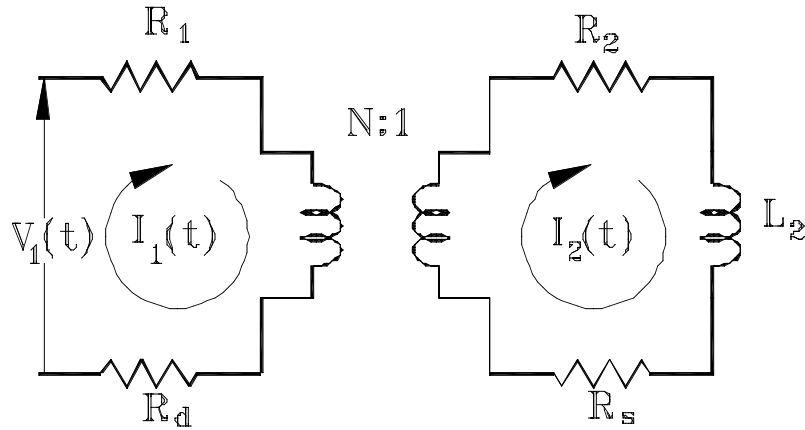$$P_{oh2} = \frac{\langle\lambda_2\rangle}{\langle\lambda_1\rangle}(P_g - P_{sh} - P_{oh1})$$

Circuit Model

A gun circuit model which incorporates the above voltage and power features can now be made. It is simply a primary circuit consisting of 2 resistors to represent P_1 and P_{oh1} and an ideal transformer with turns ratio N such that $NI_1 = I_2$ and $NV_2 = V_1$. Similarly, in the secondary circuit there are 2 resistors to represent the power losses $P_2 = P_{oh2} + P_3$, and an inductor to represent the toroidal flux Φ_2, $\langle\lambda_2\rangle\Phi_2 = \mu_o I_2$. Recall that in our heuristic model,

$$P_1 = \int \kappa_1 j_1{}^2 \frac{|\nabla\lambda_1|^2}{\lambda_1{}^2} dV_1 \qquad\qquad P_{oh2} + P_3 = \int \kappa_2 j_2{}^2 \frac{|\nabla\lambda_2|^2}{\lambda_2{}^2} dV_2$$

Since these integrals contain j^2 one might expect them to be proportional to I_1^2 and I_2^2 respectively. So we define the primary circuit powers $P_{oh1} + P_1 = I_1^2(R_1 + R_d)$ and the secondary circuit powers $P_{oh2} + P_3 = I_2^2(R_2 + R_s)$, with the ohmic powers $P_{oh1,2}$ determined from $R_{1,2}$ respectively.

An inductor L_2 in the secondary can be defined from L and the turns ratio. Dividing by N transforms a primary voltage such as $V_{L1} = \frac{\partial}{\partial t}(LI_2)$ to a secondary voltage, so we find $L_2 = \frac{\mu_0}{N<\lambda_2>}$. The power to that inductor is $I_2V_2 = I_2 \frac{\partial}{\partial t}(L_2I_2) = I_1 \frac{\partial}{\partial t}(LI_2)$. The circuit described here is shown below.

One issue worth understanding here is the amplification of the toroidal field over the edge poloidal field, or the relationship between I_2 and I_1. To increase I_2 and B_2 beyond the values created in the bubble burst is one goal of our experiments. However, Ampere's law relates the line integral of the toroidal field, around the magnetic axis, to all of the current crossing the area encircled by this line integral. The current in question is all poloidal current from the geometric axis out to the magnetic axis, a sum including I_1 and all the poloidal current penetrating the annular (mid plane) surface from the separatrix to the magnetic axis. If the fields are essentially those of the force-free Taylor equilibrium, then all fields and current ratios are known in a given geometry and there is a specific field limit defined by the Grad-Shafranov equilibrium.

As shown by Hooper[5], when the value $\Delta\lambda \rightarrow 0$ a resonance is approached and the resulting equilibria force the layer of open field lines into increasingly smaller volumes, raising the value of N. The resonance is approached from above by reducing the driving current I_1 (which specifies a value of $<\lambda_1>$ for a given gun flux), so that $<\lambda_1>$ approaches the resonant value $<\lambda_2>$ for the flux conserver. From

equilibria studies for various λ-profiles in a given geometry, one can model the relationship between N and $\Delta\lambda$.

Values in the circuit model can be determined if we know the equilibria at all times, including the λ-profiles. From these we determine N, L_2, and the current and field profiles. If the temperature and density profiles are measured or estimated, Spitzer resistivities can be calculated. The resistances, $R_{1,2}$ are calculated and the powers $P_1 + P_3$ are determined at each time step.

To use the model we take I_1 to be the independent variable in the problem (using the gun current data as input). Then I_2 is calculated from N, the resistor and inductor values are calculated, and the gun voltage waveform $V_1(t)$ is constructed from the circuit model and compared with the data. By fitting the voltage calculations to the data one can then improve upon various of the estimates used as input to the model.

We would like to know the ratio of R_s to R_d, since we know only the sum $R_d + N^2 R_s$ from the power sum P_1 and P_3. From their definitions, assuming all quantities are constant, the power ratio contains $\kappa j^2 \xi^2 V$, where ξ is the length $\lambda|\nabla\lambda|^{-1}$. So,

$$\frac{P_1}{P_3} = \frac{R_d}{N_f^2 R_s} = \frac{\kappa_1 j_1^2 \xi_1^2 V_1}{\kappa_2 j_2^2 \xi_2^2 V_2}$$

While there is no serious way to estimate each of the quantities, the hyper resistivity in particular, one might make educated guesses as to ratios, and hence make educated guesses on the resistance ratios. In summary, the sum $N_f^2 R_s + R_d$ is calculated from other known quantities, while the ratio of these two terms must be estimated.

There are additions that can be made to this model. If a fraction "f" of the gun current flows directly across the gun channel we can modify the above circuit with an appropriate resistor across the gun terminals. Its resistance can be calculated from f (as determined from the equilibria) and from the impedance of the remaining circuit. The reduced current flowing around the spheromak is now used to calculate ohmic power and dymamo power dissipation in the primary, and to determine the turns ratio from the equilibria. Now, the turns ratio is still N:1, but $I_2 = N(1 - f)I_1$.

Another small modification to the circuit would account for storing a fraction of the energy at a given moment into the waves involved in the dynamo action. A parallel circuit of the dynamo resistances $R_{s,d}$ with a high-Q L-C circuit could represent the oscillations in gun voltage seen in the experiments. Values of L,C would be calculated by the demand of high-Q, and the frequency and amplitude of the voltage oscillations. Both these additions are indicated below.
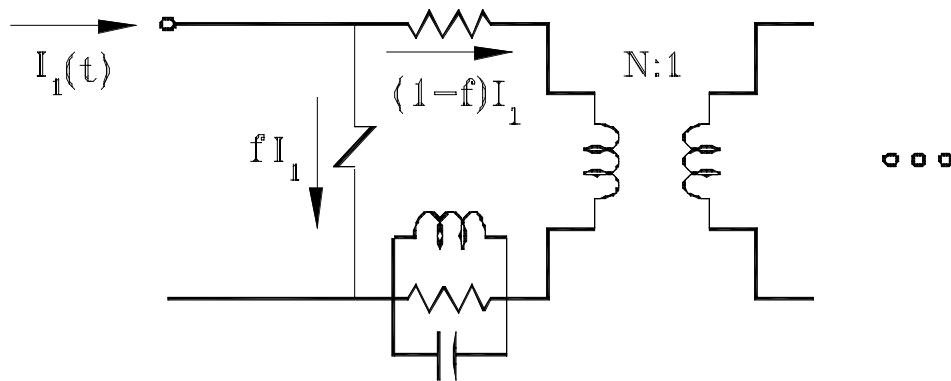
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