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Differential Emitter Geolocation

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Abstract

A technique of locating a ground-based radio frequency (RF) emitter using receivers on a constellation of satellites is described and analyzed. A reference emitter near to the emitter of interest is required so that differences in carrier phase of the emitter of interest and the reference emitter can be measured. This technique is related to differential carrier phase techniques used in Global Positioning System (GPS) based high accuracy surveying applications.

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Contents

1 Introduction

The first step in most Global Positioning System (GPS) receivers is to demodulate the signal, losing all of the information in the carrier signal. One of the exceptions to this rule is in high precision surveying, where the carrier phase information is used to do relative positioning. Relative positioning is where a reference receiver is put in the field, near a rover receiver we would like to locate. The location of the rover receiver is located relative to the position of the reference receiver. By using the carrier phase information, it is possible to determine relative positions to a small fraction of the wavelength of the carrier wave (which is about 0.2 meters). In a typical setting, the reference receiver is about 10 km away from the other receiver, and the surveys often take up to 15 minutes.

The purpose of this LDRD was to investigate if carrier phase information could be used to improve the accuracy of low bandwidth emitter geolocation systems used at Sandia.

The systems we are concerned with would differ from the systems used for doing GPS-based surveying in four significant ways. They would:

- Locate emitters rather than receivers.
- Have signal durations of seconds rather than minutes or hours.
- Use fewer satellites, and possibly use altitude information.
- Often have much lower bandwidth and therefore less geolocation accuracy than GPS, ie give answers to many multiples, rather than a small fraction, of the wavelength.

We have thought about the first of these points, and believe that it is not a significant factor. Once one understands CPRP (Carrier Phase Relative Positioning), it is clear that the second and third of these points are severe restrictions. In particular, using a 1 second signal compared to a 15 minute signal makes the integer ambiguity resolution problem (to soon be discussed) much harder. The last point on our list appears as though it should make the problem easier. At first sight this does not appear to help us. The way CPRP is usually done, it appears as though you can get extremely accurate answers if you can resolve the integer ambiguity, but no increase in precision is obtained if the integer ambiguity cannot be resolved. One of the main conclusions of our work is that this is not the case when carrier phase information is applied to low-bandwidth emitter systems. That is, it is possible to use the carrier phase information to improve our solutions (giving answers to multiples of the wavelength) even when the integer ambiguities cannot be resolved.

In CPRP, distances are measured using the carrier phase information. This results in expressions for distances of the form

$$d = (\phi + n)\lambda_e \tag{1}$$

where ϕ is phase, in units of cycles, that can be measured very accurately ,n is an integer ambiguity, and λ_e is the emitter wavelength. Here we have not made clear what distances we are referring to, but it will become clearer in §2.

When doing CPRP we need to solve for both the relative position of the rover and $N-1$ integer ambiguities. Here N is the number of satellites used in the survey. This results in a system of equations (see §2) of the form

$$\mathbf{Az} + \mathbf{Bn} = \mathbf{b} \tag{2}$$

where $\mathbf{z}$ is the vector giving the relative position of the rover relative to the reference, $\mathbf{n}$ is a vector of the integer ambiguities, and $\mathbf{b}$ is the vector of phase measurements. This results in an overdetermined system of equations. If we fix the vector $\mathbf{n}$ we can solve this as an overdetermined system of equations for $\mathbf{z}$. Once the least squares solution to this system is found, we can compute the error as a function of $\mathbf{n}$. This is a quadratic function of $\mathbf{n}$, which in §3 we show can be written as

$$E(\mathbf{n}) = (\mathbf{n} - \mathbf{n}_c)^T \mathbf{M} (\mathbf{n} - \mathbf{n}_c) + E_0 \tag{3}$$

Most discussions of CPRP assume that it is clear that we should try to minimize the function $E(\mathbf{n})$ over all values of the integers. Though this is plausible, we believe that a careful examination of why this is a good thing to do is crucial to understanding the limitations on CPRP.

A crucial step in understanding the limitations of CPRP comes from the realization that the system of equations (2) is under determined if there is only a single epoch. In §3 we show that if we only take measurements at a single point in time, this system of equations has a three dimensional nullspace. If the satellite baseline is non-zero, then we have an overdetermined system of equations. The only way these statements can be compatible with each other is if the condition number of our system goes to infinity as the baseline goes to zero. The result of this is that the matrix $\mathbf{M}$ in Eqn. (3) will have three eigenvalues that are very small for small values of the satellite baseline. In §7 we will show that these eigenvalues are proportional to the square of the satellite baseline.

One of the first conclusions from this observation is that if the baseline is large enough, the system of equations in Eqn. (2) will be well conditioned. In this case any value of $\mathbf{n}$ that gives low residuals must be close to $\mathbf{n}_c$. In this case we would get accurate answers if we considered Eqn. (2) as an overdetermined system of equations without imposing the fact that $\mathbf{n}$ had to be an integer. Though this is in agreement with common knowledge in the CPRP community, we do not know of any presentation that clearly outlines why this is so.

When the satellite baseline is small, as a result of thethree small eigenvalues of $\mathbf{M}$, there is a three dimensional surface imbedded in $N-1$ dimensional space on which we will give small residuals. Since the true values $\mathbf{n}_0$ of the integer ambiguities will give a small residual, this implies that this low dimensional surface must pass by very close to the point $\mathbf{n}_0$. However, it would be somewhat of a fluke for this surface to pass through some other value of the integers. For this reason, it is unlikely that other integers will give low residuals.

In order to understand the limitations on CPRP it is necessary to make the argument in the last paragraph more precise. To do this we first realize that the sheet we have described has a finite thickness that is proportional to the measurement errors of our system. That is, if we want to find other integers that have errors as small as the true solution, it is not necessary that they be on our sheet, but only within a distance of $\delta\phi$ of it, where $\delta\phi$ is the size of the phase measurement errors in our system. Furthermore, the extent of the sheet is not infinite. The extent of the sheet depends both on the errors $\delta\phi$ in our system, and on the size of the three small eigenvalues of $\mathbf{M}$. For example, if one of the small eigenvalues is μ_1 (with eigenvector $\mathbf{u}_1$), then if we go a distance L away from $\mathbf{n}_c$ in the direction $\mathbf{u}_1$, the error will be $L^2\mu_1$. When this gets to be as large as $\delta\phi^2$, the points further than this that are on our sheet will be giving residuals larger than those for $\mathbf{n}_0$.

From the discussion in the last paragraph, we see that our sheet has a volume in $N-1$ dimensional space. This volume will grow as the baseline decreases. When ths volume gets to be on the order of the unit cell in $N-1$ dimensional space, it will become very likely that integers other than $\mathbf{n}_0$ will be inside of this volume. When this is the case, it will be nearly impossible to resolve the integer ambiguity in CPRP.

It is well known in the CPRP literature that the integer ambiguity problem gets harder as the baseline gets shorter. One of the basic questions we had going into this LDRD was if as the baseline gets extremely short, it merely becomes computationally difficult to resolve the integer ambiguity, or if it actually becomes impossible to resolve the ambiguity. The arguments in the last few paragraphs show that it is in fact meaningless to try to resolve the integer ambiguity when the satellite baseline is too short. In particular, even if we could search over integers infinitely rapidly, we would end up finding so many integers that gave small values of the residual, that we would have no way of distinguishing these from the true solution.

In §6 we show that the integer ambiguity problem becomes unsolvable when

$$\frac{|\,d\mathbf{s}\,|}{|\,\mathbf{s}\,|} \approx K\,(\delta\phi)^{(2N+4)/6} \tag{4}$$

where K is a constant depending on the geometry. Here $|\,d\mathbf{s}\,|$ is the length that the satellites have moved during the survey, and $|\,\mathbf{s}\,|$ is the distance that the satellites are away from the center of the earth.

The main conclusion of this report is that despite the fact that it is not possible to do integer ambiguity resolution when the satellite baseline is short, in low bandwidth systems it is still possible to get very significant improvements. This results from the fact that in such systems the time of arrival (TOA) system of equations is typically well conditioned but will have very large TOA errors in terms of wavelengths. This is opposed to systems where we use the carrier phase. In this case we have poorly conditioned systems that have very small range errors. As we shall see, for many of the system parameters of concern, it is highly desirable to use the carrier phase information. Furthermore in §5 we will show that in such situations it is possible to use altitude information without seriously degrading our measurements errors.

The errors we expect to get out of carrier phase geo-location can be written as

$$\mid d\mathbf{x} \mid = C\delta\phi \frac{\mid \mathbf{s} \mid}{\mid d\mathbf{s} \mid}\lambda_e \tag{5}$$

here the quantity $\lambda_e\delta\phi$ gives the size of the measurement errors in meters, the term $\mid \mathbf{s} \mid / \mid d\mathbf{s} \mid$ gives the amplification of the errors due to the poor conditioning, and C is a constant depending on the satellite geometry.

2 Formulation

In this section we formulate the problem of determining the position of a radio frequency (RF) emitter on earth using a constellation of satellite-borne receivers. In this RF emitter geolocation application the space-based receivers measure the instantaneous carrier phases of an emitter of interest (EOI) and a nearby reference emitter (REF). To estimate the position of the EOI we find the EOI location and a set of whole cycles of phase that most closely predict the differences of observed phases taken at two or more time epochs. It is significant that the phase difference in the present, higher accuracy approach, is between emitters, rather than between time epochs as was the case with the technique developed in [1].

To better understand the differential emitter location technique consider the following equation for the carrier phase, measured in cycles, at time epoch t_0 for the nth satellite located at $\mathbf{s}_n(t_0)$ which has a phase-locked loop (PLL) that is tracking the carrier phase of the EOI.

$$\varphi_n(t_0) = \frac{\|\mathbf{x}_e - \mathbf{s}_n(t_0)\|}{\lambda_e} + I_n - L_n \tag{6}$$

Here λ_e is the wavelength of the EOI, $\mathbf{x}_e$ is the location of the EOI, I_n is undesired phase contributions from the ionosphere and L_n is the integer number of wavelengths in the path. The measured phase is wrapped, i.e. a value less than one cycle, and L_n is the unknown integer cycles of phase along the propagation path. In other words ignoring any phase constants introduced by the transmitter or receiver which will be removed by differencing operations the measured fractional phase (in cycles) plus the integral number of cycles is equal to the number of wavelengths in the propagation path plus the ionosphere effects.

The preffered way to measure carrier phase is to lock a PLL to the reconstructed carrier and then measure the phase of the PLL numerically-controlled oscillator (NCO) which has a much higher signal-to-noise (SNR) than the signal itself.

In typical cases where the wavelength, or equivalently the center frequency, is not known accurately enough, it will be necessary to estimate the emitter frequency from a frequency of arrival (FOA), the satellite position and velocity and an estimate of the emitter position. This makes the determination of emitter position and frequency an iterative process.

Writing a similar equation for phase of the reference emitter at the same satellite and epoch gives

$$\theta_n(t_0) = \frac{\|\mathbf{x}_r - \mathbf{s}_n(t_0)\|}{\lambda_r} + I_n - M_n \tag{7}$$

where λ_r is the wavelength of the REF, M_n is the unknown integer cycles of phase in the path and $\mathbf{x}_r$ is the known location of the REF. Reference emitter wavelength can be determined from REF position and a REF FOA. Note that we have assumed the same ionosphere effects in both (6) and (7). This is an approximation justified by the assumptions that the EOI and REF locations and the wavelengths are not too different.

We now difference equations (6) and (7) which eliminates the ionosphere perturbation giving

$$\delta_n(t_0) = \frac{\|\mathbf{x}_e - \mathbf{s}_n(t_0)\|}{\lambda_e} - \frac{\|\mathbf{x}_r - \mathbf{s}_n(t_0)\|}{\lambda_r} + P_n \tag{8}$$

where $\delta_n(t_0) = \varphi_n(t_0) - \theta_n(t_0)$, and $P_n = M_n - L_n$. We now do a second difference to eliminate any differences in phase introduced by the RF hardware. This second difference is between $N-1$ satellites with respect to an arbitrarily chosen reference satellite. The doubly-differenced equations are of the form

$$\Delta_n(t_0) = \frac{\|\mathbf{x}_e - \mathbf{s}_n(t_0)\|}{\lambda_e} - \frac{\|\mathbf{x}_r - \mathbf{s}_n(t_0)\|}{\lambda_r} - \frac{\|\mathbf{x}_e - \mathbf{s}_N(t_0)\|}{\lambda_e} + \frac{\|\mathbf{x}_r - \mathbf{s}_N(t_0)\|}{\lambda_r} + K_n \tag{9}$$

where $\Delta_n(t_0) = \delta_n(t_0) - \delta_N(t_0)$, and $K_n = P_n - P_N$, for $n = 1$ to $N-1$. This gives $N-1$ equations but there are 3 more unknowns than equations since the $N-1$ integers, K_n, and 3 elements of $\mathbf{x_e}$ are all unknown. We need more equations than unknowns so we continue to track phase with the PLL and write equations analogous to the above at additional time epochs. Even a single additional epoch will suffice since we can write the equations in terms of the same integer wavelength variables as in the equations for the first epoch. If we have counted the cycles of phase tracked by the PLLs from t_0 to t_1 we can write

$$\varphi_n(t_1) = (f_e - f_n)t_1 + \frac{\|\mathbf{x}_e - \mathbf{s}_n(t_1)\|}{\lambda_e} + I_n - L_n \tag{10}$$

where we have taken time to be measured from $t_0 = 0$, and where $f_e = c/\lambda_e$ is the carrier frequency of the EOI (c is the speed of light) and f_n is the frequency that the nth satellite will convert to baseband. Similarly

$$\theta_n(t_1) = (f_r - f_n)t_1 + \frac{\|\mathbf{x}_r - \mathbf{s}_n(t_1)\|}{\lambda_r} + I_n - M_n \tag{11}$$

where $f_r = c/\lambda_r$ is the REF carrier frequency. Note that $\varphi_n(t_1)$ and $\theta_n(t_1)$ are not wrapped phase measurements like $\varphi_n(t_0)$ and $\theta_n(t_0)$, ie we count cycles of phase after the initial epoch. Subtracting (11) from (10) gives

$$\delta_n(t_1) = (f_e - f_r)t_1 + \frac{\|\mathbf{x}_e - \mathbf{s}_n(t_1)\|}{\lambda_e} - \frac{\|\mathbf{x}_r - \mathbf{s}_n(t_1)\|}{\lambda_r} + P_n \tag{12}$$

Taking a second difference with respect to the Nth satellite gives the doubly-differenced equations at this epoch as

$$\Delta_n(t_1) = \frac{\|\mathbf{x}_e - \mathbf{s}_n(t_1)\|}{\lambda_e} - \frac{\|\mathbf{x}_r - \mathbf{s}_n(t_1)\|}{\lambda_r} - \frac{\|\mathbf{x}_e - \mathbf{s}_N(t_1)\|}{\lambda_e} + \frac{\|\mathbf{x}_r - \mathbf{s}_N(t_1)\|}{\lambda_r} + K_n \tag{13}$$

Our overdetermined system consists of (9) and (13) for $n = 1$ to $N-1$ which gives $2N-2$ equations in the $N+2$ unknowns $\mathbf{x}_e$ and K_n. Using more epochs would give an even more highly overdetermined system. This system of non-linear systems of equations can be solved in the least-squares sense using the iterative Gauss-Newton procedure. This technique refines an initial guess which could be obtained by estimating emitter position from time or frequency of arrival measurements, or simply using the reference emitter location. This procedure would produce a float solution, ie where K_n is a floating point solution that approximates the desired integer. In cases where better accuracy is achievable, e.g. long signal durations, the so-called fixed solution (true integer) can be found from the float solution using the LAMBDA algorithm [2].

We digress breifly to discuss the error associated with uncertainty in the emitter and reference wavelengths. A phase error on a path of length R due to a deviation in carrier frequency of Δf is $R \cdot \Delta f / c$ where c is the speed of light. Using the maximum GPS path length of 26000 km and a frequency uncertainty of 0.1 Hz gives a phase error of about 3 degrees. Fortunately the second difference will remove most of this. Since the phase from the emitter to satellite N is subtracted from the phase of the emitter to the other satellites the residual phase error will be given by $\Delta R \cdot \Delta f / c$ where ΔR is the path length difference which has a maximum value of 5600 km for GPS. This reduces the phase error to 0.67 degrees for each of the two frequency estimates that must be made. Note that it would be best to choose satellite N to best cancel the errors, e.g. select it such that its range is close to the average range.

In many situations of interest it is possible to linearize equations (9) and (13). In particular, suppose that we define

$$\mathbf{z} = \frac{\mathbf{x}_e - \mathbf{x}_r}{\lambda_e} \tag{14}$$

In this case linearizing Eqn. (9) assuming $\mathbf{z}$ is small linearizing Eqn. (9) we get

$$\left(\frac{\mathbf{x}_r - \mathbf{s}_n(t_0)}{\mid \mathbf{x}_r - \mathbf{s}_n(t_0) \mid} - \frac{\mathbf{x}_r - \mathbf{s}_N(t_0)}{\mid \mathbf{x}_r - \mathbf{s}_N(t_0) \mid} \right) \cdot \mathbf{z} + K_n = c_n \tag{15}$$

where

$$c_n = \Delta_n(t_0) - \frac{\mid \mathbf{x}_r - \mathbf{s}_n(t_0) \mid}{\lambda_e} + \frac{\mid \mathbf{x}_r - \mathbf{s}_n(t_0) \mid}{\lambda_r} + \frac{\mid \mathbf{x}_r - \mathbf{s}_N(t_0) \mid}{\lambda_e} - \frac{\mid \mathbf{x}_r - \mathbf{s}_N(t_0) \mid}{\lambda_r} \tag{16}$$

Linearizing Eqn. (13) we get

$$\left(\frac{\mathbf{x}_r - \mathbf{s}_n(t_1)}{\mid \mathbf{x}_r - \mathbf{s}_n(t_1) \mid} - \frac{\mathbf{x}_r - \mathbf{s}_N(t_1)}{\mid \mathbf{x}_r - \mathbf{s}_N(t_1) \mid} \right) \cdot \mathbf{z} + K_n = d_n \tag{17}$$

where

$$d_n = \Delta_n(t_1) - \frac{|\,\mathbf{x}_r - \mathbf{s}_n(t_1)\,|}{\lambda_e} + \frac{|\,\mathbf{x}_r - \mathbf{s}_n(t_1)\,|}{\lambda_r} + \frac{|\,\mathbf{x}_r - \mathbf{s}_N(t_1)\,|}{\lambda_e} - \frac{|\,\mathbf{x}_r - \mathbf{s}_N(t_1)\,|}{\lambda_r} \tag{18}$$

Combining these equations we end up with a linear system of equations of the form

$$\mathbf{Az} + \mathbf{Bn} = \mathbf{b} \tag{19}$$

where $\mathbf{n}$ is the array of $N-1$ integer ambiguities.

In this report we will be mainly concerned with the case where we only have two epochs. In this case we have

$$\mathbf{A} = \begin{pmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \end{pmatrix} \tag{20}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{I} \\ \mathbf{I} \end{pmatrix} \tag{21}$$

where $\mathbf{I}$ is the $N-1$ dimensional identity matrix.

3 The Three Dimensional Low Residual Surface

3.1 The Covariance Matrix

If we fix $\mathbf{n}$ in Eqn. (19) and solve the over determinied system for $\mathbf{z}$, we will get residual errors in our equations. That is, we solve the overdetermined system of equations

$$\mathbf{Az} = \mathbf{b} - \mathbf{Bn} \tag{22}$$

which has the least squares solution (using unit weighting)

$$\mathbf{z}_n = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T(\mathbf{b} - \mathbf{Bn}) \tag{23}$$

For a given of $\mathbf{n}$, the residual is given by

$$\mathbf{e}(\mathbf{n}) = \mathbf{Az}_n + \mathbf{Bn} - \mathbf{b} = \mathbf{Hb} - \mathbf{Gn} \tag{24}$$

where

$$\mathbf{G} = \mathbf{HB} \tag{25}$$

$$\mathbf{H} = \left(\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T - \mathbf{I}\right) \tag{26}$$

the squared error can be written

$$E(\mathbf{n}) = (\mathbf{Gn} - \mathbf{Hb})^T(\mathbf{Gn} - \mathbf{Hb}) \tag{27}$$

The error $\mathbf{E}(\mathbf{n})$ is a quadratic function of $\mathbf{n}$. This can be written as

$$E(\mathbf{n}) = (\mathbf{n} - \mathbf{n}_c)^T\mathbf{M}(\mathbf{n} - \mathbf{n}_c) + E_0 \tag{28}$$

where

$$\mathbf{M} = \mathbf{G}^T\mathbf{G}, \tag{29}$$

$\mathbf{n}_c$ is the vector that minimizes $E(\mathbf{n})$, and E_0 is $E(\mathbf{n}_c)$.

We would like to choose $\mathbf{n}$ so that we minimize $E(\mathbf{n})$. In the next section we will elaborate on why restricting $\mathbf{n}$ to be an integer helps us get more accurate solutions. Assuming that we want to minimize $E(\mathbf{n})$ subject to the constraint that $\mathbf{n}$ is an integer, we should be able to determine $\mathbf{n}$ by doing a brute force search through all possible values of $\mathbf{n}$. The lambda method is a method for doing this as efficiently as possible.

It should be noted that we have enough equations so that we have an overdetermined system for $\mathbf{z}$ and $\mathbf{n}$ when we treat the elements of $\mathbf{n}$ as continuous (not necessarily integer) variables. Solving this system gives us a good enough estimate for $\mathbf{n}$ that we do not need to search over a huge range of ambiguities (in each variable n_i) Although, due to the fact that we are searching over an $N-1$ dimensional space (if we have N satellites), this could lead to a very expensive computation if we are not clever about our search.

3.2 Results Assuming G is Well Conditioned

In order to understand why the integer ambiguity process only makes sense if $\mathbf{G}$ is poorly conditioned, we begin by assuming that it is not.

We know that if we substitute the true values for the integer ambiguities $\mathbf{n} = \mathbf{n}_0$, and the true solution $\mathbf{z} = \mathbf{z}_0$ into our equations, then we will get a low residual. That is, the correct solution will satisfy

$$\mathbf{A}\mathbf{z}_0 = \mathbf{b} - \mathbf{B}\mathbf{n}_0 + \delta\mathbf{b} \tag{30}$$

where $\delta\mathbf{b}$ is an error vector. We will write

$$\mid \delta\mathbf{b} \mid = \delta\phi \tag{31}$$

We will suppose that $\delta\phi$ is small enough so that when we multiply it by the wavelength λ we get a number that is on the order of $1cm$. In particular, $\delta\phi$ will be on the order of .01. This shows that if we solve the overdetermined system of equations

$$\mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} = \mathbf{b} \tag{32}$$

where we treat $\mathbf{u}$ as a continuous variable, then we are guaranteed of finding a solution that has a residual as small as $\delta\phi$. Let $(\mathbf{z}_c, \mathbf{u}_c)$ be the least squares solution to this set of equations.

We now consider two cases where we solve the overdetermined system for $\mathbf{z}$ with $\mathbf{n}$ having two different values. When we set $\mathbf{n} = \mathbf{u}_c$, we get the residual $\mathbf{e}(\mathbf{u}_c)$ which we know will have a norm less than $\delta\phi$. Also , when solve the system with $\mathbf{n} = \mathbf{n}_0$, we will also get a residual that is on the order of $\delta\phi$. It follows that we must have

$$\mathbf{e}(\mathbf{u}_c) - \mathbf{e}(\mathbf{n}_0) = \mathbf{G}(\mathbf{u}_c - \mathbf{n}_0) \tag{33}$$

If we square both sides, we get

$$(\mathbf{e}(\mathbf{u}_c) - \mathbf{e}(\mathbf{n}_0))^T(\mathbf{e}(\mathbf{u}_c) - \mathbf{e}(\mathbf{n}_0)) = (\mathbf{u}_c - \mathbf{n}_0)^T\mathbf{M}(\mathbf{u}_c - \mathbf{n}_0) \tag{34}$$

We know that the left hand side of this last equation is small (on the order of $(\delta\phi)^2$. If the matrix **M** does not have any small eigenvalues, then in order to get a small quantity on the right we must have

$$\mathbf{n}_0 \approx \mathbf{u}_c \tag{35}$$

To be more precise, we have

$$\mid \mathbf{u}_c - \mathbf{n}_0 \mid < \delta\phi/\sqrt{\mu} \tag{36}$$

where μ is the smallest eigenvalue of **M**. If **M** is not close to being singular, then the integer ambiguity problem is extremely simple, and probably not even necessary. In particular , we will get an excellent guess to $\mathbf{n}_0$ by merely using $\mathbf{u}_c$.

The difference between the solution $\mathbf{z}_c$ where we let **n** be a continuous variable, and $\mathbf{z}_0$ where we constrain **n** to be an integer is the solution to the overdetermined system of equations

$$\mathbf{A}(\mathbf{z} - \mathbf{z}_0) = \mathbf{B}(\mathbf{u}_c - \mathbf{n}_0) \tag{37}$$

Assuming that this least squares system is well conditioned, the answers for $\mathbf{z}_c$ and $\mathbf{z}_0$ will be close to each other.

3.3 Why is M Nearly Singular

When carrying out simulations on the matrices that would arise from doing relative positioning, it is found that the matrix **M** has three eigenvalues that are small. The eigenvalues get to be smaller the closer the satellite positions at the first and last epochs re to each other.

When doing relative positioning, we have N different satellites. Each of these satellites has M different epochs. The position of each satellite at the different epochs are all closely clustered around the position of the satellite at the first epoch (relative to the distance that the satellite is away from the point being located). We will now show that if the positions of the satellites are the same at all epochs, then the matrix **M** will in fact have three eigenvalues that are identically zero. This shows that when we perturb the satellites away from this degenerate situation, we will have three eigenvalues that are small.

From Eqn. (29) we see that a vector **u** is a null vector of **M** if and only if it is a null vector of **G**. A vector **u** will be a null vector of **G** if for that value of **u** we can solve the overdetermined system of equations

$$\mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} = 0 \tag{38}$$

To see this note that the least squares solution to $\mathbf{A}\mathbf{z} = -\mathbf{B}\mathbf{u}$ is given by

$$\mathbf{z}(\mathbf{u}) = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{B}\mathbf{u} \tag{39}$$

the error in solving this equation is $\mathbf{Az}(\mathbf{u})+\mathbf{Bu}$. This error will be equal to zero iff and only if $\mathbf{Gu}=0$. If the satellites are all at the same position at all epochs, then the equation $\mathbf{Az}+\mathbf{Bu}=0$, gives us M copies of a single set of equations. In particular, this equation is just M copies of the equation

$$\mathbf{A}_0\mathbf{z}+\mathbf{u}=0 \tag{40}$$

where this last set of equations gives the equations in Eqn. (38) that correspond to the first epoch. Note that the matrix $\mathbf{B}$ gets replaced by the identity matrix in Eqn. (40). For any vector $\mathbf{z}_1$ in our three dimensional space of vectors $\mathbf{z}$ we can trivially find a vector $\mathbf{u}_1$ such that $\mathbf{Az}_1+\mathbf{u}_1=0$. that is, we merely choose $\mathbf{u}_1=-\mathbf{Az}_1$. Since $\mathbf{z}$ is a three dimensional space, this will give us three linearly independent vectors $(\mathbf{z}_k,\mathbf{u}_k)$ that will satisfy Eqn. (38), and hence we will have three linearly independent vectors $\mathbf{u}_k, k=1,4$ that satisfy $\mathbf{Gu}_k=0$¡ and hence that satisfy $\mathbf{Mu}_k=0$.

In the relative positioning problem, we know that if we choose $\mathbf{n}=\mathbf{n}_0$ (where $\mathbf{n}_0$ are the true but unknown ambiguities, we will get a low residual. However, since $\mathbf{M}$ is nearly singular, we see that there will be values of $\mathbf{u}$ that are not necessarily extremely close to $\mathbf{u}_c$ that will also give a small residual. In particular, if we set

$$\mathbf{u}=\mathbf{n}_0+\sum_{k=1}^{3}a_k\mathbf{u}_k \tag{41}$$

where $\mathbf{u}_k$ are the eigenvectors of $\mathbf{M}$ associated with the small eigenvalues of $\mathbf{M}$, then we will also get small residuals, even if the values of a_k are not small (this can be made more quantitative). We see that there are many different values of the vector $\mathbf{u}$ that are close (but not too close) to $\mathbf{n}_0$ that will give us small residuals. However, if we have more than four satellites, it is unlikely that any of these vectors will actually take on integer values except for the case where we set $a_k=0$. Thus in this case, if we search over integer values of $\mathbf{u}$ we expect to only find one value where we get a low residual.

This already gives insight into the integer ambiguity problem. If we have 7 satellites, it might at first appear that we need to search over a six dimensional space of integers in order to find the best set. This could be extremely time consuming even if we have restricted our set of integers to a range of about 100 integers in each dimensional. However, we see that we in fact only have to search over a three dimensional space. in particular, we can vary the parameters a_k (a three dimensional space), and try to find values that are close to an integer.

3.4 Some Simulations

Here we present some simulations to illustrate the points we have been making. These simulations were carried out using 7 satellites. They were placed randomly (although visible from the points being located) at a distance of 4 earth radii away. The reference point was put at $\mathbf{p}_0^T=(0,0,1)$ (measured in earth radii) , and the relative position of the point being located was $\mathbf{r}^T=(2.e-$

3,0,0). Random directions for the satellite velocities were given, that were perpendicular to the vector from the center of the earth to the satellites.

Table (1) gives the unit vector $\mathbf{e}_1$ pointing from the center of the earth to the satellite, and the vector $\mathbf{e}_2$ giving the direction of the initial satellite velocity. We do not believe that there is anything special about this configuration, but given the values in the table for the sake of completeness.

In these simulations, we use 100 equally spaced epochs. The angle $\delta\alpha$ gives the change in the angle swept out by the satellites during the survey. In tables (2) and (3) we give results for sweep angles that vary between those for a 4 hour survey, and just a bit under one second. In particular, we set

$$\delta\alpha = \frac{16\alpha_0}{4^k} \tag{42}$$

where α_0 is the sweep angle for a 15 minute survey.

Table (2) shows how the condition number of $\mathbf{G}^T\mathbf{G}$ varies as we change the sweep angle. We clearly see the condition number increasing as the sweep angle diminishes. We also show the eigenvalues of $\mathbf{G}^T\mathbf{G}$. As predicted, there are three small eigenvalues, and three that are order unity. As the sweep angle gets smaller, the small eigenvalues get smaller, and the ones that are order one stay nearly the same.

Table (3) show how the errors in or relative position vary as we change the sweep angle. The errors e_{cont} show how the error varies when we treat $\mathbf{n}$ as a continuous variable. In this case we see that for a 4 hour survey little additional accuracy is given by using the fact that $\mathbf{n}$ must have integer values. In this case, the additional accuracy obtained by assuming $\mathbf{n}$ has integer values could be obtained by rounding the continuous value to the nearest integer.

In Table (3) we also show the values of e_{disc}. This is the error we get if we assume that we can some how resolve the integer ambiguity. As we see ,this error does not grow as the sweep angle diminishes. This shows that the ill conditioning in our problem is only involved with finding the integer ambiguity. That is, if we know the integer ambiguity, our problem does not get worse conditioned as the sweep angle goes to zero.

In table (3) we also show how far away the true integer ambiguity is from that calculated by treating the vector $\mathbf{n}$ as a continuous variable. In particular, ΔN_{cont} gives the largest component (in absolute value) of the vector $\mathbf{u}-\mathbf{n}$, where $\mathbf{u}$ is the vector obtained ‘by treating $\mathbf{n}$ as a continuous variable in our least squares problem, and $\mathbf{n}$ is the true integer ambiguity. In this table we also show how this quantity varies after transforming our equation using the transformation found by the lambda algorithm. The results is given by ΔN_{lambda}. We see that the lambda algorithm significantly decreases the number of integers we need to search over.

N	$\mathbf{e}_1$	$\mathbf{e}_2$
1	(3.88e-01, 5.01e-01 ,7.74e-01)	(7.62e-01, 2.98e-01, -5.75e-01)
2	(-1.50e-01, 3.18e-02 ,9.88e-01)	(8.19e-01, -5.56e-01, 1.42e-01)
3	(3.32e-01, -1.06e-02 ,9.43e-01)	(-9.32e-01, -1.56e-01, 3.26e-01)
4	(2.29e-01, 3.60e-01 ,9.05e-01)	(-9.36e-01, 3.37e-01, 1.03e-01)
5	(1.39e-01, 2.01e-01 ,9.70e-01)	(-2.87e-01, 9.45e-01, -1.55e-01)
6	(2.22e-01, -5.03e-01 ,8.35e-01)	(-9.28e-01, -3.73e-01, 2.15e-02)
7	(1.32e-01, -1.24e-01 ,9.83e-01)	(-9.77e-01, -1.82e-01, 1.08e-01)

Table 1. . This gives the positions and the directions of the velocities used in showing how the condition number of our problem varies with the baseline. The vector $\mathbf{e}_1$ gives the unit vector from the center of the earth to the satellite. The vector $\mathbf{e}_2$ gives the direction of the velocity.

k	$\delta\alpha$	$cond(\mathbf{G}^T\mathbf{G}$	$\underline{\lambda}$
0	2.17e+00	2.04e+01	(4.91e+00, 1.26e+01 , 3.14e+01 1.00e+02, 1.00e+02 , 1.00e+02)
1	5.41e-01	1.25e+02	(8.02e-01, 7.37e+00 , 4.94e+01 1.00e+02, 1.00e+02 , 1.00e+02)
2	1.35e-01	1.06e+03	(9.39e-02, 1.65e+00 , 1.04e+01 1.00e+02, 1.00e+02 , 1.00e+02)
3	3.38e-02	1.48e+04	(6.74e-03, 1.37e-01 , 5.11e-01 1.00e+02, 1.00e+02 , 1.00e+02)
4	8.46e-03	2.30e+05	(4.35e-04, 8.97e-03 , 3.00e-02 1.00e+02, 1.00e+02 , 1.00e+02)
5	2.11e-03	3.65e+06	(2.74e-05, 5.66e-04 , 1.85e-03 1.00e+02, 1.00e+02 , 1.00e+02)
6	5.29e-04	5.83e+07	(1.72e-06, 3.55e-05 , 1.15e-04 1.00e+02, 1.00e+02 , 1.00e+02)
7	1.32e-04	9.32e+08	(1.07e-07, 2.22e-06 , 7.20e-06 1.00e+02, 1.00e+02 , 1.00e+02)

Table 2. This shows how the condition number of $\mathbf{G}^T\mathbf{G}$varies with the baseline. The baseline is proportional to $\delta\alpha$. We have chosen $\delta\alpha = 16\alpha_0/4^k$, where α_0 s the sweep angle for a fifteen minute survey. The vector $\underline{\lambda}$ gives the eigenvalues of $\mathbf{G}$. We see that there are three small eigenvalues that get smaller as we decrease $\delta\alpha$. The other three eigenvalues stay nearly constant as we change $\delta\alpha$.

k	$\delta\alpha$	e_{cont}	e_{disc}	ΔN_{cont}	ΔN_{lambda}
0	2.17e+00	5.76e-02	2.62e-02	1.16e-02	1.16e-02
1	5.41e-01	1.21e-02	1.96e-02	7.53e-03	1.58e-02
2	1.35e-01	5.71e-02	1.65e-02	1.84e-02	2.52e-02
3	3.38e-02	2.55e-01	1.69e-02	6.12e-02	3.59e-02
4	8.46e-03	1.04e+00	1.71e-02	2.28e-01	8.37e-02
5	2.11e-03	4.19e+00	1.71e-02	8.93e-01	1.22e-01
6	5.29e-04	1.68e+01	1.71e-02	3.55e+00	3.13e-01
7	1.32e-04	6.71e+01	1.71e-02	1.42e+01	4.51e-01

Table 3. . This shows how the errors vary as we change the baseline. The baseline is proportional to $\delta\alpha$. We have chosen $\delta\alpha = 16\alpha_0/4^k$, where α_0 is the sweep angle for a fifteen minute survey. The error e_{cont} is the error in the relative position **r** (in meters) we get if we treat the integer ambiguities as a continuous variable. The error e_{disc} is what we get if we use the correct value for the vector **n** when solving for the relative position **r**. The number ΔN_{cont} gives the largest difference between the true integer ambiguity and the one calculated treating **n** as a continuous variable. The number ΔN_{lambda} gives the same quantity , but after one has applied the transformation (found by the lambda method) to our system of equations.

4 Float Solutions and Carrier Phase Geo-Location

Classically integer ambiguity resolution is used as a crucial element in carrier phase geo-location. The float solution, which is obtained by treating the integer ambiguities as continuous variables, is used to get an initial estimate of the integer ambiguities. One then uses a technique such as the lambda algorithm to determine the integer ambiguities, and this information is then used to determine the position of the object being located to a fraction of a wavelength.

We will now show that when locating emitters, it is possible to use the carrier phase information to improve the accuracy even when the satellite baselines are quite short. We do this without resolving the integer ambiguity, and hence obtain solutions that are only accurate to quite a few multiples of the wavelength. However, due to the fact that our location without using carrier phase is much less accurate than this, this can result in a significant improvement in accuracy.

In this section we will discuss how to arrive at Eqn. (5) in the introduction that gives the error in the float solution when using carrier phase geo-location. We will discuss the errors for the case of four satellites, where the problem is precisely determined (not over-determined).

As discussed in §3, Eqn. (19) is underdetermined when the satellite baseline is identically zero. In the case of four satellites, this will result in a precisely determined but ill-conditioned system when the baseline is small. Using an argument almost identical to that in §3, our linearized equations will have three small eigenvalues. Using an analysis almost identical to that in §7, it can be shown that the three small eigenvalues of our linear system can be written as

$$\mu_k = c_k \frac{|\, d\mathbf{s} \,|}{|\, \mathbf{s} \,|}, k = 1, 2, 3 \tag{43}$$

where c_k are constants depending on the geometry, and $|\, d\mathbf{s} \,|$ is the change in satellite position, and $|\, \mathbf{s} \,|$ is the distance from the satellite to the emitter. Here we are using one of the four satellites to determine this number, the distance for the other satellites can be considered as contributing to the calculation of c_k.

As discussed in more detail in §5, when solving a linear system $\mathbf{Cp} = \mathbf{c}$, the error in the solution can be computed by expanding both the solution $\mathbf{p}$ and the error vector $\mathbf{c}$ in terms of the eigenfunctions of $\mathbf{C}$. When we do this we find that if $\mathbf{C}$ has some small eigenvalues, then the errors in $\mathbf{c}$ will be amplified by $1/\mu_k$ where μ_k are the small eigenvalues. For the case of carrier phase geo-location, the errors on the right hand side of Eqn. (19) are proportional to

$$|\, \delta \mathbf{b} \,| = \lambda_e \delta \phi \tag{44}$$

where λ_e is the wavelength of the emitter, and $\delta \phi$ is the size of the errors in the phase measurements. Combining Eqn. (43) with Eqn. (44) we arrive at the expression in Eqn. (5) for the error in the float solution.

It should be emphasized that if we were to use TOA measurements, we would have well conditioned systems (assuming the GDOP is good), but very large measurement errors. When using carrier phase information, we have poorly conditioned systems, depending on the ratio $|\,d\mathbf{s}\,| \,/\, |\,\mathbf{s}\,|$, but very low measurement errors. This can result in significant improvements over the TOA measurement errors depending on the details of the system parameters.

5 The Effect of the Altitude Constraint

Here we are concerned with the question of whether it is possible to use a low precision altitude constraint along with very high precision carrier phase information without seriously degrading the solution error. When no altitude constraint is used, we have seen that for small baselines we get a poorly conditioned system of equations that has small errors in the data. We will be particularly interested in comparing the case where we have four satellites to the case where we have three satellites and an altitude constraint. When we replace one of the satellites with an altitude constraint, we still have nearly dependent vectors for our equations giving the phase information. The question is if the data from the altitude constraint (which has much larger errors) will pollute the solution so that we end up getting poor answers. We will formulate this problem in general abstract terms, and show that the answer is no.

5.1 Formulation of Problem

We will consider problems of the form

$$\mathbf{C}(\varepsilon)\mathbf{p} = \mathbf{c}(\varepsilon) \tag{45}$$

where

$$\mathbf{C}(\varepsilon) = \mathbf{C}_0 + \varepsilon\mathbf{C}_1 \tag{46}$$

and

$$\mathbf{C}_0 = \begin{pmatrix} \mathbf{R}_0 \\ \mathbf{s}^T \end{pmatrix} \tag{47}$$

$$\mathbf{C}_1 = \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{0}^T \end{pmatrix} \tag{48}$$

$$\mathbf{c}(\varepsilon) = \begin{pmatrix} \varepsilon\mathbf{a} \\ \alpha \end{pmatrix} \tag{49}$$

We will assume that $\mathbf{R}_0$ has a three non-trivial left null vectors.

$$\mathbf{q}_k^T\mathbf{R}_0 = 0 \quad k = 1,3 \tag{50}$$

In this problem $\mathbf{c}(\varepsilon)$ represents the errors in the data, and $\mathbf{p}(\varepsilon)$ represents the error in the solution. The parameter ε represents a small parameter representing the baseline. The matrices

$\mathbf{R}_k, k=0,1$ represent the equations associated with the phase data, and the vector $\mathbf{s}^T$ is the linearized equation associated with the altitude date. The fact that there are three left null vectors corresponds to the fact that when the baseline is zero, we have 3 linearly independent null vectors. We have scaled our error vector so that the errors associated with the phase measurements are on the order of ε, but the errors in the altitude are on the order of unity. We want to know if as ε goes to zero, we get errors that are on the order of unity. We will see that this is in fact the case.

5.2 Solution Using Left and Right Eigenvectors

Our arguments will depend on solving systems of linear equations using the left and right eigenvectors of our matrix. We will begin by reviewing this solution process. Suppose we have a system of precisely determined linear equations with m equations in m unknowns.

$$\mathbf{Cp} = \mathbf{c} \tag{51}$$

Generically we expect the matrix $\mathbf{C}$ to have m linearly independent eigenvectors

$$\mathbf{C}\mathbf{u}_k = \mu_k \mathbf{u}_k, \quad k = 1, m \tag{52}$$

We will also have left eigenvectors satisfying

$$\mathbf{v}_k^T \mathbf{C} = \mu_k \mathbf{v}_k^T, \quad k = 1, m \tag{53}$$

Generically these can be normalized so that

$$\langle \mathbf{v}_i, \mathbf{u}_j \rangle = \bar{\mathbf{v}}_i^T \mathbf{u}_j = \delta_{ij} \tag{54}$$

When solving the equation

$$\mathbf{Cp} = \mathbf{c} \tag{55}$$

we can write the solution by expanding both $\mathbf{p}$ and $\mathbf{c}$ in terms of the eigenvectors $\mathbf{u}_k$.

$$\mathbf{p} = \sum p_k \mathbf{u}_k \tag{56}$$

$$\mathbf{c} = \sum c_k \mathbf{u}_k \tag{57}$$

Substituting these expressions into Eqn. (55) and using the fact that $\mathbf{u}_k$ is an eigenvector of $\mathbf{C}$, we get

$$p_k = \frac{c_k}{\mu_k} = \frac{\langle \mathbf{v}_k, \mathbf{c} \rangle}{\mu_k} \tag{58}$$

5.3 Perturbation Expansion for Small Values of ε

We now do a small ε perturbation expansion of the solution $\mathbf{p}(\varepsilon)$ for the problem formulated in §5.1. In particular, we assume we are solving the problem where we have three satellites and an altitude contraint. This will give us a system of 5 equations in 5 unknowns. We will see that the solution is order one as ε goes to zero. For small values of ε we can write

$$\mu_k(\varepsilon) = \mu_k \varepsilon + .., k = 1,3 \tag{59}$$

Eqn. (58) shows that even though the eigenvalues μ_k are order ε, we will get solutions that are order ε provided the right hand side is on the order of ε. However, in the problem formulated in §5.1, the last component of $\mathbf{c}(\varepsilon)$ is order one. It thus appears as though we might get answers that are on the order of $1/\varepsilon$ in this case. We will now see that this is not the case.

In particular we consider the solutions to

$$\mathbf{C}(\varepsilon)\mathbf{p}(\varepsilon) = \mathbf{e}_5 \tag{60}$$

where

$$\mathbf{e}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{61}$$

We will see that the solutions to this equation will be order one because the inner product of $\mathbf{e}_5$ with the left eigenvectors $\mathbf{v}_k(\varepsilon)$ associated with the three small eigenvalues is order ε. To show this note that the left eigenvectors $\mathbf{q}_k$ in Eqn. (50) imply that if $\mathbf{v}_k(\varepsilon)$ are the left eigenvectors of $\mathbf{C}(\varepsilon)$ then

$$\mathbf{v}_k(0) = \begin{pmatrix} \mathbf{q}_k \\ 0 \end{pmatrix} \quad k = 1,3 \tag{62}$$

We will suppose that these are the only left null vectors that $\mathbf{C}_0$ has. For a random vector $\mathbf{s}$ this will generically be the case. We will once again have three eigenvalues that are zero for $\varepsilon = 0$, and they will generically be linear in ε for small values of ε.

Note that

$$\mathbf{v}_k^T(0)\mathbf{e}_5 = 0, k = 1,3 \tag{63}$$

It follows that

$$\langle \mathbf{v}_k(\varepsilon), \mathbf{e}_5 \rangle = O(\varepsilon), k = 1,3 \tag{64}$$

Using Eqn. (58), this shows that the solution $\mathbf{p}(\varepsilon)$ to Eqn. (60) will be order ε as ε goes to zero.

6 The Limits to Integer Ambiguity Resolution

Here we will give a very simple argument explaining why it is not possible to resolve the integer ambiguity when the satellite baseline is too small. We have shown that the residual error can be written as

$$E(\mathbf{n}) = (\mathbf{n} - \mathbf{n}_c)^T \mathbf{M} (\mathbf{n} - \mathbf{n}_c) + E_0 \tag{65}$$

where $\mathbf{n}_c$ is float solution. We have shown that the matrix $\mathbf{M}$ has several eigenvalues that are nearly unity, and three that are small, and get smaller as the baseline decreases.

We will suppose that the phase measurement errors are on the order of $\delta\phi$, and the size of the three small eigenvalues of $\mathbf{M}$ are on the order of ε. In particular, we suppose that when we substitute the true solution $\mathbf{n}_0$ we get the residual

$$E(\mathbf{n}_0) \approx \delta\phi^2 \tag{66}$$

We know that the true answer $\mathbf{n}_0$ must be an integer, and must line on the low residual surface. If

$$\delta\phi << \varepsilon \tag{67}$$

When ε is small, we will get residuals that are on the order of those given by $\mathbf{n}_0$ if

$$\mathbf{n} = \mathbf{n}_0 + \sum_{k=1}^{3} \alpha_k \mathbf{a}_k \tag{68}$$

and

$$\varepsilon \sum_{k=1}^{3} \alpha_k^2 \leq \delta\phi^2 \tag{69}$$

The three dimensional volume of this sheet will be on the order of

$$V_s = \left(\frac{\delta\phi}{\sqrt{\varepsilon}} \right)^3 \tag{70}$$

Suppose we have N satellites. We can stray from this low residual surface in the other $N-1$ directions by $\delta\phi$ and still have a residuals that is as low as $\delta\phi^2$. With this in mind, the low residual surface actually becomes a low residual shell that has a finite thickness. The thickness of this shell (in $N+2$ dimensional space is given by

$$t_S = \delta\phi^{N-1} \tag{71}$$

The total volume of this shell where we get residuals as low as $\delta\phi^2$ is given by

$$V = V_s \times t_S = \frac{\delta\phi^{N+2}}{\varepsilon^{3/2}} \tag{72}$$

A reasonable conjecture is that the probability of having an integer lie on this low residual shell will get to be very close to unity when the volume of this shell gets to be on the same order of magnitude as the unit cell of integers in $N+2$ dimensional space.

this will be the case when

$$\varepsilon = \delta\phi^{(2N+4)/3} \tag{73}$$

In §7 we show that

$$\varepsilon \approx \frac{|\, d\mathbf{s} \,|^2}{|\, \mathbf{s} \,|^2} \tag{74}$$

where $d\mathbf{s}$ is the distance that the satellites move, and $\mathbf{s}$ is the distance from the satellites to the point bieng located. With this in mind, we see that a rough estimate for how much the satellites can move and still resolve the integer ambiguity is given by

$$\frac{|\, d\mathbf{x} \,|}{|\, \mathbf{s} \,|} = K\delta\phi^{(2N+4)/6} \tag{75}$$

Here K is a constant that is order unity, and depends on the particular geometry of our satellites.

7 The Eigenvalues as a Function of the Baseline

7.1 Problem Setup

We will consider the case of two epochs. The overdetermined system of equations can be written as

$$\mathbf{Ax} + \mathbf{Bu} = \mathbf{b} \tag{76}$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \end{pmatrix} \tag{77}$$

and

$$\mathbf{B} = \begin{pmatrix} \mathbf{I}_0 \\ \mathbf{I}_0 \end{pmatrix} \tag{78}$$

In this system, the vector $\mathbf{x}$ represents the relative position divided by the wavelength, and $\mathbf{u}$ gives the integer ambiguities. The matrix $\mathbf{I}_0$ is the $N-1$ by $N-1$ dimensional identity matrix.

If the satellite baseline is short, then $\mathbf{S}_1$ and $\mathbf{S}_2$ will be nearly equal to each other. We will write

$$\mathbf{S}_1 = \mathbf{S}_0 + d\mathbf{S} \quad , \mathbf{S}_2 = \mathbf{S}_0 - d\mathbf{S} \tag{79}$$

where

$$\mathbf{S}_0 = \frac{\mathbf{S}_1 + \mathbf{S}_2}{2} \tag{80}$$

$$d\mathbf{S} = \frac{\mathbf{S}_1 - \mathbf{S}_2}{2} \tag{81}$$

For short satellite baselines, the matrix $d\mathbf{S}$ will be nearly linearly proportional to the time dt between the epochs.

We are interested in the the eigenvalues of the matrix

$$\mathbf{J} = (\mathbf{HB})^T \mathbf{HB} \tag{82}$$

where

$$\mathbf{H} = \mathbf{I} - \mathbf{A}\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T \tag{83}$$

where

$$\mathbf{I} = \begin{pmatrix} \mathbf{I}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_0 \end{pmatrix} \tag{84}$$

$\mathbf{H}$ is symmetric and idempotent, which means it satisfies

$$\mathbf{H}^2 = \mathbf{H} \quad \mathbf{H}^T = \mathbf{H} \tag{85}$$

Using this, and the fact that $\mathbf{H} = \mathbf{H}^T$, we can write

$$\mathbf{J} = \mathbf{B}^T\mathbf{H}\mathbf{B} \tag{86}$$

It follows that we can write

$$\mathbf{J} = 2\mathbf{I} - 4\mathbf{S}_0^T\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{S}_0 \tag{87}$$

This can further be written as

$$\mathbf{J} = 2\left(\mathbf{I} - \mathbf{S}_0\left(\mathbf{S}_0^T\mathbf{S}_0 + d\mathbf{S}^T d\mathbf{S}\right)^{-1}\mathbf{S}_0^T\right) \tag{88}$$

7.2 Perturbation Theory

If $d\mathbf{S} = 0$, then Eqn. (88) shows that for any vector $\mathbf{z}$ we have

$$\mathbf{J}\mathbf{S}_0\mathbf{z} = 0 \tag{89}$$

The matrix $\mathbf{S}_0$ is an $(N-1)\times 3$ dimensional matrix, and will typically have a three dimensional range, so there will be a three dimensional null space of $\mathbf{J}$. That is, $\mathbf{J}$ will have three zero eigenvalues. The matrix $\mathbf{S}_0^T$ will typically have an $N-4$ dimensional null space. If $\mathbf{q}$ is any vector in this null space, then Eqn. (88) shows that if $d\mathbf{S} = 0$, we will have $\mathbf{J}\mathbf{q} = 2\mathbf{q}$. This shows that the matrix $\mathbf{J}$ will have $N-4$ eigenvalues that are equal to 2. When $d\mathbf{S} \neq 0$, the zero eigenvalues will become small eigenvalues. We are particularly interested in knowing how these eigenvalues scale as we change the time dt between the epochs.

We will use the fact that if ε is a small parameter, then to first order in ε we have $(\mathbf{I}+\varepsilon\mathbf{C})^{-1} \approx \mathbf{I} - \varepsilon\mathbf{C}$. Using this, and assuming $d\mathbf{S}$ is small we can write

$$\left(\mathbf{S}_0^T\mathbf{S}_0 + d\mathbf{S}^T d\mathbf{S}\right)^{-1} = \left(\mathbf{I} - \left(\mathbf{S}_0^T\mathbf{S}_0\right)^{-1} d\mathbf{S}^T d\mathbf{S}\right)\left(\mathbf{S}_0^T\mathbf{S}_0\right)^{-1} \tag{90}$$

It follows that we can write

$$\mathbf{J} = \mathbf{J}_0 + d\mathbf{J} + \ldots \tag{91}$$

where

$$\mathbf{J}_0 = 2\left(\mathbf{I} - \mathbf{S}_0\left(\mathbf{S}_0^T\mathbf{S}_0\right)^{-1}\mathbf{S}_0^T\right) \tag{92}$$

and

$$d\mathbf{J} = 2\mathbf{S}_0\left(\mathbf{S}_0^T\mathbf{S}_0\right)^{-1}\left(d\mathbf{S}^T d\mathbf{S}\right)\left(\mathbf{S}_0^T\mathbf{S}_0\right)^{-1}\mathbf{S}_0^T \tag{93}$$

We would like to find out how the zero eigenvalues change when we add the perturbation $d\mathbf{J}$ to our matrix. When $d\mathbf{J} = 0$, the eigenvector can be written as

$$\mathbf{u}_0 = \mathbf{S}_0\mathbf{z} \tag{94}$$

where $\mathbf{z}$ is an arbitrary vector. When we add a small perturbation to $\mathbf{J}$ we get

$$\mathbf{u} = \mathbf{S}_0\mathbf{z} + d\mathbf{u} \tag{95}$$

Here $d\mathbf{u}$ is an infinitesimally small vector. Collecting the lowest order terms in the expression

$$\left(\mathbf{J}_0 + d\mathbf{J}\right)\left(\mathbf{S}_0\mathbf{z} + d\mathbf{u}\right) = \lambda\left(\mathbf{S}_0\mathbf{z} + d\mathbf{u}\right) \tag{96}$$

we get

$$d\mathbf{J}\mathbf{S}_0\mathbf{z} - \lambda\mathbf{S}_0\mathbf{z} = -\mathbf{J}_0 d\mathbf{u} \tag{97}$$

Since the matrix $\mathbf{J}_0$ is singular, this can only have a solution if the quantity on the left hand side of Eqn. (97) is orthogonal to all the vectors in the null space of $\mathbf{J}^T$. This is equivalent to requiring that

$$\mathbf{S}_0^T\left(d\mathbf{J}\mathbf{S}_0\mathbf{z} - \lambda\mathbf{S}_0\mathbf{z}\right) = 0 \tag{98}$$

This can be written as

$$\left(2d\mathbf{S}^T d\mathbf{S} - \lambda\mathbf{S}_0^T\mathbf{S}_0\right)\mathbf{z} = 0 \tag{99}$$

This is a three dimensional eigenvalue problem for determining how the three zero egeinvalues of $\mathbf{J}$ change under a perturbation. It should be noted that the entries in the matrix $\mathbf{S}_0$ are on the order of unity, while the entries in the matrix $d\mathbf{S}$ are on the order of $\mid d\mathbf{s}_k \mid / \mid \mathbf{s}_k \mid$ where $\mathbf{s}_k$ is the position of a satellite. This shows us that the eigenvalues λ will be on the order of

$$\lambda \approx \frac{\mid d\mathbf{s}_k \mid^2}{\mid \mathbf{s}_k \mid^2} \tag{100}$$

That is, the eigenvalues λ will be on the order of the square of the distance the satellites have moved divided by the distance to the satellites.

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