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Spherical Heat Conduction Verification Problem

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What follows is the derivation of an analytic solution for a pure heat conduction problem which should be useful for verification purposes. Consider a sphere of radius R at a constant temperature T_0. I seek a solution to the homogeneous heat diffusion equation in spherical coordinates (exterior to the hot sphere)

$$\rho C \frac{\partial T}{\partial t} - \frac{1}{r^2}\frac{\partial}{\partial r}(\kappa r^2 \frac{\partial T}{\partial r}) = 0 \qquad [1]$$

subject to the initial and boundary conditions

$$\begin{aligned} &T(t=0, r>R) = 0 \\ &T(t, r \le R) = T_0 \quad . \\ &T(t, r=\infty) = 0 \end{aligned} \qquad [2]$$

In Eq.1, C is the specific heat, ρ is the density, and κ is the conduction coefficient. Specify temperature dependent forms for the specific heat and conduction coefficients as

$$\begin{aligned} &\kappa = \kappa_0 T^n \\ &C = C_0 T^n \end{aligned} \qquad [3]$$

where κ_0 and C_0 are constants and n is some exponent not necessarily an integer. If we substitute Eq.3 into Eq.1 and define

$$\Phi = T^{n+1} \qquad [4]$$

we have

$$\rho C_0 \frac{\partial \Phi}{\partial t} - \frac{1}{r^2}\frac{\partial}{\partial r}(\kappa_0 r^2 \frac{\partial \Phi}{\partial r}) = 0 . \qquad [5]$$

Take the Laplace transform of Eq.5 to get

$$\frac{\partial}{\partial r}(r^2 \frac{\partial \Phi(s,r)}{\partial r}) - r^2 s \frac{\rho C_0}{\kappa_0}\Phi(s,r) = 0 . \qquad [6]$$

Now the transformed boundary conditions are

$$\begin{aligned} &\Phi(s, r=R) = \frac{\Phi_0}{s} = \frac{T_0^{n+1}}{s} \quad . \\ &\Phi(s, r=\infty) = 0 \end{aligned} \qquad [7]$$

This ODE has the solution (for r>R)

$$\Phi(s,r) = \frac{\Phi_0 R}{sr}\exp\{-(r-R)\sqrt{\frac{s\rho C_0}{\kappa_0}}\} \quad . \qquad [8]$$

Now perform the Laplace inversion to get

$$T(t,r) = \begin{Bmatrix} T_0 \\ T_0[\frac{R}{r} Erfc\{0.5(r-R)\sqrt{\frac{\rho C_0}{\kappa_0 t}}\}]^{(\frac{1}{n+1})} \end{Bmatrix} \quad \text{for} \quad \begin{matrix} r \le R \\ r > R \end{matrix} \qquad [9]$$

For $n = 0$ this is an easy computational problem. But at large n it will strain the diffusion codes ability to accurately resolve the gradients in the material properties.