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Spherical Heat Conduction Verification Problem

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What follows is the derivation of an analytic solution for a pure heat conduction problem which should be useful for verification purposes. Consider a sphere of radius R at a constant temperature T_0 . I seek a solution to the homogeneous heat diffusion equation in spherical coordinates (exterior to the hot sphere)

$$\rho C \frac{\partial T}{\partial t} - \frac{1}{r^2} \frac{\partial}{\partial r} (\kappa r^2 \frac{\partial T}{\partial r}) = 0 \quad [1]$$

subject to the initial and boundary conditions

$$\begin{aligned} T(t=0, r > R) &= 0 \\ T(t, r \leq R) &= T_0 \\ T(t, r = \infty) &= 0 \end{aligned} \quad [2]$$

In Eq.1, C is the specific heat, ρ is the density, and κ is the conduction coefficient. Specify temperature dependent forms for the specific heat and conduction coefficients as

$$\begin{aligned} \kappa &= \kappa_0 T^n \\ C &= C_0 T^n \end{aligned} \quad [3]$$

where κ_0 and C_0 are constants and n is some exponent not necessarily an integer. If we substitute Eq.3 into Eq.1 and define

$$\Phi = T^{n+1} \quad [4]$$

we have

$$\rho C_0 \frac{\partial \Phi}{\partial t} - \frac{1}{r^2} \frac{\partial}{\partial r} (\kappa_0 r^2 \frac{\partial \Phi}{\partial r}) = 0. \quad [5]$$

Take the Laplace transform of Eq.5 to get

$$\frac{\partial}{\partial r} (r^2 \frac{\partial \Phi(s, r)}{\partial r}) - r^2 s \frac{\rho C_0}{\kappa_0} \Phi(s, r) = 0. \quad [6]$$

Now the transformed boundary conditions are

$$\begin{aligned} \Phi(s, r = R) &= \frac{\Phi_0}{s} = \frac{T_0^{n+1}}{s} \\ \Phi(s, r = \infty) &= 0 \end{aligned} \quad [7]$$

This ODE has the solution (for $r > R$)

$$\Phi(s, r) = \frac{\Phi_0 R}{sr} \exp\left\{-(r-R) \sqrt{\frac{s \rho C_0}{\kappa_0}}\right\}. \quad [8]$$

Now perform the Laplace inversion to get

$$T(t, r) = \begin{cases} T_0 \\ T_0 \left[\frac{R}{r} \operatorname{Erfc}\left\{0.5(r-R) \sqrt{\frac{\rho C_0}{\kappa_0 t}}\right\} \right]^{\left(\frac{1}{n+1}\right)} \end{cases} \quad \text{for} \quad \begin{cases} r \leq R \\ r > R \end{cases} \quad [9]$$

For $n = 0$ this is an easy computational problem. But at large n it will strain the diffusion codes ability to accurately resolve the gradients in the material properties.