A Simple Harmonic Universe

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We explore simple but novel bouncing solutions of general relativity that avoid singularities. These solutions require curvature $k = +1$, and are supported by a negative cosmological term and matter with $-1 < w < -1/3$. In the case of moderate bounces (where the ratio of the maximal scale factor a_+ to the minimal scale factor a_- is $\mathcal{O}(1)$), the solutions are shown to be classically stable and cycle through an infinite set of bounces. For more extreme cases with large a_+/a_-, the solutions can still oscillate many times before classical instabilities take them out of the regime of validity of our approximations. In this regime, quantum particle production also leads eventually to a departure from the realm of validity of semiclassical general relativity, likely yielding a singular crunch. We briefly discuss possible applications of these models to realistic cosmology.

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Two questions have recurred again and again in theoretical cosmology, starting with [1–3]: 1) is the Universe eternal, or did it have a beginning at some definite time in the past?, and 2) is it possible to make Universes which enjoy one or more "bounces" where the scale factor first crunches, and then bangs? [18].

The answers to these two questions are deeply intertwined with the subject matter of the singularity theorems of Penrose and Hawking (discussed comprehensively in [4]). These theorems show that, given an energy condition of the form

$$T_{\mu\nu} v^\mu v^\nu \geq 0 \tag{1}$$

for a suitable class of vectors v^μ, where $T_{\mu\nu}$ is the stress-energy tensor of the sources supporting the Universe, one can prove that the Universe must be geodesically incomplete ("singular"). Even in scenarios where the current ΛCDM cosmology was preceeded by a phase of slow-roll inflation [5], with eternal inflation occurring on even larger scales, it is a striking result [6] that the initial singularity remains.

It is instructive to discuss which energy conditions need to be assumed to prove existence of a cosmological singularity for the FLRW cosmologies

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2(\theta) d\phi^2) \right). \tag{2}$$

For $k = -1, 0$ the only condition that must be assumed is the null energy condition (NEC), i.e. eqn. (1) where v^μ is a future-pointing null vector field. The NEC is reasonable and in agreement with the known macroscopic matter and energy sources in our Universe [19].

For $k = +1$, however, the *strong* energy condition (SEC) must be assumed [20]. We are essentially certain that this condition is violated by macroscopic sources in our world, as well as in many completely consistent theoretical toy models. The goal of this paper is to explore the two questions above for $k = +1$ Universes with sources satisfying the NEC (but violating the SEC). We will find that one can make classical cosmologies that live eternally, undergoing an infinite sequence of non-singular bounces, and remaining within the regime of validity of general relativity. When the ratio between maximal and minimal scale factors is not too large, these cosmologies are stable to small perturbations. In the opposite regime, when the ratio is large, we instead find both classical and quantum pathologies; classically there are growing modes (which can be tuned away), and quantum mechanically, particle production backreacts significantly after some number of cycles, likely causing a singular crunch [21].

Solutions. The FRW equations for the metric eqn. (2) are

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3} G\rho - \frac{k}{a^2} \ , \ \ \frac{\ddot{a}}{a} = -\frac{4\pi}{3} G\,(\rho + 3p) \tag{3}$$

where ρ is the energy density and p is the pressure. We want oscillatory solutions, namely those with two extrema ($\dot{a} = 0$) such that at the smaller (where $a \equiv a_-$) $\ddot{a} > 0$, and at the larger (where $a \equiv a_+$) $\ddot{a} < 0$. It is easy to see that these requirements, along with the NEC, only allow solutions for a when there is positive curvature, $k = +1$. The minimal model which oscillates has three components: positive curvature, a negative cosmological constant (energy density = $\Lambda < 0$), and a "matter" source with equation of state in the range

$$p = w\rho, \ \ -1 < w < -1/3 \tag{4}$$

(we will see a bit later that it is important that this source *not* be a perfect fluid). For this content the energy density is $\rho = \Lambda + \rho_0\, a^{-3(1+w)}$ where ρ_0 is a constant parametrizing the density of the "matter." Then the solution to eqns. (3) is oscillatory.

In the special case that $w = -\frac{2}{3}$ these equations just describe a constrained simple harmonic oscillator and the

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solution (setting $k=+1$) is

$$a=\frac{\rho_0}{2|\Lambda|}+a_0\cos\left(\omega t+\psi\right) \tag{5}$$

where ψ is an arbitrary phase and

$$\omega\equiv\sqrt{\frac{8\pi}{3}G|\Lambda|}\ ,\ a_0\equiv\frac{1}{2|\Lambda|}\sqrt{\frac{3\Lambda}{2\pi G}+\rho_0^2}. \tag{6}$$

This requires $\rho_0^2\geq\frac{3}{2\pi}\frac{|\Lambda|}{G}$ for positivity of the radicand. Note that the Universe is static when this condition is saturated, though this requires a fine-tuning. In the opposite limit, $\frac{\rho_0^2}{\Lambda}\to\infty$, the ratio of the maximum to the minimum sizes a_+/a_- of the Universe goes to infinity.

It is useful to switch to conformal time η, where $d\eta^2=dt^2/a(t)^2$. Defining

$$\gamma\equiv\frac{3|\Lambda|}{2\pi G\rho_0^2} \tag{7}$$

the solution for the scale factor (5) becomes

$$a(\eta)=\frac{1}{\omega}\frac{\sqrt{\gamma}}{1-\sqrt{1-\gamma}\cos(\eta)}\ . \tag{8}$$

Here ω is the frequency of oscillations given in (5), and we have set $\psi=0$. Notice that $\gamma\approx 4a_-/a_+$ for small γ.

Stability. There are several simple stability issues we discuss here. (See e.g. [7] for a discussion of the corresponding stability issues in the Einstein static Universe.) First of all, the "matter" source in Eqn. (4) may itself present dangers. In fact the canonical source which behaves this way, a perfect fluid, would present a serious problem. To see this, recall that for scalar perturbations, one considers a more general metric

$$ds^2=a(\eta)^2\left[-(1+2\Phi(\eta,x))d\eta^2+(1-2\Psi(\eta,x))d\Omega_3^2\right]. \tag{9}$$

For perfect fluids, $\Phi=\Psi$, $\delta p=c_s^2\delta\rho$, and

$$\Psi''+3\mathcal{H}(1+c_s^2)\Psi'\ +\ \left[2\mathcal{H}'+(1+3c_s^2)(\mathcal{H}^2-k)\right]\Psi - c_s^2\nabla_{S^3}^2\Psi=0\ . \tag{10}$$

The derivatives are with respect to conformal time, and $\mathcal{H}=a'/a$. An important point, clear from the *sign* of the $\nabla_{S^3}^2$ term in (10), is that if c_s^2 is negative, there are disastrous short-distance (high-momentum) instabilities.

Now, a perfect fluid with $w<-1/3$ would have negative c_s^2. However, as explained in [8], one can find matter sources supporting equations of state of the form (4) but with $c_s^2>0$ (and in fact comparable to the speed of light), if one considers a "solid" with elastic resistance to shear deformations. A canonical example which they discuss is a frustrated network of domain walls, which in the leading approximations gives precisely the $w=-\frac{2}{3}$ case with the simplest behavior of $a(t)$. For our purposes, the crucial point is simply that once we have achieved c_s^2 sufficiently positive, it is easy to check that the scalar perturbations above are stable. Further discussion of this point will appear in [9].

In addition to the above scalar perturbations, we need to consider tensor perturbations. These are governed by an equation whose form is identical to that of (13) below, and will be analyzed there. Next, homogenous but anisotropic perturbations are given by the Bianchi type IX metric [10] $ds^2=-dt^2+\sum_{i=1}^3 a_i^2(t)\,\sigma_i^2$, where σ_i are the Maurer-Cartan forms on S^3. It is useful to parametrize the a_i by an overall $a(t)$ and two 'shape' deformations $\beta_\pm(t)$,

$$a_1=a\,e^{\frac{\beta_++\beta_-}{2}}\ ,\ a_2=a\,e^{\frac{\beta_+-\beta_-}{2}}\ ,\ a_3=a\,e^{-\beta_+}\,. \tag{11}$$

Linearizing the FRW equations for $\beta_\pm\ll 1$ then obtains

$$\beta_\pm''+2\mathcal{H}\beta_\pm'+8k\beta_\pm=0\,. \tag{12}$$

These modes will be analyzed momentarily.

Another potential source present in our Universe, and indeed in any cosmological scenario, is gravity itself. That is, the Universe may respond to a produced gas of gravitons. The dynamics of massless particles may be described by a probe scalar field, with equation of motion

$$\phi''+2\mathcal{H}\phi'-\nabla_{S^3}^2\phi\ =\ 0\,. \tag{13}$$

We note as a curiosity that because of the periodicity of a, (10) and (13) can be recast as a Schrödinger problem characterising motion of electrons in a particular 1d periodic potential (where Bloch's theorem applies).

The three types of perturbations (10), (12) and (13) have a similar structure; in fact, the anisotropic perturbation (12) is just a particular case of (13). Tensor modes of the metric are also described by eqn. (13). We denote a generic linearized mode by u, and expand in spherical harmonics, $\nabla_{S^3}^2u_l=-l(l+2)u_l$. We now summarize the results of our numerical analysis of perturbations.

There are three regimes of momenta where we expect (and shall find) different behavior. It is important to distinguish Universes with $\gamma\sim\mathcal{O}(1)$ from those with $\gamma\ll 1$; we shall describe the behavior in both limits.

• $l=0$ homogeneous mode: we expect that shifting such a mode should be analogous to shifting the homogeneous mode of the scale factor, which would simply move us in the space of periodic solutions and lead to a linear growth of the perturbation in naive perturbation theory (since e.g. two sinusoidal functions with slightly different frequency will perturbatively grow apart at a linear rate, as they get out of phase). This is borne out by the numerics for both $\gamma\ll 1$ and $\gamma\sim 1$. Thus, although this looks like a growing perturbation, that is likely just a failure of perturbation theory. Certainly a homogeneous, isotropic perturbation to the metric (as opposed to ϕ) just moves us to a different one of our solutions and is not a dangerous instability.

• modes with momentum $2 \leq l \lesssim \frac{1}{\sqrt{\gamma}}$ on the S^3: these have long enough wavelength to detect the difference between our cosmology and Minkowski space. For $\gamma \sim 1$, i.e. a Universe which is "quivering" around a mean size, we find that they have oscillatory behavior and are thus stable. In contrast, for $\gamma \ll 1$, they can be unstable; we shall discuss bounds derived from their behavior below.

• modes with $l \gg \frac{1}{\sqrt{\gamma}}$: these have small enough wavelength that they should barely detect the departures of our metric from flat space. As expected, they behave more or less like typical Minkowski space scalar field modes for times smaller than the period of oscillation of the Universe, for both $\gamma \ll 1$ and $\gamma \sim 1$.

The $l = 1$ mode is special. The perturbations governed by (13) are stable for $\gamma \sim 1$; on the other hand, the gravitational instabilities sourced by (10) are always unstable for $l = 1$. For the case of a single-component perfect fluid, on which we have focused so far, this mode is absent from the physical spectrum: $\partial_i \Psi_{l=1}$ generates a global rotation on the S^3 and hence is pure gauge. However, in multi-component systems there will generically be entropy perturbations; these contribute an inhomogeneous term to (10) and can source a physical $l = 1$ mode. We find that the corresponding metric scalar mode $\Psi_{l=1}$ grows for all γ, unlike the case of modes with $l \geq 2$. [22]

However, we point out that even in these cases the $l = 1$ growing mode may be absent due to different mechanisms. A simple variant of our setup would be to orbifold the S^3 by a freely acting group in order to project out this mode. Orbifolding does not change the equations of motion (only local quantities appear there) but will project out modes from the spectrum. We have also not included the effects of non-gravitational damping modes. For example, a gas of gravitons will be subject to collisionless damping from free streaming, which damps growth at a rate proportional to the frequency ω_k of the mode. There is a range of γ for which the growth rate of the $l = 1$ mode predicted by (10) is smaller than the corresponding free streaming damping rate, thus killing this mode. The other fluids in the setup, including the domain wall network, may also have other collisional forms of damping that can reduce the growth of this mode. We will discuss these points further in [9]. In what follows we will assume that the $l = 1$ growing mode is absent.

To summarize, the Universes with $\gamma \sim 1$ are classically stable at the linearized level and live forever. The Universes with $\gamma \ll 1$ suffer from exponential growth (as a function of cycle number) of the finite momentum modes with $l \ll \frac{1}{\sqrt{\gamma}}$. We show the numerical analysis of the modes of eqn. (13) in Figure 1 for all three regimes of momenta and various values of γ. The exponential growth whose beginning is shown in the middle figure would not be present for $\gamma \sim 1$. The metric scalar perturbations Ψ behave in a qualitatively similar way, although they exhibit a faster rate of growth due to the gravitational backreaction included in eqn. (10) [23].

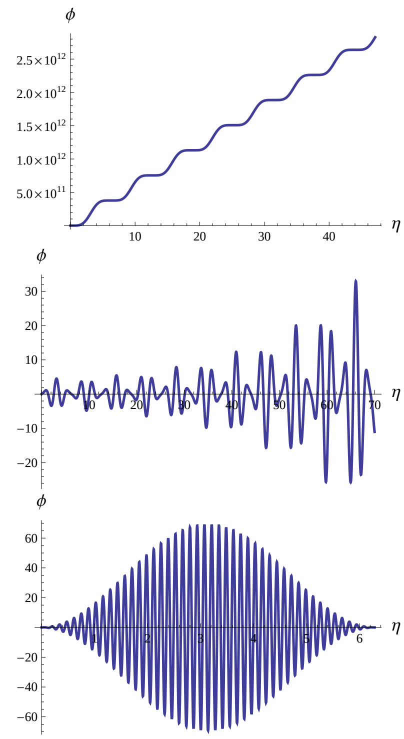

FIG. 1: Massless scalar field evolution in conformal time, for different values of momenta. The first plot shows the homogeneous ($l = 0$) solution with $\gamma = 10^{-5}$. The second plot corresponds to $l = 2$ and $\gamma = 0.225$; three cycles are included, showing the exponential growth in the amplitude. The third plot has $l = 45$ and $\gamma = 0.01$, and shows a single cycle. The initial conditions are $\phi(0) = 0$ and $\phi'(0) = 1$.

Classical and quantum destruction of the Universe. For $\gamma \sim 1$, the Universes we are studying are classically stable. For $\gamma \ll 1$, the exponential growth of the modes with $0 < l < \frac{1}{\sqrt{\gamma}}$ clearly indicates that we should expect such a Universe to have a bounded lifetime (at least until our approximations break down). Can we tune this to allow a large number of oscillations within our regime of computational control?

The cross-over from exponential to oscillatory behavior in the numerical solutions at $l \sim l_c = \frac{1}{\sqrt{\gamma}}$, together with basic attempts to fit the growing solutions, suggest a rough form for the growing modes

$$u_l(N) \sim u_0 \exp\left(c\sqrt{1 - \frac{l^2}{l_c^2}} \times N\right) \tag{14}$$

where $c \sim O(1)$, and $u_l(N)$ denotes the value of the lth momentum mode after N oscillations, with starting vev u_0. The important physical question is: when does the energy density in these modes become large enough that they compete with the dominant energy sources present in our background geometry? The ratio of the energy density in the scalar perturbation to the cosmological constant is given by

$$\sum_l \frac{a^2 l(l+2) u_l^2}{a^4|\Lambda|} \sim \frac{\gamma}{M_P^2} \int^{l_c} dl\ l^2 u_l^2 \ . \qquad (15)$$

Using (14), and evaluating the resulting integral in a saddle-point approximation, we find the dominant l is $l_{saddle}^2 \sim l_c^2/N$, and the energy ratio is thus

$$\frac{\epsilon_u}{|\Lambda|} \sim \gamma l_c^2 \frac{u_0^2}{M_P^2} \exp\left(O(N) - O(\log N)\right) \ . \qquad (16)$$

So, backreaction from the classical scalar field becomes important after a number of cycles N_c given by

$$N_c \sim \log\left(\frac{M_P}{u_0}\right) \ . \qquad (17)$$

Classically, by tuning the initial state to have sufficiently small u_0 in Planck units, we can obtain an arbitrarily large lifetime even for the systems with $\gamma \ll 1$.

Quantum mechanics is expected to induce an RMS value of u_0, preventing a classical tune from saving the Universe for $\gamma \ll 1$. Let us show this explicitly for the scalar field (13). To quantize the field, we impose canonical commutation relations on the canonically normalized scalar $\chi \equiv a(\eta)\phi$,

$$[\chi(\theta), \partial_\eta \chi(\theta')] = i\delta^{(3)}(\theta - \theta') , \qquad (18)$$

where θ coordinatizes the three-sphere. This implies that in the instantaneous ground state characterizing the scalar at a time when the Universe has scale factor a, $a^2\phi_0^2 \sim 1$. Now $a_+ = \frac{2}{\omega\sqrt{\gamma}}$, while $a_- = \frac{\sqrt{\gamma}}{2\omega}$. We are free to choose, as our initial quantum state, the instantaneous vacuum associated to any value of the scale factor. Choosing, for instance, the "natural" quantum vacuum associated with $a = a_+$ (where the Universe is large and smooth and we have a natural expectation for the vacuum state), gives $\phi_0 \sim \omega\sqrt{\gamma}$. This gives a bound on the number of cycles

$$N_c \sim \log\left(\frac{M_P}{\omega\sqrt{\gamma}}\right) \ . \qquad (19)$$

This can be made parametrically large for small values of Λ (or, in this particular case, for sufficiently small γ).

For $\gamma \sim 1$ the solutions to (13) are oscillatory, so the RMS values for various fields induced by quantum mechanics will not cause instabilities. Hence for these values of γ, the universe is stable against perturbative classical and quantum instabilities.

Conclusions and Questions. Our model with $\gamma \sim 1$ seems to provide an example of an eternal universe without singularities. It avoids the singularity theorems by having positive curvature and violating the SEC, though not the NEC. This universe is both classically and quantum mechanically stable against small perturbations at linearized level. Possibly, however, the background "solid" could have microscopic dynamics that produce entropy, leading to a singularity even in our seemingly eternal models. This is an interesting, but model-dependent, question. We have focused on model-independent bounds here. This raises the question, can we find assumptions (weaker than the SEC) allowing proof of a 'quantum singularity theorem' that applies to closed Universes, extending the results of [4] to this physically important case without assuming unphysical energy conditions? We intend to pursue these and other questions in the future [9].

The cyclic nature of these cosmologies strongly suggests searching for exactly periodic quantum states in our geometry, perhaps characterizing special choices of the wavefunction of the Universe. Could some of these special quantum states be eternal, and provide "natural" boundary conditions for certain closed cosmologies, in analogy with [11]? Perhaps this could even allow compactification of the time dimension.

How realistic are these models? In particular, can we embed realistic ΛCDM cosmologies, with a preceding phase of inflation, into the expansion phase of one of our cycles in the $\gamma \ll 1$ case? This would require a transition from radiation (and then matter) dominance during expansion to curvature (and our "solid") dominance near the following bounce. Given their relative scalings with a, this may require the radiation and matter modes to be "Higgsed" above a large energy scale. As we have seen, such a universe with $\gamma \ll 1$ appears unstable. However, we were maximally pessimistic in ignoring free streaming; could this effect vitiate the growth of inhomogeneous perturbations? Alternatively, for the stable, eternal $\gamma \sim 1$ cosmologies, can we envision a Universe which begins in such a phase, persists there for a long period, and then tunnels to a realistic inflationary Universe? Could either of these possibilities demonstrate that our observed universe might not have emerged from an initial singularity?

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[18] We will only discuss solutions where the minimal size attained during the bounce can be made parametrically larger than the Planck length. Ambitious cosmological scenarios that invoke crunches which require a boundary condition at the singularity provided by the as yet unknown high energy theory appear in e.g. [12, 13].

[19] Interesting cosmological scenarios which attain a smooth bounce by violating the NEC can be found in [14].

[20] More generally, the power of the singularity theorems is that they apply for much less symmetric space-times than those allowed by the FLRW ansatz. In these generic cases as well, the SEC must be assumed to prove a theorem. So our results for $k = +1$ FLRW may be reflective of phenomena that can occur in less symmetric space-times.

[21] Other, unrelated discussions of attempts to make non-singular cosmologies using $k = +1$ appear in [15–17].

[22] The metric scalar fluctuations are classified into adiabatic (or curvature) perturbations, and entropy (or isocurvature) perturbations. The later are described by (10) plus an inhomogeneous term, and satisfy the initial condition $\Psi = \Psi' = 0$. This term can act as a source of $\Psi_{l=1}$, which grows exponentially fast for all γ.

[23] As a check we note that the homogeneous equation can be solved exactly, exhibiting the expected linear growth. The other behaviors are similarly as one expects, and the crossover between the linearly growing, exponentially growing, and well-behaved short-wavelength modes occurs smoothly, giving no indication of numerical glitches.