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SOME CHARACTERIZATIONS OF INNER PRODUCT SPACES VIA SOME GEOMETRICAL INEQUALITIES

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In this paper, we explore some geometrical inequalities. We present versions for inner product spaces and we prove that this inequalities can characterize the inner product spaces.

1. INTRODUCTION

It is known that any inner product space is also a normed space, but the reverse is not necessary true. More, the reverse implication represented, more time, one of the important problems of mathematical analysis. In 1935, Fréchet [**6**] solved this problem and obtained the first characterization of inner product spaces.

Proposition 1.1. (Fréchet) *A complex normed space* $(X, \|\cdot\|)$ *is an inner product space if and only if*

$$\|x+y+z\|^2 + \|x\|^2 + \|y\|^2 + \|z\|^2 = \|x+y\|^2 + \|y+z\|^2 + \|z+y\|^2 ,$$

for all $x, y, z \in X$.

In the same year, Jordan and von Neumann presented a new result [**8**], known as *the parallelogram law.* This result had a decisive impact on the development of this research direction.

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Proposition 1.2. (Jordan-von Neumann) *A complex normed space* $(X, \|\cdot\|)$ *is an inner product space if and only if*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

for all $x, y \in X$.

Later, many mathematicians have obtained other such kind of characterizations. Some recent examples are included in the references [**11**] or [**12**].

Hence, transforming Jordan's result, a new idea appears: the characterizations of inner product space in terms of inequalities. In this direction, the first known result, due to Schoenberg [**14**], is contained in the following proposition.

Proposition 1.3. *Let* $(X, \|\cdot\|)$ *be a normed space. The norm* $\|\cdot\|$ *is deduced by an inner product if and only if*

$$\|x+y\|^2 + \|x-y\|^2 \approx 2\|x\|^2 + 2\|y\|^2,$$

for all $x, y \in X$, *where the symbol "*$\approx$*" denotes exactly one and only one of the symbol "*$\leq$*" or "*$\geq$*".*

Today, its known more results. The references [**7**] or [**13**] contain some recent examples. In this context, the aim of our paper is to present new characterizations of the inner product space involving inequalities. We mention that our main results have a starting point some geometrical inequalities, proposed in various journals of elementary mathematics.

In the second section we recall some results included in the references [**2**] or [**15**] and we present and prove similar version for the inner product space (Theorem 2.1, 2.2 and 2.3). The proof of the initial results are connected, fact that is maintained for its analogues on the inner product space results.

The third section is reserved for new characterizations of the inner product space (Theorem 3.1, 3.2 and 3.3). These theorems shows that some results from elementary mathematics are more important than they seem at first view.

2. SOME GEOMETRICAL INEQUALITIES

In [**2**], Ballieu proved that, for any triangle ABC, the following inequality is true:

$$\sin\frac{A}{2} \leq \frac{a}{b+c}. \tag{1}$$

The equality holds if and only if $b = c$.

By using geometrical interpretation of some relation from normed or inner product space (for example see [**1**]). the inequality (1) for inner product space is following:

Theorem 2.1. *Let X a real or complex normed space and $x, y, z \in X$ such that $x + y + z = 0$. Then*

$$(\|y\| + \|z\|)\sqrt{\|x\|^2 - (\|y\| - \|z\|)^2} \le 2\|x\| \cdot \sqrt{\|y\| \cdot \|z\|}. \tag{2}$$

The equality holds when $\|y\| = \|z\|$ or there exists $\lambda \ge 0$ such that $y = \lambda z$.

Proof. We have

$$\begin{aligned}
& (\|y\| + \|z\|) \cdot \sqrt{\|x\|^2 - (\|y\| - \|z\|)^2} \le 2\|x\| \cdot \sqrt{\|y\| \cdot \|z\|} \\
\Leftrightarrow\ & (\|y\| + \|z\|)^2 \cdot \left(\|x\|^2 - (\|y\| - \|z\|)^2\right) \le 4\|x\|^2 \cdot \|y\| \cdot \|z\| \\
\Leftrightarrow\ & (\|y\| + \|z\|)^2 \|x\|^2 - (\|y\| + \|z\|)^2 (\|y\| - \|z\|)^2 \le 4\|x\|^2 \cdot \|y\| \cdot \|z\| \\
\Leftrightarrow\ & 0 \le (\|y\| + \|z\|)^2 (\|y\| - \|z\|)^2 - (\|y\| - \|z\|)^2 \|x\|^2 \\
\Leftrightarrow\ & 0 \le (\|y\| - \|z\|)^2 \left((\|y\| + \|z\|)^2 - \|x\|^2\right).
\end{aligned}$$

The last inequality is true due the fact that

$$\|x\| = \|-y - z\| = \|y + z\| \le \|y\| + \|z\|.$$

The equality holds if $(\|y\| + \|z\|)^2 = 0$ or $(\|y\| + \|z\|)^2 - \|x\|^2 = 0$, also $y = z$ or or there exists $\lambda \ge 0$ such that $y = \lambda z$. Now the proof is complete. □

We remark that, under the condition $y, z \in X \setminus \{0\}$, the inequality (2) becomes

$$\frac{\sqrt{\|x\|^2 - (\|y\| - \|z\|)^2}}{2 \cdot \sqrt{\|y\| \cdot \|z\|}} \le \frac{\|x\|}{\|y\| + \|z\|},$$

also a closer form to Ballieu inequality.

Based to (1), it can prove the following inequality, known as *the cevian inequality*. We recall that, for any triangle ABC and for any point $D \in (BC)$ such that $\dfrac{BD}{BC} = t$, we have

$$AD \ge (tb + (1 - t)c)\cos\frac{A}{2}. \tag{3}$$

If $t = \dfrac{1}{2}$, we obtain

$$2m_a \ge (b + c)\cos\frac{A}{2}, \tag{4}$$

also an inequality that involves the median of the triangle.

The following theorem extend these two inequalities for the case of the inner product space.

Theorem 2.2 *Let X a real or complex inner product space. Then:*

a) *For any $x, y \in X$ and $t \in [0,1]$, we have*

$$(t\|x\| + (1-t)\|y\|) \cdot \sqrt{(\|x\| + \|y\|)^2 - \|x-y\|^2} \leq 2\|tx + (1-t)y\| \cdot \sqrt{\|x\| \cdot \|y\|}; \tag{5}$$

b) *For any $x, y \in X$, we have*

$$(\|x\| + \|y\|) \cdot \sqrt{(\|x\| + \|y\|)^2 - \|x-y\|^2} \leq 2 \cdot \|x+y\| \cdot \sqrt{\|x\| \cdot \|y\|}. \tag{6}$$

Proof. *a*) We use the equality

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2. \tag{7}$$

The most general form of (7) can be found in [**12**]. We apply Theorem 2.1 for $tx + (1-t)y$, $-tx$ and $-(1-t)y$ and we obtain

$$(\|tx\| + \|(1-t)y\|) \cdot \sqrt{\|tx + (1-t)y\|^2 - (\|tx\| - \|(1-t)y\|)^2} \leq 2\|tx + (1-t)y\| \cdot \sqrt{\|tx\| \cdot \|(1-t)y\|} \tag{8}$$

Hence

$$\begin{aligned}
&\|tx + (1-t)y\|^2 - (\|tx\| - \|(1-t)y\|)^2 \\
&= t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2 - t^2\|x\|^2 - (1-t)^2\|y\|^2 \\
&\qquad + 2t(1-t)\|x\|\|y\| \\
&= t(1-t)\|x\|^2 + t(1-t)\|y\|^2 + 2t(1-t)\|x\|\|y\| \\
&\qquad - t(1-t)\|x-y\|^2 \\
&= t(1-t)\left(\left(\|x\|^2 + \|y\|^2\right) - \|x-y\|^2\right),
\end{aligned}$$

then (8) becomes

$$\begin{aligned}
&(\|tx\| + \|(1-t)y\|) \cdot \sqrt{t(1-t)\left(\left(\|x\|^2 + \|y\|^2\right) - \|x-y\|^2\right)} \\
&\leq 2\|tx + (1-t)y\| \cdot \sqrt{t\|x\| \cdot (1-t)\|y\|}.
\end{aligned}$$

Now, we obtain the conclusion after the simplify with $\sqrt{t(1-t)}$.

b) We choose $t = \frac{1}{2}$ in (5) and we obtain (6). □

Based to the previous elementary results, Tsintsifas [**15**] proved that the following inequality

$$\frac{(b+c)^2}{4bc} \leq \frac{m_a}{w_a}, \tag{9}$$

holds for any triangle.

Further, we consider an inner product space X and $x, y \in X$. If we accept that $(0, x, y)$ represents a triangle, then $\frac{x+y}{2}$ denotes the "*midpoint*" of the "*segment*" xy and $\frac{\|y\|}{\|x+y\|} \cdot x + \frac{\|y\|}{\|x+y\|} \cdot y$ denotes the point where the "*bisector*" of the "*angle*" xOy bisects the "*line*" xy. Then (9) becomes:

Theorem 2.3. *Let X a real or complex inner product space. Then*

$$(\|x\| + \|y\|) \cdot \| \|y\| \cdot x + \|x\| \cdot y \| \leq 2 \|x\| \cdot \|y\| \cdot \|x+y\|, \tag{10}$$

for any $x, y \in X$.

Proof. The inequality is trivial if $x = 0$ or $y = 0$. We can assume that $x, y \in X \setminus \{0\}$. We denote

$$t = \frac{\|y\|}{\|x\| + \|y\|}.$$

By applying (7), we obtain

$$\begin{aligned} \left\| \frac{\|y\|}{\|x\| + \|y\|} \cdot x + \frac{\|x\|}{\|x\| + \|y\|} \cdot y \right\|^2 = \frac{\|y\|}{\|x\| + \|y\|} \cdot \|x\|^2 + \frac{\|x\|}{\|x\| + \|y\|} \cdot \|y\|^2 \\ - \frac{\|x\| \cdot \|y\|}{(\|x\| + \|y\|)^2} \cdot \|x-y\|^2, \end{aligned}$$

which is equivalent with

$$\| \|y\| \cdot x + \|x\| \cdot y \|^2 = \|x\| \cdot \|y\| \cdot (\|x\| + \|y\|)^2 - \|x\| \cdot \|y\| \cdot \|x-y\|^2.$$

We obtain

$$\| \|y\| \cdot x + \|x\| \cdot y \| = \sqrt{\|x\| \cdot \|y\|} \cdot \sqrt{(\|x\| + \|y\|)^2 - \|x-y\|^2}.$$

Now, we apply (6) and we have

$$\begin{aligned} &(\|x\| + \|y\|) \cdot \| \|y\| \cdot x + \|x\| \cdot y \| \\ &= (\|x\| + \|y\|) \cdot \sqrt{\|x\| \cdot \|y\|} \cdot \sqrt{(\|x\| + \|y\|)^2 - \|x-y\|^2} \\ &\leq 2 \cdot \|x+y\| \cdot \sqrt{\|x\| \cdot \|y\|} \cdot \sqrt{\|x\| \cdot \|y\|} \\ &= 2 \cdot \|x\| \cdot \|y\| \cdot \|x+y\| \end{aligned}$$

and the proof is complete. □

An interesting fact is that the inequality (10) is connected with another know inequality, respectively Dunkl-William inequality (see [**4**]). They proved that

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x-y\|}{\|x\|+\|y\|} \tag{11}$$

hold for any $x, y \in X \backslash \{0\}$, where X represent a real or complex inner space. This connection is more general because the following theorem proved the equivalence between these two inequalities hold for any normed space.

Theorem 2.4. *Let X a real or complex normed space. The inequalities (10) and (11) are equivalents.*

Proof. First, we apply (10) for x and $-y$ and we obtain

$$(\|x\|+\|y\|) \cdot \|\|y\| x - \|x\| y\| \leq 2\|x\| \cdot \|y\| \cdot \|x-y\|.$$

This is equivalent with

$$\frac{\|\|y\| \cdot x - \|x\| \cdot y\,\|}{\|x\| \cdot \|y\|} \leq \frac{2\|x-y\|}{\|x\|+\|y\|}$$

and we find (11). A similar reasoning led us from (11) to (10). □

We conclude this section with a consequence of the previous results. We obtain a short solution for the inequality Dadipour-Moslehian from [**3**].

Corrolary 2.5. (Dadipour-Moslehian) *Let X a real or complex inner product space. Let $x, y \in X \backslash \{0\}$. Then:*

a) *For any $t \in conv \left\{ \frac{\|y\|}{\|x\|+\|y\|}, \frac{1}{2} \right\}$, we have*

$$\|tx + (1-t)y\| \leq 2t(1-t)\|x+y\|;$$

b) *For any $s \in [0,1]$, we have*

$$\left\| \frac{x}{\|x\|^s} + \frac{y}{\|y\|^s} \right\| \leq \frac{2\|x+y\|}{\|x\|^s + \|y\|^s}.$$

Proof. *a*) Let the function $f : [0,1] \to \mathbb{R}$, defined for any $s \in [0,1]$ by

$$f(s) = \|sx + (1-s)y\| - 2s(1-s)\|x+y\|.$$

This is a convex function as a sum of two convex functions. Moreover, $f\left(\frac{1}{2}\right) = 0$.

Denote $u=\dfrac{\|y\|}{\|x\|+\|y\|}$. Let $t\in conv\left\{u,\ \dfrac{1}{2}\right\}$. Then there exists $\alpha\in[0,1]$ such that $t=\alpha u+(1-\alpha)\dfrac{1}{2}$. Then

$$\begin{aligned}
f(t) &= f\left(\alpha u+(1-\alpha)\frac{1}{2}\right)\\
&\le \alpha f(u)+(1-\alpha)f\left(\frac{1}{2}\right)\\
&= \alpha\left(\|ux+(1-u)y\|-2u(1-u)\|x+y\|\right)\\
&\le \alpha\cdot\left\|\frac{\|y\|}{\|x\|+\|y\|}\cdot x+\frac{\|x\|}{\|x\|+\|y\|}\cdot y\right\|-2\cdot\frac{\|x\|\cdot\|y\|}{(\|x\|+\|y\|)^2}\cdot\|x+y\|\\
&= \alpha\cdot\frac{(\|x\|+\|y\|)\cdot\|\|y\|\cdot x+\|x\|\cdot y\ \|-2\|x\|\cdot\|y\|\cdot\|x+y\|}{(\|x\|+\|y\|)^2}\\
&\le 0,
\end{aligned}$$

due to (10).

b) The case $\|x\|=\|y\|$ goes to equality. We assume that $\|x\|\neq\|y\|$ and define the function

$$g:[0,1]\to\mathbb{R},\, g(s)=\frac{\|y\|^s}{\|x\|^s+\|y\|^s}.$$

This function is continuous and monotone. Then

$$g([0,1])=conv\{g(0),g(1)\}=conv\left\{\frac{\|y\|}{\|x\|+\|y\|},\ \frac{1}{2}\right\},$$

also

$$\frac{\|y\|^s}{\|x\|^s+\|y\|^s}\in conv\left\{\frac{\|y\|}{\|x\|+\|y\|},\ \frac{1}{2}\right\},$$

for any $s\in[0,1]$. Now, we apply the previous assertion for $t=\dfrac{\|y\|^s}{\|x\|^s+\|y\|^s}$ and we obtain

$$\begin{aligned}
&\left\|\frac{\|y\|^s}{\|x\|^s+\|y\|^s}\cdot x+\frac{\|x\|^s}{\|x\|^s+\|y\|^s}\cdot y\right\|\le 2\cdot\frac{\|x\|^s\|y\|^s}{(\|x\|^s+\|y\|^s)^2}\cdot\|x+y\|\\
\Leftrightarrow\quad &\|\|y\|^s\cdot x+\|x\|^s\cdot y\ \|\le 2\cdot\frac{\|x\|^s\|y\|^s}{\|x\|^s+\|y\|^s}\cdot\|x+y\|\\
\Leftrightarrow\quad &\frac{\|\|y\|^s\cdot x+\|x\|^s\cdot y\ \|}{\|x\|^s\cdot\|y\|^s}\le\frac{2\|x+y\|}{\|x\|^s+\|y\|^s}
\end{aligned}$$

and the conclusion follows now. □

3. SOME NEW CHARACTERIZATIONS OF INNER PRODUCT SPACE

Some of the inequalities proved in the previous section are more important because its characterize the inner product space, also represent some conditions that a normed space to be an inner produc space. We start with the inequality (10) and we present two proof for the following theorem.

Theorem 3.1. *Let X a real or complex normed space. If the inequality (10) holds, for any $x, y \in X$, then X is an inner product space.*

First proof. In [**9**], Kirk proved that a real or complex normed space is an inner product space if and only if the inequality (11) holds. In this context, our proof is consequences of Theorem 2.4. □

Second proof. In [**10**], Lorch proved that a real or complex normed space X is an inner product space if and only if

$$\left\|\alpha x+\alpha^{-1}y\right\| \geq \|x+y\|,$$

for any $x, y \in X$ with $\|x\| = \|y\|$ and $\alpha > 0$.

Now, let $x, y \in X$ with $\|x\| = \|y\|$ and $\alpha > 0$. We apply (10) for αx and $\alpha^{-1}y$ and we obtain

$$\begin{aligned}
& \left(\|\alpha x\|+\left\|\alpha^{-1}y\right\|\right)\cdot\left\|\left\|\alpha^{-1}y\right\|\alpha x+\|\alpha x\|\alpha^{-1}y\right\| \leq 2\|\alpha x\|\cdot\left\|\alpha^{-1}y\right\|\cdot\left\|\alpha x+\alpha^{-1}y\right\| \\
\Leftrightarrow\ & \left(\alpha\|x\|+\alpha^{-1}\|y\|\right)\cdot\left\|\|y\|x+\|x\|y\right\| \leq 2\|x\|\cdot\|y\|\cdot\left\|\alpha x+\alpha^{-1}y\right\| \\
\Leftrightarrow\ & \left(\alpha+\alpha^{-1}\right)\|x\|\cdot\|x\|\|x+y\| \leq 2\|x\|^2\cdot\left\|\alpha x+\alpha^{-1}y\right\| \\
\Leftrightarrow\ & \left(\alpha+\alpha^{-1}\right)\|x+y\| \leq 2\cdot\left\|\alpha x+\alpha^{-1}y\right\|.
\end{aligned}$$

Hence $2 \leq \alpha + \alpha^{-1}$, we obtain

$$2\|x+y\| \leq \left(\alpha+\alpha^{-1}\right)\|x+y\| \leq 2\cdot\left\|\alpha x+\alpha^{-1}y\right\|,$$

also Lorch's inequality. Now, the second proof is complete. □

In the same mode, we will see that the inequlities (5) and (6) characterize the inner product space.

Theorem 3.2. *Let X a real or complex normed space. If the inequality (6) holds, for any $x, y \in X$, then X is an inner product space.*

Proof. Our proof is based on the Day's results [**5**]. He proved that a real or complex normed space X is an inner product space if and only if

$$\|x+y\|^2+\|x-y\|^2 \geq 4,$$

for any $x, y \in X$ with $\|x\| = \|y\| = 1$.

Now, let $x, y \in X$, such that $\|x\| = \|y\| = 1$. Then (6) becames

$$\begin{aligned} & 2 \cdot \sqrt{4 - \|x-y\|^2} \le 2 \cdot \|x+y\| \\ \Leftrightarrow \quad & 4 - \|x-y\|^2 \le \|x+y\|^2, \end{aligned}$$

also Day's inequality. Now the conclusion follows. □

Theorem 3.3. *Let X a real or complex normed space. If the inequality (5) holds then X is an inner product space.*

Proof. Theorem 2.2 shows that (5) implies (6). Now the conclusion follows due to the previous theorem. □

We conclude saying that the results from in this paper prove the beauty of the mathematics. In fact, more elementary results conceal deeper results from higher mathematics. Some of these have been discovered, while others are waiting to be discovered.

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