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ON THE ADJACENCY DIMENSION OF GRAPHS

A. Estrada-Moreno, Y. Ramírez-Cruz, J. A. Rodríguez-Velázquez

In this article we study the problem of finding the k-adjacency dimension of a graph. We give some necessary and sufficient conditions for the existence of a k-adjacency basis of an arbitrary graph G and we obtain general results on the k-adjacency dimension, including general bounds and closed formulae for some families of graphs.

1. INTRODUCTION

A generator of a metric space (X, d) is a set $S \subset X$ of points in the space with the property that every point of X is uniquely determined by the distances from the elements of S. Given a simple and connected graph $G = (V, E)$, we consider the function $d_G : V \times V \to \mathbb{N} \cup \{0\}$, where $d_G(x, y)$ is the length of a shortest path between u and v and $\mathbb{N}$ is the set of positive integers. Then (V, d_G) is a metric space since d_G satisfies (i) $d_G(x, x) = 0$ for all $x \in V$,(ii) $d_G(x, y) = d_G(y, x)$ for all $x, y \in V$ and (iii) $d_G(x, y) \le d_G(x, z) + d_G(z, y)$ for all $x, y, z \in V$. A vertex $v \in V$ is said to *distinguish* two vertices x and y if $d_G(v, x) \ne d_G(v, y)$. A set $S \subset V$ is said to be a *metric generator* for G if any pair of vertices of G is distinguished by some element of S. A minimum cardinality metric generator is called a *metric basis*, and its cardinality the *metric dimension* of G, denoted by $\dim(G)$.

The notion of metric dimension of a graph was introduced by Slater in [**17**], where metric generators were called *locating sets*. Harary and Melter independently introduced the same concept in [**9**], where metric generators were called *resolving sets*. Applications of this invariant to the navigation of robots in networks are discussed in [**14**] and applications to chemistry in [**12, 13**]. Several variations of metric generators, including resolving dominating sets [**1**], independent resolving

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sets [**2**], local metric sets [**15**], strong resolving sets [**16**], adjacency resolving sets [**11**], k-metric generators [**3, 4**], etc., have since been introduced and studied. In this article, we focus on the last of these issues: we are interested in the study of adjacency resolving sets and k-metric generators.

The notion of adjacency generator was first introduced by Jannesari and Omoomi in [**11**] as a tool to study the metric dimension of lexicographic product graphs. This concept has been studied further by Fernau and Rodríguez-Velázquez in [**7, 8**] where they showed that the (local) metric dimension of the corona product of a graph of order n and some non-trivial graph H equals n times the (local) adjacency dimension of H. As a consequence of this strong relation they showed that the problem of computing the adjacency dimension is NP-hard. A set $S \subset V$ of vertices in a graph $G = (V, E)$ is said to be an *adjacency generator* for G if for every two vertices $x, y \in V \setminus S$ there exists $s \in S$ such that s is adjacent to exactly one of x and y. A minimum cardinality adjacency generator is called an *adjacency basis* of G, and its cardinality the *adjacency dimension* of G, denoted by $\operatorname{adim}(G)$.

Notice that S is an adjacency generator for G if and only if S is an adjacency generator for its complement $\overline{G}$. This is justified by the fact that given an adjacency generator S for G, it holds that for every $x, y \in V \setminus S$ there exists $s \in S$ such that s is adjacent to exactly one of x and y, and this property holds in $\overline{G}$. Thus, $\operatorname{adim}(G) = \operatorname{adim}(\overline{G})$. Besides, from the definition of adjacency and metric bases, we deduce that S is an adjacency basis of a graph G of diameter at most two if and only if S is a metric basis of G. In these cases, $\operatorname{adim}(G) = \dim(G)$.

As pointed out in [**7, 8**], any adjacency generator of a graph $G = (V, E)$ is also a metric generator in a suitably chosen metric space. Given a positive integer t, we define the distance function $d_{G,t} : V \times V \to \mathbb{N} \cup \{0\}$, where

$$d_{G,t}(x, y) = \min\{d_G(x, y), t\}.$$

Then any metric generator for $(V, d_{G,t})$ is a metric generator for $(V, d_{G,t+1})$ and, as a consequence, the metric dimension of $(V, d_{G,t+1})$ is less than or equal to the metric dimension of $(V, d_{G,t})$. In particular, the metric dimension of $(V, d_{G,1})$ is equal to $|V| - 1$, the metric dimension of $(V, d_{G,2})$ is equal to $\operatorname{adim}(G)$ and, if G has diameter $D(G)$, then $d_{G,D(G)} = d_G$ and so the metric dimension of $(V, d_{G,D(G)})$ is equal to $\dim(G)$. Notice that when using the metric $d_{G,t}$ the concept of metric generator needs not be restricted to the case of connected graphs, as for any pair of vertices x, y belonging to different connected components of G we can assume that $d_G(x, y) = \infty > 2$ and so $d_{G,t}(x, y) = t$.

The concept of k-metric generator, introduced by Estrada-Moreno, Yero and Rodríguez-Velázquez in [**4, 6**], is a natural extension of the concept of metric generator. A set $S \subseteq V$ is said to be a *k-metric generator* for G if and only if any pair of vertices of G is distinguished by at least k elements of S, *i.e.*, for any pair of different vertices $u, v \in V$, there exist at least k vertices $w_1, w_2, \ldots, w_k \in S$ such that

$$d_G(u, w_i) \neq d_G(v, w_i), \text{ for every } i \in \{1, \ldots, k\}.$$

A k-metric generator of minimum cardinality in G is called a *k-metric basis*, and its cardinality the k-metric dimension of G, denoted by $\dim_k(G)$.

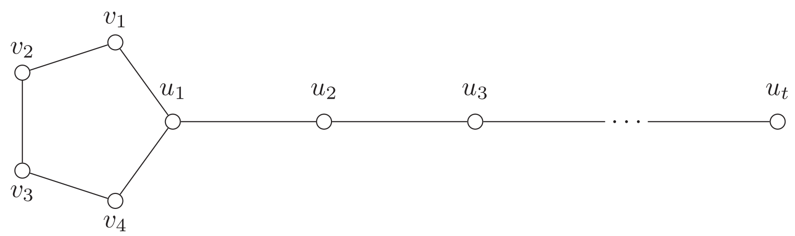

Figure 1. For $k \in \{1, 2, 3, 4\}$, $\dim_k(G) = k + 1$.

As an example we take a graph G obtained from the cycle graph C_5 and the path P_t, by identifying one of the vertices of the cycle, say u_1, and one of the extremes of P_t, as we show in Figure 1. Let $S_1 = \{v_1, v_2\}$, $S_2 = \{v_1, v_2, u_t\}$, $S_3 = \{v_1, v_2, v_3, u_t\}$ and $S_4 = \{v_1, v_2, v_3, v_4, u_t\}$. For $k \in \{1, 2, 3, 4\}$ the set S_k is k-metric basis of G.

Note that every k-metric generator S satisfies that $|S| \geq k$ and, if $k > 1$, then S is also a $(k-1)$-metric generator. Moreover, 1-metric generators are the standard metric generators (resolving sets or locating sets as defined in [**9**] or [**17**], respectively). Some basic results on the k-metric dimension of a graph have recently been obtained in [**3, 4, 5, 6, 18**]. In particular, it was shown in [**18**] that the problem of computing the k-metric dimension of a graph is NP-hard.

We say that a set $S \subseteq V(G)$ is a *k-adjacency generator* for G if for every two vertices $x, y \in V(G)$, there exist at least k vertices $w_1, w_2, \ldots, w_k \in S$ such that

$$d_{G,2}(x, w_i) \neq d_{G,2}(y, w_i), \text{ for every } i \in \{1, \ldots, k\}.$$

A minimum k-adjacency generator is called a *k-adjacency basis* of G and its cardinality, the *k-adjacency dimension* of G, is denoted by $\operatorname{adim}_k(G)$. For connected graphs, any k-adjacency basis is a k-metric basis. Hence, if there exists a k-adjacency basis of a connected graph G, then

$$\dim_k(G) \leq \operatorname{adim}_k(G).$$

Moreover, if G has diameter at most two, then $\dim_k(G) = \operatorname{adim}_k(G)$.

For the graph G shown in Figure 2 we have $\dim_1(G) = 8 < 9 = \operatorname{adim}_1(G)$, $\dim_2(G) = 12 < 14 = \operatorname{adim}_2(G)$ and $\dim_3(G) = 20 = \operatorname{adim}_3(G)$. Note that the only 3-adjacency basis of G, and at the same time the only 3-metric basis, is $V(G) - \{0, 6, 12, 18\}$.

In this article we study the problem of finding the k-adjacency dimension of a graph. The paper is organized as follows: in Section 2 we give some necessary and sufficient conditions for the existence of a k-adjacency basis of an arbitrary graph G, *i.e.,* we determine the range of k where $\operatorname{adim}_k(G)$ makes sense. Section 3 is devoted to the study of the k-adjacency dimension. We obtain general results on this

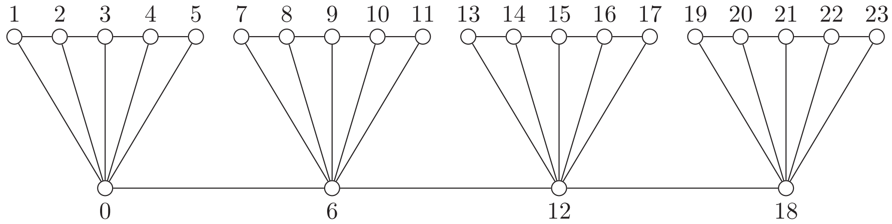

Figure 2. The set $\{2,4,6,8,10,14,16,20,21\}$ is an adjacency basis of G, while the set $\{2l+1:\ l\in\{0,\dots,11\}\}\cup\{6,12\}$ is a 2-adjacency basis and $V(G)-\{0,6,12,18\}$ is a 3-adjacency basis.

invariants including tight bounds and closed formulae for some particular families of graphs. Finally, in Section 4 we obtain closed formulae for the k-adjacency dimension of join graphs $G+H$ in terms of the k-adjacency dimension of G and H. These results concern the k-metric dimension, as join graphs have diameter two.

As we can expect, the obtained results will become important tools for the study of the k-metric dimension of lexicographic product graphs and corona product graphs. Moreover, we would point out that several results obtained in this article, like those in Remark 9 and subsequent, until Theorem 13, need not be restricted to the metric $d_{G,2}$, they can be expressed in a more general setting, for instance, by using the metric $d_{G,t}$ for any positive integer t.

We will use the notation K_n, $K_{r,s}$, C_n, N_n and P_n for complete graphs, complete bipartite graphs, cycle graphs, empty graphs and path graphs, respectively. We use the notation $u\sim v$ if u and v are adjacent and $G\cong H$ if G and H are isomorphic graphs. For a vertex v of a graph G, $N_G(v)$ will denote the set of neighbours or *open neighborhood* of v in G, *i.e.*, $N_G(v)=\{u\in V(G):\ u\sim v\}$. The *closed neighborhood*, denoted by $N_G[x]$, equals $N_G(x)\cup\{x\}$. If there is no ambiguity, we will simply write $N(x)$ or $N[x]$. We also define $\delta(v)=|N(v)|$ as the degree of vertex v, as well as, $\delta(G)=\min_{v\in V(G)}\{\delta(v)\}$ and $\Delta(G)=\max_{v\in V(G)}\{\delta(v)\}$. The subgraph induced by a set S of vertices will be denoted by $\langle S\rangle$, the diameter of a graph will be denoted by $D(G)$ and the girth by $\mathsf{g}(G)$. For the remainder of the paper, definitions will be introduced whenever a concept is needed.

2. *k*-ADJACENCY DIMENSIONAL GRAPHS

We say that a graph G is *k-adjacency dimensional* if k is the largest integer such that there exists a k-adjacency basis of G. Notice that if G is a k-adjacency dimensional graph, then for each positive integer $r\le k$, there exists at least one r-adjacency basis of G.

Given a connected graph G and two different vertices $x,y\in V(G)$, we denote by $\mathcal{C}_G(x,y)$ the set of vertices that distinguish the pair x,y with regard to the metric $d_{G,2}$, *i.e.*,

$$\mathcal{C}_G(x,y)=\{z\in V(G):\ d_{G,2}(x,z)\neq d_{G,2}(y,z)\}.$$

Then a set $S \subseteq V(G)$ is a k-adjacency generator for G if $|\mathcal{C}_G(x,y) \cap S| \geq k$ for all $x, y \in V(G)$. Notice that two vertices x, y are twins if and only if $\mathcal{C}_G(x,y) = \{x, y\}$.

Since for every $x, y \in V(G)$ we have that $|\mathcal{C}_G(x,y)| \geq 2$, it follows that the whole vertex set $V(G)$ is a 2-adjacency generator for G and, as a consequence, we deduce that every graph G is k-adjacency dimensional for some $k \geq 2$. On the other hand, for any graph G of order $n \geq 3$, there exists at least one vertex $v \in V(G)$ such that $|N_G(v)| \geq 2$ or $|V(G) - N_G(v)| \geq 2$, so for any pair $x, y \in N_G(v)$ or $x, y \in V(G) - N_G(v)$, we deduce that $v \notin \mathcal{C}_G(x,y)$ and, as a result, there is no n-adjacency dimensional graph of order $n \geq 3$.

We define the following parameter

$$\mathcal{C}(G) = \min_{x,y \in V(G)} \{|\mathcal{C}_G(x,y)|\}.$$

Theorem 1. *A graph G is k-adjacency dimensional if and only if $k = \mathcal{C}(G)$. Moreover, $\mathcal{C}(G)$ can be computed in $O(|V(G)|^3)$ time.*

Proof. First we shall prove the equivalence. (Necessity) If G is a k-adjacency dimensional graph, then for any k-adjacency basis B and any pair of vertices $x, y \in V(G)$, we have $|B \cap \mathcal{C}_G(x,y)| \geq k$. Thus, $k \leq \mathcal{C}(G)$. Now we suppose that $k < \mathcal{C}(G)$. In such a case, for every $x', y' \in V(G)$ such that $|B \cap \mathcal{C}_G(x', y')| = k$, there exists $z_{x'y'} \in \mathcal{C}_G(x', y') - B$ such that $d_{G,2}(z_{x'y'}, x') \neq d_{G,2}(z_{x'y'}, y')$. Hence, the set

$$B \cup \left(\bigcup_{x', y' \in V(G):\, |B \cap \mathcal{C}_G(x', y')| = k} \{z_{x'y'}\} \right)$$

is a $(k+1)$-adjacency generator for G, which is a contradiction. Therefore, $k = \mathcal{C}(G)$.

(Sufficiency) Let $a, b \in V(G)$ such that $\min_{x,y \in V(G)} |\mathcal{C}_G(x,y)| = |\mathcal{C}_G(a,b)| = k$. Since no set $S \subseteq V(G)$ satisfies $|S \cap \mathcal{C}_G(a,b)| > k$ and $V(G)$ is a k-adjacency generator for G, we conclude that G is a k-adjacency dimensional graph.

Now, we assume that the graph G is represented by its adjacency matrix $\mathbf{A}$. We recall that $\mathbf{A}$ is a symmetric $(n \times n)$-matrix given by

$$\mathbf{A}(i,j) = \begin{cases} 1, & \text{if } u_i \sim u_j, \\ 0, & \text{otherwise.} \end{cases}$$

Now observe that for every $z \in V(G) - \{x, y\}$ we have that $z \in \mathcal{C}_G(x,y)$ if and only if $\mathbf{A}(x,z) \neq \mathbf{A}(y,z)$. Considering this, we can compute $|\mathcal{C}_G(x,y)|$ in linear time for each pair $x, y \in V(G)$. Therefore, the overall running time for determining $\mathcal{C}(G)$ is dominated by the cubic time of computing the value of $|\mathcal{C}_G(x,y)|$ for $\binom{|V(G)|}{2}$ pairs of vertices x, y of G. $\square$

As Theorem 1 shows, given a graph G and a positive integer k, the problem of deciding if G is k-adjacency dimensional is easy to solve. Even so, we would point out some useful particular cases.

Remark 2. A graph G is 2-adjacency dimensional if and only if there are at least two vertices of G belonging to the same twin equivalence class.

Note that by the previous remark we deduce that graphs such as the complete graph K_n and the complete bipartite graph $K_{r,s}$ are 2-adjacency dimensional.

If $u, v \in V(G)$ are adjacent vertices of degree two and they are not twin vertices, then $|\mathcal{C}_G(u,v)| = 4$. Thus, for any integer $n \geq 5$, C_n is 4-adjacency dimensional and we can state the following more general remark.

Remark 3. Let G be a twins-free graph of minimum degree two. If G has two adjacent vertices of degree two, then G is 4-adjacency dimensional.

For any hypercube Q_r, $r \geq 2$, we have $|\mathcal{C}_{Q_r}(u,v)| = 2r$ if $u \sim v$, $|\mathcal{C}_{Q_r}(u,v)| = 2r-2$ if $d_{Q_r}(u,v) = 2$ and $|\mathcal{C}_{Q_r}(u,v)| = 2r+2$ if $d_{Q_r}(u,v) \geq 3$. Hence, $\mathcal{C}(Q_r) = 2r-2$.

Remark 4. For any integer $r \geq 2$ the hypercube Q_r is $(2r-2)$-adjacency dimensional.

It is straightforward that for any graph G of girth $\mathsf{g}(G) \geq 5$ and minimum degree $\delta(G) \geq 2$, $\mathcal{C}(G) \geq 2\delta(G)$. Hence, the following remark is immediate.

Remark 5. Let G be a k-adjacency dimensional graph. If $\mathsf{g}(G) \geq 5$ and $\delta(G) \geq 2$, then $k \geq 2\delta(G)$.

An end-vertex of a graph G is a vertex of degree one, and its neighbour is its support vertex. If there is an end-vertex u in G whose support vertex v has degree two, then $|\mathcal{C}_G(u,v)| = |N_G[v]| = 3$. Hence, we deduce the following result.

Remark 6. Let G be a twins-free graph. If there exists an end-vertex whose support vertex has degree two, then G is 3-adjacency dimensional.

The case of trees is summarized in the following remark. Before stating it, we need some additional terminology. Let T be a tree. A vertex of degree at least 3 is called a *major vertex* of T. A leaf u of T is said to be a *terminal vertex of a major vertex* v of T if $d_T(u,v) < d_T(u,w)$ for every other major vertex w of T. The *terminal degree* of a major vertex v is the number of terminal vertices of v. A major vertex v of T is an *exterior major vertex* of T if it has positive terminal degree.

Remark 7. Let T be a k-adjacency dimensional tree of order $n \geq 3$. Then $k \in \{2,3\}$ and $k = 2$ if and only if there are two leaves sharing a common support vertex.

Proof. By Remark 2 we conclude that $k = 2$ if and only if there are two leaves sharing a common support vertex. Also, if T is a path different from P_3, then by Remark 6 we have that $k = 3$.

If T is not a path, then there exists at least one exterior major vertex u of terminal degree greater than one. Then, either u is the support vertex of all its terminal vertices, in which case Remark 2 leads to $k = 2$, or u has at least one terminal vertex whose support vertex has degree two, in which case Remark 6 leads to $k = 3$ if there are no leaves of T sharing a common support vertex. □

Since $|\mathcal{C}_G(x,y)| \le \delta(x) + \delta(y) + 2$, for all $x, y \in V(G)$, the following remark immediately follows.

Remark 8. If G is a k-adjacency dimensional graph, then

$$k \le \min_{x,y \in V(G)} \{\delta(x) + \delta(y)\} + 2.$$

This bound is achieved, for instance, for any graph G constructed as follows. Take a cycle C_n whose vertex set is $V(C_n) = \{u_1, u_2, \dots, u_n\}$ and an empty graph N_n whose vertex set is $V(N_n) = \{v_1, v_2, \dots, v_n\}$ and then, for $i = 1$ to n, connect by an edge u_i to v_i. In this case, G is 4-adjacency dimensional. Also, a trivial example is the case of graphs having two isolated vertices, which are 2-adjacency dimensional.

As defined in [**3**], a connected graph G is *k-metric dimensional* if k is the largest integer such that there exists a k-metric basis. Since any k-adjacency generator is a k-metric generator, the following result is straightforward.

Remark 9. If a graph G is k-adjacency dimensional and k'-metric dimensional, then $k \le k'$. Moreover, if $D(G) \le 2$, then $k' = k$.

3. k-ADJACENCY DIMENSION. BASIC RESULTS

In this section we present some results that allow us to compute the k-adjacency dimension of several families of graphs. We also give some tight bounds on the k-adjacency dimension of a graph.

Theorem 10 (Monotony)**.** *Let G be a k-adjacency dimensional graph and let k_1, k_2 be two integers. If $1 \le k_1 < k_2 \le k$, then* $\operatorname{adim}_{k_1}(G) < \operatorname{adim}_{k_2}(G)$.

Proof. Let B be a k-adjacency basis of G. Let $x \in B$. Since $|B \cap \mathcal{C}_G(y,z)| \ge k$, for all $y, z \in V(G)$, we have that $B - \{x\}$ is a $(k-1)$-adjacency generator for G and, as a consequence, $\operatorname{adim}_{k-1}(G) \le |B - \{x\}| < |B| = \operatorname{adim}_k(G)$. By analogy we deduce that $\operatorname{adim}_{k-2}(G) < \operatorname{adim}_{k-1}(G)$ and, repeating this process until we get $\operatorname{adim}(G) < \operatorname{adim}_2(G)$, we obtain the result.

Corollary 11. *Let G be a k-adjacency dimensional graph of order n.*

(i) *For any $r \in \{2, \dots, k\}$,* $\operatorname{adim}_r(G) \ge \operatorname{adim}_{r-1}(G) + 1$.

(ii) *For any $r \in \{1, \dots, k\}$,* $\operatorname{adim}_r(G) \ge \operatorname{adim}(G) + (r-1)$.

(iii) *For any $r \in \{1, \dots, k-1\}$,* $\operatorname{adim}_r(G) < n$.

For instance, for the Petersen graph we have $\operatorname{adim}_6(G) = \operatorname{adim}_5(G) + 1 = \operatorname{adim}_4(G) + 2 = \operatorname{adim}_3(G) + 3 = 10$ and $\operatorname{adim}_2(G) = \operatorname{adim}_1(G) + 1 = 4$.

In order to continue presenting our results, we need to define a new parameter:

$$\mathcal{C}_k(G) = \bigcup_{|\mathcal{C}_G(x,y)| = k} \mathcal{C}_G(x,y).$$

For any k-adjacency basis A of a k-adjacency dimensional graph G, it holds that every pair of vertices $x, y \in V(G)$ satisfies $|A \cap \mathcal{C}_G(x,y)| \geq k$. Thus, for every $x, y \in V(G)$ such that $|\mathcal{C}_G(x,y)| = k$ we have that $\mathcal{C}_G(x,y) \subseteq A$, and so $\mathcal{C}_k(G) \subseteq A$. The following result is a direct consequence of this.

Remark 12. If G is a k-adjacency dimensional graph and A is a k-adjacency basis, then $\mathcal{C}_k(G) \subseteq A$ and, as a consequence,

$$\operatorname{adim}_k(G) \geq |\mathcal{C}_k(G)|.$$

Theorem 13. *Let G be a k-adjacency dimensional graph of order $n \geq 2$. Then* $\operatorname{adim}_k(G) = n$ *if and only if* $\mathcal{C}_k(G) = V(G)$.

Proof. Assume that $\mathcal{C}_k(G) = V(G)$. Since every k-adjacency dimensional graph G satisfies that $\operatorname{adim}_k(G) \leq n$, by Remark 12 we obtain that $\operatorname{adim}_k(G) = n$.

Suppose that there exists at least one vertex x such that $x \notin \mathcal{C}_k(G)$. In such a case, for any $a, b \in V(G)$ such that $x \in \mathcal{C}_G(a,b)$, we have that $|\mathcal{C}_G(a,b)| > k$. Hence, $|\mathcal{C}_G(a,b) - \{x\}| \geq k$, for all $a, b \in V(G)$ and, as a consequence, $V(G) - \{x\}$ is a k-adjacency generator for G, which leads to $\operatorname{adim}_k(G) < n$. Therefore, if $\operatorname{adim}_k(G) = n$, then $\mathcal{C}_k(G) = V(G)$. $\square$

As we will show in Propositions 32 and 33, $\operatorname{adim}_3(P_n) = n$ for $n \in \{4, \ldots, 8\}$ and $\operatorname{adim}_4(C_n) = n$ for $n \geq 5$. These are examples of graphs satisfying conditions of Theorem 13.

Corollary 14. *Let G be a graph of order $n \geq 2$. Then* $\operatorname{adim}_2(G) = n$ *if and only if every vertex of G belongs to a non-singleton twin equivalence class.*

Since $\mathcal{C}_G(x,y) = \mathcal{C}_{\overline{G}}(x,y)$ for all $x, y \in V(G)$, we deduce the following result, which was previously observed for $k = 1$ by Jannesari and Omoomi in [**11**].

Remark 15. For any nontrivial graph G and $k \in \{1, 2, \ldots, \mathcal{C}(G)\}$,

$$\operatorname{adim}_k(G) = \operatorname{adim}_k(\overline{G}).$$

Now we consider the limit case of the trivial bound $\operatorname{adim}_k(G) \geq k$. The case $k = 1$ was studied in [**11**] where the authors showed that $\operatorname{adim}_1(G) = 1$ if and only if $G \in \{P_2, P_3, \overline{P}_2, \overline{P}_3\}$.

Proposition 16. *If G is a graph of order $n \geq 2$, then* $\operatorname{adim}_k(G) = k$ *if and only if $k \in \{1, 2\}$ and $G \in \{P_2, P_3, \overline{P}_2, \overline{P}_3\}$*

Proof. The case $k = 1$ was studied in [**11**]. On the other hand, by performing some simple calculations, it is straightforward to see that $\operatorname{adim}_2(G) = 2$ for $G \in \{P_2, P_3, \overline{P}_2, \overline{P}_3\}$.

Now, suppose that $\operatorname{adim}_k(G) = k$ for some $k \geq 2$. By Corollary 11 we have $k = \operatorname{adim}_k(G) \geq \operatorname{adim}_1(G) + k - 1$ and, as a consequence, $\operatorname{adim}_1(G) = 1$. Hence, $G \in \{P_2, P_3, \overline{P}_2, \overline{P}_3\}$. Finally, since the graphs in $\{P_2, P_3, \overline{P}_2, \overline{P}_3\}$ are 2-adjacency dimensional, the proof is complete. $\square$

According to the result above, it is interesting to study the graphs where $\mathrm{adim}_k(G) = k + 1$. To begin with, we state the following remark.

Remark 17. If G is a graph of order $n \geq 7$, then $\mathrm{adim}_1(G) \geq 3$.

Proof. Suppose, for purposes of contradiction, that $\mathrm{adim}_1(G) \leq 2$. By Proposition 16 we deduce that $\mathrm{adim}_1(G) = 2$. Let $B = \{u, v\}$ be an adjacency basis of G. Then for any $w \in V(G) - B$ the distance vector $(d_{G,2}(u, w), d_{G,2}(v, w))$ must belong to $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Since $|V(G)-B| \geq 5$, by Dirichlet's box principle at least two elements of $V(G)-B$ have the same distance vector, which is a contradiction. Therefore, $\mathrm{adim}_1(G) \geq 3$. $\square$

By Corollary 11 (ii) and Remark 17 we obtain the following result.

Theorem 18. *For any graph G of order $n \geq 7$ and $k \in \{1, \dots, \mathcal{C}(G)\}$,*

$$\mathrm{adim}_k(G) \geq k + 2.$$

From Remark 17 and Theorem 18, we only need to consider graphs of order $n \in \{3, 4, 5, 6\}$ to determine those satisfying $\mathrm{adim}_k(G) = k + 1$. If $n = 3$, then by Proposition 16 we conclude that $\mathrm{adim}_1(G) = 2$ or $\mathrm{adim}_2(G) = 3$ if and only if $G \in \{K_3, N_3\}$. For $k \in \{1, 2\}$ and $n \in \{4, 5, 6\}$ the graphs satisfying $\mathrm{adim}_k(G) = k + 1$ can be determined by a simple calculation. Here we just show some of these graphs in Figure 3. Finally, the cases $\mathrm{adim}_3(G) = 4$ and $\mathrm{adim}_5(G) = 5$ are studied in the following two remarks.

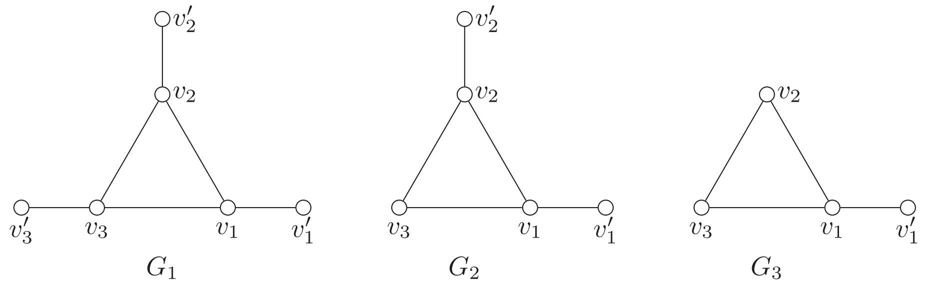

Figure 3. Any graph belonging to the families $\mathcal{G}_B(G_1)$, $\mathcal{G}_B(G_2)$ or $\{K_1 \cup K_3, G_3\}$, where $B = \{v_1, v_2, v_3\}$, satisfies $\mathrm{adim}_2(G) = 3$. The reader is referred to Subsection 3.1 for the construction of the families $\mathcal{G}_B(G_i)$.

The set of nontrivial distinctive vertices of a pair $x, y \in V(G)$, with regard to the metric $d_{G,2}$, will be denoted by $\mathcal{C}^*_G(x, y) = \mathcal{C}_G(x, y) - \{x, y\}$. Notice that two vertices x, y are twins if and only if $\mathcal{C}^*_G(x, y) = \emptyset$.

Remark 19. A graph G of order greater than or equal to four satisfies $\mathrm{adim}_3(G) = 4$ if and only if $G \in \{P_4, C_5\}$.

Proof. If $G \in \{P_4, C_5\}$, then it is straightforward to check that $\mathrm{adim}_3(G) = 4$. Assume that $B = \{v_1, \dots, v_4\}$ is a 3-adjacency basis of G. Since for any pair of vertices $v_i, v_j \in B$, there exists $v_l \in B \cap \mathcal{C}^*(v_i, v_j)$, by inspection we can check that $\langle B \rangle \cong P_4$. We assume that $v_i \sim v_{i+1}$ for $i \in \{1,2,3\}$. If $V(G) - B = \emptyset$, then $G \cong P_4$. Suppose that there exists $v \in V(G) - B$. If $v \sim v_2$, then the fact that $|B \cap \mathcal{C}^*(v, v_1)| \geq 2$ leads to $v \sim v_3$ and $v \sim v_4$. Since $|B \cap \mathcal{C}^*(v, v_4)| \geq 2$ and $v \sim v_3$, it follows that $v \sim v_1$. Thus, v is connected to any vertices in B, which leads to $|B \cap \mathcal{C}^*(v, v_2)| = |\{v_4\}| = 1$, contradicting the fact that B is a 3-adjacency basis of G. Analogously if $v \sim v_3$, then we arrive at the same contradiction. Thus, $v \sim v_1$ or $v \sim v_4$. If $v \sim v_1$ and $v \not\sim v_4$, then $|B \cap \mathcal{C}^*(v, v_2)| = |\{v_3\}| = 1$, contradicting the fact that B is a 3-adjacency basis of G. Now, if $v \sim v_1$ and $v \sim v_4$, then $G \cong C_5$. If $|V(G)| \geq 6$, then there exist $u, v \in V(G) - B$. Since $|B \cap \mathcal{C}(u, v)| \geq 3$, then either $|B \cap N(u)| \geq 2$ or $|B \cap N(v)| \geq 2$. Suppose that $|B \cap N(u)| \geq 2$. As discussed earlier, $B \cap N(u) = \{v_1, v_4\}$. Since $|B \cap \mathcal{C}(u, v)| \geq 3$, it follows that either $v \sim v_2$ or $v \sim v_3$, which, as we saw earlier, contradicts the fact that B is a 3-adjacency basis of G. □

By Corollary 11 (i) and Remark 19 we deduce that $\mathrm{adim}_4(G) \geq 6$ for any graph G of order at least five such that $G \ncong C_5$. Since $\mathrm{adim}_4(C_5) = 5$, we obtain the following result.

Remark 20. A graph G of order $n \geq 5$ satisfies that $\mathrm{adim}_4(G) = 5$ if and only if $G \cong C_5$.

From Corollary 11 (i) and Remark 20, it follows that any 4-adjacency dimensional graph G of order six satisfies $\mathrm{adim}_4(G) = 6$, as the case of C_6.

3.1. Large families of graphs having a common k-adjacency generator

Given a k-adjacency basis B of a graph $G = (V, E)$, we say that a graph $G' = (V, E')$ belongs to the family $\mathcal{G}_B(G)$ if and only if $N_{G'}(x) = N_G(x)$, for every $x \in B$. Figure 4 shows some graphs belonging to the family $\mathcal{G}_B(G)$ having a common 2-adjacency basis $B = \{v_2, v_3, v_4, v_5\}$.

Notice that if $B \neq V(G)$, then the edge set of any graph $G' \in \mathcal{G}_B(G)$ can be partitioned into two sets E_1, E_2, where E_1 consists of all edges of G having at least one vertex in B and E_2 is a subset of edges of a complete graph whose vertex set is $V(G) - B$. Hence, $\mathcal{G}_B(G)$ contains $2^{\frac{|V(G)-B|(|V(G)-B|-1)}{2}}$ different graphs.

With the above notation in mind we can state our next result.

Theorem 21. *Any k-adjacency basis B of a graph G is a k-adjacency generator for any graph $G' \in \mathcal{G}_B(G)$, and as a consequence,*

$$\mathrm{adim}_k(G') \leq \mathrm{adim}_k(G).$$

Proof. Assume that B is a k-adjacency basis of a graph $G = (V, E)$. Let $G' = (V, E')$ such that $N_{G'}(x) = N_G(x)$, for every $x \in B$. We will show that B is a k-adjacency generator for any graph G'. To this end, we take two different vertices $u, v \in V$. Since B is a k-adjacency basis of G, there exists $B_{uv} \subseteq B$ such that $|B_{uv}| \geq k$ and for every $x \in B_{uv}$ we have that $d_{G,2}(x, u) \neq d_{G,2}(x, v)$.

Now, since for every $x \in B_{uv}$ we have that $N_{G'}(x) = N_G(x)$, we obtain that $d_{G',2}(u,x) = d_{G,2}(u,x) \neq d_{G,2}(v,x) = d_{G',2}(v,x)$. Hence, B is a k-adjacency generator for G' and, in consequence, $|B| = \mathrm{adim}_k(G) \geq \mathrm{adim}_k(G')$. □

By Proposition 16 we have that if G is a graph of order $n \geq 2$, then $\mathrm{adim}_k(G) = k$ if and only if $k \in \{1,2\}$ and $G \in \{P_2, P_3, \overline{P}_2, \overline{P}_3\}$. Thus, for any graph H of order greater than three, $\mathrm{adim}_k(H) \geq k+1$. Therefore, the next corollary is a direct consequence of Theorem 21.

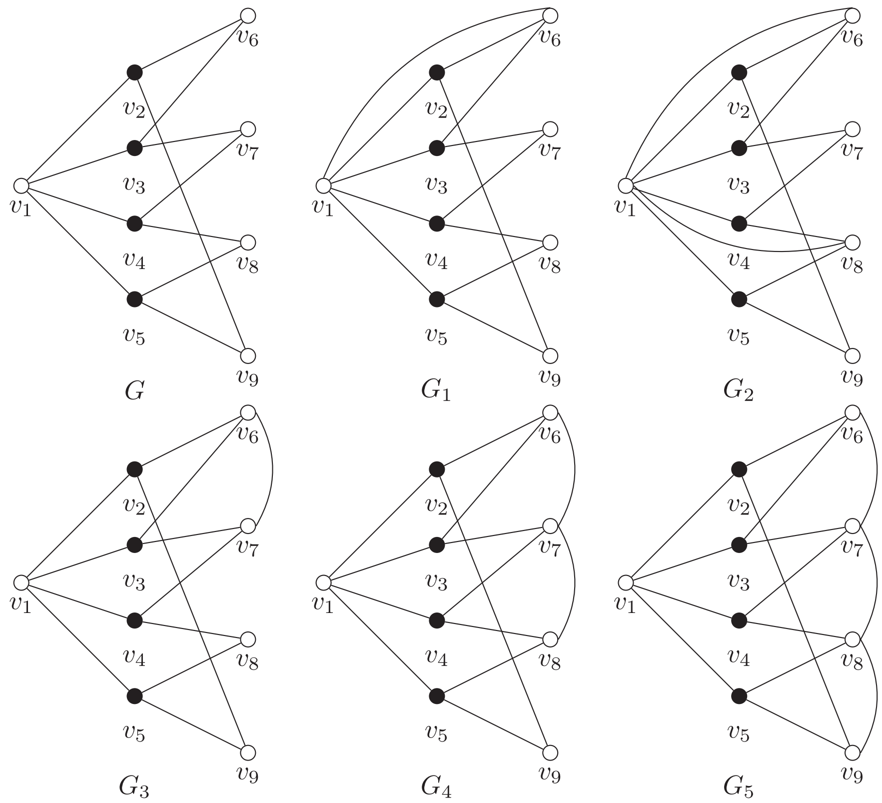

Figure 4. $B = \{v_2, v_3, v_4, v_5\}$ is a 2-adjacency basis of G and $\{G, G_1, G_2, G_4, G_5\}$ is a subfamily of $\mathcal{G}_B(G)$.

Corollary 22. *Let B be a k-adjacency basis of a graph G of order $n \geq 4$ and let $G' \in \mathcal{G}_B(G)$. If $\mathrm{adim}_k(G) = k+1$, then $\mathrm{adim}_k(G') = k+1$.*

Our next result immediately follows from Theorems 18 and 21.

Theorem 23. *Let B be a k-adjacency basis of a graph G of order $n \geq 7$ and let $G' \in \mathcal{G}_B(G)$. If $\mathrm{adim}_k(G) = k+2$, then $\mathrm{adim}_k(G') = k+2$.*

An example of application of the result above is shown in Figure 4, where

$\mathrm{adim}_2(G') = 4$ for all $G' \in \mathcal{G}_B(G)$. In this case $\mathcal{G}_B(G)$ contains $2^{10} = 1024$ different graphs.

4. THE k-ADJACENCY DIMENSION OF JOIN GRAPHS

The *join* $G + H$ of two vertex-disjoint graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph with vertex set $V(G + H) = V_1 \cup V_2$ and edge set

$$E(G + H) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$$

Note that $D(G + H) \leq 2$ and so for any pair of graphs G and H,

$$\dim_k(G + H) = \mathrm{adim}_k(G + H).$$

4.1. The particular case of $K_1 + H$

The following remark is a particular case of Corollary 14.

Remark 24. Let H be a graph of order n. Then $\mathrm{adim}_2(K_1 + H) = n + 1$ if and only if $\Delta(H) = n-1$ and every vertex $v \in V(H)$ of degree $\delta(v) < n-1$ belongs to a non-singleton twin equivalence class.

For any graph H, if $x, y \in V(H)$, then $\mathcal{C}_{K_1+H}(x, y) = \mathcal{C}_H(x, y)$. Also, if $x \notin V(H)$ then $\mathcal{C}_{K_1+H}(x, y) = \{x\} \cup (V(H) - N_H(y))$. Hence,

$$\mathcal{C}(K_1 + H) = \min\{\mathcal{C}(H), n - \Delta(H) + 1\}.$$

Proposition 25. *Let H be a graph of order $n \geq 2$ and $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$. Then*

$$\mathrm{adim}_k(K_1 + H) \geq \mathrm{adim}_k(H).$$

Proof. Let A be a k-adjacency basis of $K_1 + H$, $A_H = A \cap V(H)$ and let $x, y \in V(H)$ be two different vertices. Since $\mathcal{C}_{K_1+H}(x, y) = \mathcal{C}_H(x, y)$, it follows that $|A_H \cap \mathcal{C}_H(x, y)| = |A \cap \mathcal{C}_{K_1+H}(x, y)| \geq k$, and as a consequence, A_H is a k-adjacency generator for H. Therefore, $\mathrm{adim}_k(K_1 + H) = |A| \geq |A_H| \geq \mathrm{adim}_k(H)$.

Theorem 26. *For any nontrivial graph H, the following assertions are equivalent:*

(i) *There exists a k-adjacency basis A of H such that $|A - N_H(y)| \geq k$, for all $y \in V(H)$.*

(ii) $\mathrm{adim}_k(K_1 + H) = \mathrm{adim}_k(H)$.

Proof. Let A be a k-adjacency basis of H such that $|A - N_H(y)| \geq k$, for all $y \in V(H)$. By Proposition 25 we have that $\mathrm{adim}_k(K_1 + H) \geq \mathrm{adim}_k(H)$. It remains to prove that $\mathrm{adim}_k(K_1 + H) \leq \mathrm{adim}_k(H)$. We will prove that A is a k-adjacency generator for $K_1 + H$. We differentiate two cases for two vertices $x, y \in V(K_1 + H)$. If $x, y \in V(H)$, then the fact that A is a k-adjacency basis of H leads to $k \leq |A \cap \mathcal{C}_H(x, y)| = |A \cap \mathcal{C}_{K_1+H}(x, y)|$. On the other hand, if x is the vertex of K_1 and $y \in V(H)$, then the fact that $\mathcal{C}_{K_1+H}(x,y) = \{x\} \cup (V(H) - N_H(y))$ and $|A - N_H(y)| \geq k$ leads to $|A \cap \mathcal{C}_{K_1+H}(x,y)| \geq k$. Therefore, A is a k-adjacency generator for $K_1 + H$, and as a consequence, $\mathrm{adim}_k(H) = |A| \geq \mathrm{adim}_k(K_1 + H)$.

On the other hand, let B be a k-adjacency basis of $K_1 + H$ such that $|B| = \mathrm{adim}_k(H)$ and let $B_H = B \cap V(H)$. Since for any $h_1, h_2 \in V(H)$ the vertex of K_1 does not belong to $\mathcal{C}_{K_1+H}(h_1, h_2)$, we conclude that B_H is a k-adjacency generator for H. Thus, $|B_H| = \mathrm{adim}_k(H)$ and, as a consequence, B_H is a k-adjacency basis of H. If there exists $h \in V(H)$ such that $|B_H - N_H(h)| < k$, then $|B \cap \mathcal{C}_{K_1+H}(v,h)| = |B_H - N_H(h)| < k$, which is a contradiction. Therefore, the result follows. □

Our next result on graphs of diameter grater than or equal to six, is a direct consequence of Theorem 26.

Corollary 27. *For any graph H of diameter $D(H) \geq 6$ and $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$,*

$$\mathrm{adim}_k(K_1 + H) = \mathrm{adim}_k(H).$$

Proof. Let S be a k-adjacency basis of H. We will show that $|S - N_H(x)| \geq k$, for all $x \in V(H)$. Suppose, for the purpose of contradiction, that there exists $x \in V(H)$ such that $|S \cap (V(H) - N_H(x))| < k$. Let $F(x) = S \cap N_H[x]$. Notice that $|S| \geq k$ and hence $F(x) \neq \emptyset$.

From the assumptions above, if $V(H) = F(x) \cup \{x\}$, then $D(H) \leq 2$, which is a contradiction. If for every $y \in V(H) - (F(x) \cup \{x\})$ there exists $z \in F(x)$ such that $d_H(y,z) = 1$, then $d_H(v,v') \leq 4$ for all $v, v' \in V(H) - (F(x) \cup \{x\})$. Hence $D(H) \leq 4$, which is a contradiction. So, we assume that there exists a vertex $y' \in V(H) - (F(x) \cup \{x\})$ such that $d_H(y',z) > 1$, for every $z \in F(x)$, *i.e*, $N_H(y') \cap F(x) = \emptyset$. If $V(H) = F(x) \cup \{x, y'\}$, then by the connectivity of H we have $y' \sim x$ and, as consequence, $D(H) = 2$, which is also a contradiction. Hence, $V(H) - (F(x) \cup \{x, y'\}) \neq \emptyset$. Now, for any $w \in V(H) - (F(x) \cup \{x, y'\})$ we have that $|\mathcal{C}_H(y', w) \cap S| \geq k$ and, since $|S \cap (V(H) - N_H(x))| < k$ and $N_H(y') \cap F(x) = \emptyset$, we deduce that $N_H(w) \cap F(x) \neq \emptyset$. From this fact and the connectivity of H, we obtain that $d_H(y', w) \leq 5$. Hence $D(H) \leq 5$, which is also a contradiction. Therefore, if $D(H) \geq 6$, then for every $x \in V(H)$ we have that $|S \cap (V(H) - N_H(x))| \geq k$. Therefore, the result follows by Theorem 26.

Corollary 28. *Let H be a graph of girth $\mathsf{g}(H) \geq 5$ and minimum degree $\delta(H) \geq 3$. Then for any $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$,*

$$\mathrm{adim}_k(K_1 + H) = \mathrm{adim}_k(H).$$

Proof. Let A be a k-adjacency basis of H and let $x \in V(H)$ and $y \in N_H(x)$. Since $\mathsf{g}(H) \geq 5$, for any $u, v \in N_H(y) - \{x\}$ we have that $\mathcal{C}_H(u,v) \cap N_H[x] = \emptyset$. Also, since $|\mathcal{C}_H(u,v) \cap A| \geq k$, we obtain that $|A - N_H(x)| \geq k$. Therefore, by Theorem 26 we conclude the proof. □

A *fan graph* is defined as the join graph $K_1 + P_n$, where P_n is a path of order n, and a *wheel graph* is defined as the join graph $K_1 + C_n$, where C_n is a cycle graph of order n. The following closed formulae for the k-metric dimension of fan and wheel graphs were obtained in [**4, 10**]. Since these graphs have diameter two, we express the result in terms of the k-adjacency dimension.

Proposition 29. [**10**]

$$\text{(i)}\ \ \operatorname{adim}_1(K_1+P_n)=\begin{cases} 1, & \textit{if } n=1,\\ 2, & \textit{if } n=2,3,4,5,\\ 3, & \textit{if } n=6,\\ \left\lfloor \dfrac{2n+2}{5}\right\rfloor, & \textit{otherwise.}\end{cases}$$

$$\text{(ii)}\ \ \operatorname{adim}_1(K_1+C_n)=\begin{cases} 3, & \textit{if } n=3,6,\\ \left\lfloor \dfrac{2n+2}{5}\right\rfloor, & \textit{otherwise.}\end{cases}$$

Proposition 30. [**11**] *For any integer $n \geq 4$,*

$$\operatorname{adim}_1(P_n)=\operatorname{adim}_1(C_n)=\left\lfloor \frac{2n+2}{5}\right\rfloor.$$

Notice that by Propositions 29 and 30, for any $n \geq 4$, $n \neq 6$, we have that

$$\operatorname{adim}_1(P_n)=\operatorname{adim}_1(K_1+P_n)=\operatorname{adim}_1(C_n)=\operatorname{adim}_1(K_1+C_n).$$

In order to show the relationship between the k-adjacency dimension of fan (wheel) graphs and path (cycle) graphs, we state the following known results.

Proposition 31. [**4**]

$$\text{(i)}\ \ \operatorname{adim}_2(K_1+P_n)=\begin{cases} 3, & \textit{if } n=2,\\ 4, & \textit{if } n=3,4,5,\\ \left\lceil \dfrac{n+1}{2}\right\rceil, & \textit{if } n\geq 6.\end{cases}$$

$$\text{(ii)}\ \ \operatorname{adim}_2(K_1+C_n)=\begin{cases} 4, & \textit{if } n=3,4,5,6,\\ \left\lceil \dfrac{n}{2}\right\rceil, & \textit{if } n\geq 7.\end{cases}$$

$$\text{(iii)}\ \ \operatorname{adim}_3(K_1+P_n)=\begin{cases} 5, & \textit{if } n=4,5,\\ n-\left\lfloor \dfrac{n-4}{5}\right\rfloor, & \textit{if } n\geq 6.\end{cases}$$

$$\text{(iv)}\ \ \operatorname{adim}_3(K_1+C_n)=\begin{cases} 5, & \textit{if } n=5,6,\\ n-\left\lfloor \dfrac{n}{5}\right\rfloor, & \textit{if } n\geq 7.\end{cases}$$

$$\text{(v)}\ \ \operatorname{adim}_4(K_1+C_n)=\begin{cases} 6, & \textit{if } n=5,6,\\ n, & \textit{if } n\geq 7.\end{cases}$$

By Theorem 1 we have that any path graph of order at least four is 3-adjacency dimensional and any cycle graph of order at least five is 4-adjacency dimensional. From Propositions 25 and 31 we will derive closed formulae for the k-adjacency dimension of paths (for $k \in \{2,3\}$) and cycles (for $k \in \{2,3,4\}$).

Proposition 32. *For any integer $n \geq 4$,*

$$\operatorname{adim}_2(P_n) = \left\lceil \frac{n+1}{2} \right\rceil \text{ and } \operatorname{adim}_3(P_n) = n - \left\lfloor \frac{n-4}{5} \right\rfloor.$$

Proof. Let $k \in \{2,3\}$ and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$, where v_i is adjacent to v_{i+1} for every $i \in \{1, \ldots, n-1\}$.

We first consider the case $n \geq 7$. Since $\mathcal{C}_{P_n}(v_1, v_2) = \{v_1, v_2, v_3\}$ and $\mathcal{C}_{P_n}(v_{n-1}, v_n) = \{v_{n-2}, v_{n-1}, v_n\}$, we deduce that for any k-adjacency basis A of P_n and any $y \in V(T)$, $|A - N_{P_n}(y)| \geq k$. Hence, Theorem 26 leads to $\operatorname{adim}_k(K_1 + P_n) = \operatorname{adim}_k(P_n)$. Therefore, by Proposition 31 we deduce the result for $n \geq 7$.

Now, for $n = 6$, since $\mathcal{C}_{P_6}(v_1, v_2) = \{v_1, v_2, v_3\}$ and $\mathcal{C}_{P_6}(v_5, v_6) = \{v_4, v_5, v_6\}$, we deduce that $\operatorname{adim}_2(P_6) \geq 4$ and $\operatorname{adim}_3(P_6) = 6$. In addition, $\{v_1, v_3, v_4, v_6\}$ is a 2-adjacency generator for P_6 and so $\operatorname{adim}_2(P_6) = 4$.

From now on, let $n \in \{4,5\}$. By Proposition 25 we have $\dim_k(K_1 + P_n) \geq \operatorname{adim}_k(P_n)$. It remains to prove that $\operatorname{adim}_k(K_1 + P_n) \leq \operatorname{adim}_k(P_n)$.

If $n = 4$ or $n = 5$, then by Proposition 16, $\operatorname{adim}_2(P_n) \geq 3$. Note that $\{v_1, v_2, v_4\}$ and $\{v_1, v_3, v_5\}$ are 2-adjacency generators for P_4 and P_5, respectively. Thus, $\operatorname{adim}_2(P_4) = \operatorname{adim}_2(P_5) = 3$. Let A be a 3-adjacency basis of P_n, where $n \in \{4,5\}$. Since $\mathcal{C}_{P_n}(v_1, v_2) = \{v_1, v_2, v_3\}$ and $\mathcal{C}_{P_n}(v_{n-1}, v_n) = \{v_{n-2}, v_{n-1}, v_n\}$, we have that $(A \cap \mathcal{C}_{P_n}(v_1, v_2)) \cup (A \cap \mathcal{C}_{P_n}(v_{n-1}, v_n)) = V(P_n)$, and as consequence, $A = V(P_n)$. Therefore, $\operatorname{adim}_3(P_4) = 4$ and $\operatorname{adim}_3(P_5) = 5$ and, as a consequence, the result follows.

Proposition 33. *For any integer $n \geq 5$,*

$$\operatorname{adim}_2(C_n) = \left\lceil \frac{n}{2} \right\rceil, \ \operatorname{adim}_3(C_n) = n - \left\lfloor \frac{n}{5} \right\rfloor \text{ and } \operatorname{adim}_4(C_n) = n.$$

Proof. Let $k \in \{2,3,4\}$ and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$, where v_i is adjacent to v_{i+1} and the subscripts are taken modulo n.

First, consider the case $n \geq 7$. Since $\mathcal{C}_{C_n}(v_{i+3}, v_{i+4}) = \{v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\}$, we deduce that for any k-adjacency basis A of C_n, $|A - N_{C_n}(v_i)| \geq k$. Hence, Theorem 26 leads to $\operatorname{adim}_k(K_1 + C_n) = \operatorname{adim}_k(C_n)$. Therefore, by Proposition 31 we deduce the result for $n \geq 7$.

From now on, let $n \in \{5,6\}$. By Proposition 25 we have $\dim_k(K_1 + G) \geq \operatorname{adim}_k(G)$. It remains to prove that $\operatorname{adim}_k(K_1 + H) \leq \operatorname{adim}_k(H)$.

By Theorem 10, we deduce that $2 = \operatorname{adim}_1(C_5) < \operatorname{adim}_2(C_5) < \operatorname{adim}_3(C_5) < \operatorname{adim}_4(C_5) \leq 5$. Hence, $\operatorname{adim}_2(C_5) = 3$, $\operatorname{adim}_3(C_5) = 4$ and $\operatorname{adim}_4(C_5) = 5$. Therefore, for $n = 5$ the result follows.

By Theorem 10, $\mathrm{adim}_2(C_6) > \mathrm{adim}_1(C_6) = 2$ and, since $\{v_1, v_3, v_5\}$ is a 2-adjacency generator for C_6, we obtain that $\mathrm{adim}_2(C_6) = 3$. Now, let A_4 be a 4-adjacency basis of C_6. If $|A_4| \leq 5$, then there exists at least one vertex which does not belong to A_4, say v_1. Then, $|\mathcal{C}_{C_n}(v_1, v_2) \cap A_4| \leq 3$, which is a contradiction. Thus, $\mathrm{adim}_4(C_6) = |A_4| = 6$. Let $A_3^1 = \{v_1, v_2, v_3, v_4\}$, $A_3^2 = \{v_1, v_2, v_3, v_5\}$ and $A_3^3 = \{v_1, v_2, v_4, v_5\}$. Note that any manner of selecting four different vertices from C_6 is equivalent to some of these A_3^1, A_3^2, A_3^3. Since $|\mathcal{C}_{C_n}(v_5, v_6) \cap A_3^1| = |\{v_1, v_4\}| = 2 < 3$, $|\mathcal{C}_{C_n}(v_4, v_6) \cap A_3^2| = |\{v_1, v_3\}| = 2 < 3$ and $|\mathcal{C}_{C_n}(v_1, v_2) \cap A_3^3| = |\{v_1, v_2\}| = 2 < 3$, we deduce that $\mathrm{adim}_3(C_6) \geq 5 > |A_3^1| = |A_3^2| = |A_3^3| = 4$. By Theorem 10, $5 \leq \mathrm{adim}_3(C_6) < \mathrm{adim}_4(C_6) \leq 6$. Thus, $\mathrm{adim}_3(C_6) = 5$ and, as a consequence, the result follows. □

By Propositions 29, 30, 31, 32 and 33 we observe that for any $k \in \{1, 2, 3\}$ and $n \geq 7$, $\mathrm{adim}_k(K_1 + P_n) = \mathrm{adim}_k(P_n)$ and for any $k \in \{1, 2, 3, 4\}$, $\mathrm{adim}_k(K_1 + C_n) = \mathrm{adim}_k(C_n)$. The next result is devoted to characterize the trees where $\mathrm{adim}_k(K_1 + T) = \mathrm{adim}_k(T)$.

Proposition 34. *Let T be a tree. The following statements hold.*

(a) $\mathrm{adim}_1(K_1 + T) = \mathrm{adim}_1(T)$ *if and only if* $T \notin \mathcal{F}_1 = \{P_2, P_3, P_6, K_{1,n}, T'\}$, *where $n \geq 3$ and T' is obtained from $P_5 \cup \{K_1\}$ by joining by an edge the vertex of K_1 to the central vertex of P_5.*

(b) $\mathrm{adim}_2(K_1 + T) = \mathrm{adim}_2(T)$ *if and only if* $T \notin \mathcal{F}_2 = \{P_r, K_{1,n}, T'\}$, *where $r \in \{2, \ldots, 5\}$, $n \geq 3$ and T' is a graph obtained from $K_{1,n} \cup K_2$ by joining by an edge one leaf of $K_{1,n}$ to one leaf of K_2.*

(c) $\mathrm{adim}_3(K_1 + T) = \mathrm{adim}_3(T)$ *if and only if* $T \notin \mathcal{F}_3 = \{P_4, P_5\}$.

Proof. For any $k \in \{1, 2, 3\}$ and $T \in \mathcal{F}_k$, a simple inspection shows that $\mathrm{adim}_k(K_1 + T) \neq \mathrm{adim}_k(T)$. From now on, assume that $T \notin \mathcal{F}_k$, for $k \in \{1, 2, 3\}$, and let $\mathrm{Ext}(T)$ be the number of exterior major vertices of T. We differentiate the following three cases.

Case 1. $T = P_n$. The result is a direct consequence of combining Propositions 29 and 30 for $k = 1$ and Propositions 31 and 32 for $k > 1$.

In the following cases we shall show that there exists a k-adjacency basis A of T such that $|A - N_T(v)| \geq k$, for all $v \in V(T)$. Therefore, the result follows by Theorem 26.

Case 2. $\mathrm{Ext}(T) = 1$. Let u be the only exterior major vertex of T.

We first take $k = 1$. Since any two vertices adjacent to u must be distinguished by at least one vertex, we have that all paths from u to its terminal vertices, except at most one, contain at least one vertex in A. Thus, $|A - N_T(y)| \geq 1$, for all $y \in V(T) - \{u\}$. Now we shall show that $|A - N_T(u)| \geq 1$. If $u \in A$ or $A \not\subseteq N_T(u)$, then we are done, so we suppose that for any adjacency basis A of T, $u \notin A$ and $A \subseteq N_T(u)$. If there exists a leaf v such that $d_T(u, v) \geq 4$, then the support v' of v satisfies $\mathcal{C}_T(v, v') \cap A = \emptyset$, which is a contradiction. Hence, the eccentricity of u satisfies $2 \le \epsilon(u) \le 3$. If w is a leaf of T such that $d_T(u,w) = \epsilon(u)$, then the vertex $u' \in N_T(u)$ belonging to the path from u to w must belong to A and, as a consequence $A' = (A - \{u'\}) \cup \{w\}$ is an adjacency basis of T, which is a contradiction.

We now take $k = 2$. Let A be a 2-adjacency basis of T. Since any two vertices adjacent to u must be distinguished by at least two vertices in A, either all paths joining u to its terminal vertices contain at least one vertex of A or all but one contain at least two vertices of A. Thus, any vertex $y \in V(T) - \{u\}$ and any 2-adjacency basis A of T satisfy that $|A - N_T(y)| \ge 2$.

If there exist two vertices $v, v' \in V(T)$ such that $d_T(u,v) \ge 3$ and $d_T(u,v') \ge 3$, then $|A - N_T(u)| \ge 2$, as $|A \cap \mathcal{C}_2(v,v')| \ge 2$. On the other hand, if there exists only one leaf v such that $d_T(u,v) \ge 3$ and another leaf w such that $d_T(u,w) = 2$, we have that in order to distinguish v and it support as well as w and its support, $|A \cap N_T[v]| \ge 1$ and $|A \cap \{u,w\}| \ge 1$ and, as a result, $|A - N_T(u)| \ge 2$. Now, since $T \notin \mathcal{F}_2$ it remains to consider the case where u has eccentricity two. Let v, w be two leaves such that $d_T(u,v) = d_T(u,w) = 2$. If $|N_T(u)| = 3$, then the set A composed by u and its three terminal vertices is a 2-adjacency basis of T such that $|A - N_T(u)| \ge 2$. Assume that $|N_T(u)| \ge 4$. In order to distinguish v and its support vertex v', as well as w and its support vertex w', any 2-adjacency basis A of T must contain at least two vertices of $\{u, v, v'\}$ and at least two vertices of $\{u, w, w'\}$. If $u \notin A$, then $v, w \in A$, and as a consequence, $|A - N_T(u)| \ge 2$. Assume that $u \in A$. In this case, if $A - N_T[u] \ne \emptyset$, then $|A - N_T(u)| \ge 2$. Otherwise, $A \subseteq N_T[u]$ and $\{u, v', w'\} \subset A$ and, as a consequence, $A' = (A - \{v'\}) \cup \{v\}$ is a 2-adjacency basis of T and $|A' - N_T(u)| \ge 2$.
Finally, suppose that there exists exactly one leaf v such that $d_T(u,v) = 2$. Let v' be the support vertex of v. In this case, $V(T) - \{v'\}$ is a 2-adjacency basis A of T such that $|A - N_T(u)| \ge 2$.
We now take $k = 3$. In this case, there exist two leaves v, w such that $d_T(u,v) \ge 2$ and $d_T(u,w) \ge 2$. Since v and its support vertex v' must be distinguished by at least three vertices, they must belong to any 3-adjacency basis. Analogously, w and its support vertex w' must belong to any 3-adjacency basis. In general, any leaf that is not adjacent to u and its support vertex belong to any 3-adjacency basis of T. Moreover, there exists at most one terminal vertex x adjacent to u. If x exists, it must be distinguished from any vertex belonging to $N_T(u) - \{x\}$ by at least three vertices. Thus, they must belong to any 3-adjacency basis. Any vertex y different from u and any 3-adjacency basis A of T satisfy $v, v' \in A - N_T(y)$ or $w, w' \in A - N_T(y)$. If $v, v' \in A - N_T(y)$ and $w, w' \in A - N_T(y)$, then $|A - N_T(y)| \ge 3$. Otherwise, assuming without loss of generality that $v, v' \in A - N_T(y)$, there exists a terminal vertex z different from w such that $y \not\sim z$. Thus, again $|A - N_T(y)| \ge 3$. If $d_T(u,v) = 2$, then v, v' are distinguished only by u, v, v', so u must belong to any 3-adjacency basis of T. Thus, for any 3-adjacency basis A of T we have that $u, v, w \in A - N_T(u)$, and as a consequence, $|A - N_T(u)| \ge 3$. Finally, if $d_T(u,v) > 2$ and $d_T(u,w) > 2$, then $v, v', w, w' \in A - N_T(u)$. Hence $|A - N_T(u)| \ge 3$.

Case 3. $\mathrm{Ext}(T) \ge 2$. In this case, there are at least two exterior major vertices u, v

of T having terminal degree at least two. Let u_1, u_2 be two terminal vertices of u and v_1, v_2 be two terminal vertices of v. Let u'_1 and u'_2 be the vertices adjacent to u in the paths $u-u_1$ and $u-u_2$, respectively. Likewise, let v'_1 and v'_2 be the vertices adjacent to v in the paths $v-v_1$ and $v-v_2$, respectively. Notice that it is possible that $u_1 = u'_1$, $u_2 = u'_2$, $v_1 = v'_1$ or $v_2 = v'_2$. Note also that $\mathcal{C}(u'_1, u'_2) = (N_T[u'_1] \cup N_T[u'_2]) - \{u\}$ and $\mathcal{C}(v'_1, v'_2) = (N_T[v'_1] \cup N_T[v'_2]) - \{v\}$. Since for any k-adjacency basis A of T it holds that $|\mathcal{C}(u'_1, u'_2) \cap A| \geq k$ and $|\mathcal{C}(v'_1, v'_2) \cap A| \geq k$, and for any vertex $w \in V(T)$ we have that $(A - N_T(w)) \cap \mathcal{C}(u'_1, u'_2) = \emptyset$ or $(A - N_T(w)) \cap \mathcal{C}(v'_1, v'_2) = \emptyset$, we conclude that $|A - N_T(w)| \geq k$. □

From now on, we shall study some cases where $\operatorname{adim}_k(K_1 + H) > \operatorname{adim}_k(H)$. First of all, notice that by Corollary 27, if H is a connected graph and $\operatorname{adim}_k(K_1 + H) \geq \operatorname{adim}_k(H) + 1$, then $D(H) \leq 5$ and, by Corollary 28, if H has minimum degree $\delta(H) \geq 3$, then it has girth $\mathtt{g}(H) \leq 4$. We would point out the following consequence of Theorem 26.

Corollary 35. *If $\operatorname{adim}_k(K_1 + H) \geq \operatorname{adim}_k(H) + 1$, then either H is connected or H has exactly two connected components, one of which is an isolated vertex.*

Proof. Let A be a k-adjacency basis of H. We differentiate three cases for H.

Case 1. There are two connected components H_1 and H_2 of H such that $|V(H_1)| \geq 2$ and $|V(H_2)| \geq 2$. As for any $i \in \{1, 2\}$ and $u, v \in V(H_i)$, $|C_H(u, v) \cap A| = |C_{H_i}(u, v) \cap A| \geq k$ we deduce that $|A \cap V(H_1)| \geq k$ and $|A \cap V(H_2)| \geq k$. Hence, if $x \in V(H_1)$, then $|A - N_H(x)| \geq |A \cap V(H_2)| \geq k$ and if $x \in V(H) - V(H_1)$, then $|A - N_H(x)| \geq |A \cap V(H_1)| \geq k$. Thus, by Theorem 26, $\operatorname{adim}_k(K_1 + H) = \operatorname{adim}_k(H)$.

Case 2. There is a connected component H_1 of H such that $|V(H_1)| \geq 2$ and there are two isolated vertices $u, v \in V(H)$. From $C_H(u, v) = \{u, v\}$ we conclude that $k \leq 2$ and $|\{u, v\} \cap A| \geq k$. Moreover, for any $x, y \in V(H_1)$, $x \neq y$, we have that $|C_H(x, y) \cap A| = |C_{H_1}(u, v) \cap A| \geq k$ and so $|A \cap V(H_1)| \geq k$. Hence, if $x \in V(H_1)$, then $|A - N_H(x)| \geq |\{u, v\} \cap A| \geq k$ and if $x \in V(H) - V(H_1)$, then $|A - N_H(x)| \geq |A \cap V(H_1)| \geq k$. Thus, by Theorem 26, $\operatorname{adim}_k(K_1 + H) = \operatorname{adim}_k(H)$.

Case 3. $H \cong N_n$, for $n \geq 2$. In this case $k \in \{1, 2\}$, $\operatorname{adim}_1(K_1 + N_n) = \operatorname{adim}_1(N_n) = n - 1$ and $\operatorname{adim}_2(K_1 + N_n) = \operatorname{adim}_2(N_n) = n$.

Therefore, according to the three cases above, the result follows. □

By Proposition 25 and Theorem 26, $\operatorname{adim}_k(K_1 + H) \geq \operatorname{adim}_k(H) + 1$ if and only if for any k-adjacency basis A of H, there exists $h \in V(H)$ such that $|A - N_H(h)| < k$. Consider, for instance, the graph G showed in Figure 4. The only 2-adjacency basis of G is $B = \{v_2, v_3, v_4, v_5\}$ and $|B - N_G(v_1)| = 0$, so $\operatorname{adim}_2(K_1 + G) \geq \operatorname{adim}_2(G) + 1 = 5$. It is easy to check that $A = \{v_1, v_6, v_7, v_8, v_9\}$ is a 2-adjacency generator for $K_1 + G$, and so $\operatorname{adim}_2(K_1 + G) = \operatorname{adim}_2(G) + 1 = 5$. We emphasize that neither $B \cup \{v_1\}$ nor $B \cup \{x\}$ are 2-adjacency bases of $\langle x \rangle + G$.

Proposition 36. *Let H be a graph of order $n \geq 2$ and let $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$. If for any k-adjacency basis A of H, there exists $h \in V(H)$ such that $|A - N_H(h)| =$*

$k-1$ and $|A - N_H(h')| \geq k-1$, for all $h' \in V(H)$, then

$$\mathrm{adim}_k(K_1 + H) = \mathrm{adim}_k(H) + 1.$$

Proof. If for any k-adjacency basis A of H, there exists a vertex $h \in V(H)$ such that $|A - N_H(h)| = k-1$, then by Theorem 26, $\mathrm{adim}_k(K_1 + H) \geq \mathrm{adim}_k(H) + 1$.

Now, let A be a k-adjacency basis of H and let v be the vertex of K_1. Since $|A - N_H(h')| \geq k-1$, for all $h' \in V(H)$, the set $A \cup \{v\}$, is a k-adjacency generator for $K_1 + H$ and, as a consequence, $\mathrm{adim}_k(K_1 + H) \leq |A \cup \{v\}| = \mathrm{adim}_k(H) + 1$. □

The graph H shown in Figure 5 has six 3-adjacency basis. For instance, one of them is $B = \{1,2,3,4,5,8,9\}$ and the remaining ones can be found by symmetry. Notice that for any 3-adjacency basis, say A, there are two vertices i, j such that $|A - N_H(i)| = 2$, $|A - N_H(j)| = 2$ and $|A - N_H(l)| \geq 3$, for all $l \neq i, j$. In particular, for the basis B we have $i = 3$ and $j = 4$. Therefore, Proposition 36 leads to $\mathrm{adim}_3(K_1 + H) = \mathrm{adim}_3(H) + 1 = 8$.

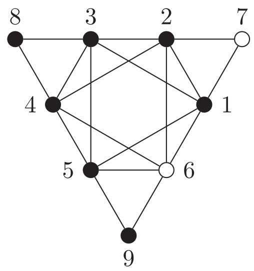

Figure 5. The set $B = \{1,2,3,4,5,8,9\}$ is a 3-adjacency basis of this graph.

By Theorem 26 and Proposition 36 we deduce the following result previously obtained in [**11**].

Proposition 37. [**11**] *Let H be graph of order $n \geq 2$. If for any adjacency basis A of H, there exists $h \in V(H) - A$ such that $A \subseteq N_H(h)$, then*

$$\mathrm{adim}_1(K_1 + H) = \mathrm{adim}_1(H) + 1,$$

otherwise,

$$\mathrm{adim}_1(K_1 + H) = \mathrm{adim}_1(H).$$

Theorem 38. *For any nontrivial graph H,*

$$\mathrm{adim}_2(K_1 + H) \leq \mathrm{adim}_2(H) + 2.$$

Proof. Let A be a 2-adjacency basis of H and let u be the vertex of K_1. Notice that there exists at most one vertex $x \in V(H)$ such that $A \subseteq N_H(x)$. Now, if $|A - N_H(v)| \geq 1$ for all $v \in V(H)$, then we define $X = A \cup \{u\}$ and, if there exists $x \in V(H)$ such that $A \subseteq N_H(x)$, then we define $X = A \cup \{x, u\}$. We claim that X is a 2-adjacency generator for $K_1 + H$. To show this, we first note that for any $y \in V(H)$ we have that $|\mathcal{C}_{K_1+H}(u, y) \cap X| = |((A - N_H(y)) \cup \{u\}) \cap X| \geq 2$. Moreover, for any $a, b \in V(H)$ we have that $\mathcal{C}_{K_1+H}(a, b) = \mathcal{C}_H(a, b)$. Therefore, X is a 2-adjacency generator for $K_1 + H$ and, as a consequence, $\mathrm{adim}_2(K_1 + H) \leq \mathrm{adim}_2(H) + 2$. □

We would point out that if for any 2-adjacency basis A of a graph H, there exists a vertex x such that $A \subseteq N_H(x)$, then not necessarily $\mathrm{adim}_2(K_1 + H) = \mathrm{adim}_2(H) + 2$. To see this, consider the graph G shown Figure 4, where $\{v_2, v_3, v_4, v_5\}$ is the only 2-adjacency basis of G and $\{v_2, v_3, v_4, v_5\} \subseteq N_H(v_1)$. However, $\{v_1, v_6, v_7, v_8, v_9\}$ is a 2-adjacency basis of K_1+G and so $\mathrm{adim}_2(K_1+H) = \mathrm{adim}(H)+1$. Now, we prove some results showing that the inequality given in Theorem 38 is tight.

Theorem 39. *Let H be a nontrivial graph. If there exists a vertex x of degree $\delta(x) = |V(H)| - 1$ not belonging to any 2-adjacency basis of H, then*

$$\mathrm{adim}_2(K_1 + H) = \mathrm{adim}_2(H) + 2.$$

Proof. Let u be the vertex of K_1 and let $x \in V(H)$ be a vertex of degree $\delta(x) = |V(H)| - 1$ not belonging to any 2-adjacency basis of H. In such a case, $\mathcal{C}_{K_1+H}(x, u) = \{x, u\}$ and, as a result, both x and u must belong to any 2-adjacency basis X of K_1+H. Since $X-\{u\}$ is a 2-adjacency generator for H and $x \in X-\{u\}$ we conclude that $|X - \{u\}| \geq \mathrm{adim}_2(H) + 1$ and so $\mathrm{adim}_2(K_1 + H) = |X| \geq \mathrm{adim}_2(H) + 2$. By Theorem 38 we conclude the proof. □

Examples of graphs satisfying the premises of Theorem 39 are the fan graphs $F_{1,n} = K_1+P_n$ and the wheel graphs $W_{1,n} = K_1+C_n$ for $n \geq 7$. For these graphs we have $\mathrm{adim}_2(K_1+F_{1,n}) = \mathrm{adim}_2(F_{1,n})+2$ and $\mathrm{adim}_2(K_1+W_{1,n}) = \mathrm{adim}_2(W_{1,n})+2$.

Theorem 40. *Let H be a graph having an isolated vertex v and a vertex u of degree $\delta(x) = |V(H)| - 2$. If for any 2-adjacency basis B of H, neither u nor v belongs to B, then*

$$\mathrm{adim}_2(K_1 + H) = \mathrm{adim}_2(H) + 2.$$

Proof. Let u be the vertex of K_1. Since $\mathcal{C}_{K_1+H}(x, u) = \{x, u, v\}$, at least two vertices of $\{x, u, v\}$ must belong to any 2-adjacency basis X of $K_1 + H$. Then we have that $x \in X - \{u\}$ or $v \in X - \{u\}$. Since $X - \{u\}$ is a 2-adjacency generator for H, we conclude that if $|X \cap \{x, v\}| = 1$, then $\mathrm{adim}_2(K_1 + H) > |X - \{u\}| \geq \mathrm{adim}_2(H) + 1$, whereas if $|X \cap \{x, v\}| = 2$, then $\mathrm{adim}_2(K_1 + H) \geq |X - \{u\}| \geq \mathrm{adim}_2(H) + 2$. Hence, $\mathrm{adim}_2(K_1 + H) = |X| \geq \mathrm{adim}_2(H) + 2$. By Theorem 38 we conclude the proof. □

For instance, we take a family of graphs $\mathcal{G} = \{G_1, G_2, \ldots\}$ such that for any $G_i \in \mathcal{G}$, every vertex in $V(G_i)$ belongs to a non-singleton true twin equivalence class. Then $X = \bigcup_{G_i \in \mathcal{G}} V(G_i)$ is the only 2-adjacency basis of $H = K_1 \cup (K_1 + \bigcup_{G_i \in \mathcal{G}} G_i)$. Therefore, $\mathrm{adim}_2(K_1 + H) = \mathrm{adim}_2(H) + 2$.

Proposition 41. *Let H be graph and $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$. If there exists a vertex $x \in V(H)$ and a k-adjacency basis A of H such that $A \subseteq N_H(x)$, then*

$$\mathrm{adim}_k(K_1 + H) \leq \mathrm{adim}_k(H) + k.$$

Proof. Let u be the vertex of K_1 and assume that there exists a vertex $v_1 \in V(H)$ and a k-adjacency basis A of H such that $A \subseteq N_H(v_1)$. Since $k \leq |V(H)| - \Delta(H) + 1$, we have that $|V(H) - N_H(v_1)| \geq k - 1$. With this fact in mind, we shall show that $X = A \cup \{u\} \cup A'$ is a k-adjacency generator for $K_1 + H$, where $A' = \emptyset$ if $k = 1$ and $A' = \{v_1, v_2, \ldots, v_{k-1}\} \subset V(H) - N_H(v_1)$ if $k \geq 2$. To this end we only need to check that $|\mathcal{C}_{K_1+H}(u, v) \cap X| \geq k$, for all $v \in V(H)$. On one hand, $|\mathcal{C}_{K_1+H}(u, v_1) \cap X| = |\{u\} \cup A'| = k$. On the other hand, since $A \subseteq N_H(v_1)$, for any $v \in V(H) - \{v_1\}$ we have that $|A - N_H(v)| \geq k$ and, as a consequence, $|\mathcal{C}_{K_1+H}(u, v) \cap X| \geq k$. Therefore, X is a k-adjacency generator for $K_1 + H$ and, as a result, $\operatorname{adim}_k(K_1 + H) \leq |X| = \operatorname{adim}_k(H) + k$. □

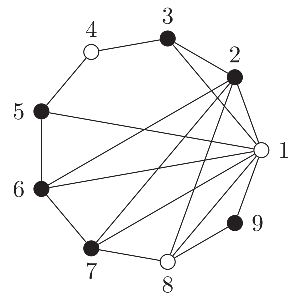

Figure 6. The set $A = \{2, 3, 5, 6, 7, 9\}$ is the only 3-adjacency basis of H and $A \subset N_H(1)$.

The bound above is tight. It is achieved, for instance, for the graph shown in Figure 6. In this case $\operatorname{adim}_3(K_1 + H) = \operatorname{adim}_3(H) + 3 = 9$. The set $\{2, 3, 5, 6, 7, 9\}$ is the only 3-adjacency basis of H, whereas $\langle u \rangle + H$ has four 3-adjacency bases, *i.e.*, $\{1, 2, 3, 4, 5, 6, 7, 8, u\}$, $\{1, 2, 3, 4, 5, 6, 7, 9, u\}$ $\{1, 2, 3, 4, 5, 7, 8, 9, u\}$ and $\{1, 2, 3, 4, 6, 7, 8, 9, u\}$.

Conjecture 42. *Let H be graph of order $n \geq 2$ and $k \in \{1, \ldots, \mathcal{C}(K_1 + H)\}$. Then*

$$\operatorname{adim}_k(K_1 + H) \leq \operatorname{adim}_k(H) + k.$$

We have shown that Conjecture 42 is true for any graph H and $k \in \{1, 2\}$, and for any H and k satisfying the premises of Proposition 41. Moreover, in order to assess the potential validity of Conjecture 42, we explored the entire set of graphs of order $n \leq 11$ and minimum degree two by means of an exhaustive search algorithm. This search yielded no graph H such that $\operatorname{adim}_k(K_1 + H) > \operatorname{adim}_k(H) + k$, $k \in \{3, 4\}$, a fact that empirically supports our conjecture.

4.2. The k-adjacency dimension of $G + H$ for $G \not\cong K_1$ and $H \not\cong K_1$

Two different vertices u, v of $G + H$ belong to the same twin equivalence class if and only if at least one of the following three statements hold.

(a) $u, v \in V(G)$ and u, v belong to the same twin equivalence class of G.

(b) $u, v \in V(H)$ and u, v belong to the same twin equivalence class of H.

(c) $u \in V(G)$, $v \in V(H)$, $N_G[u] = V(G)$ and $N_H[v] = V(H)$.

The following two remarks are direct consequence of Corollary 14.

Remark 43. Let G and H be two graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively. Then $\operatorname{adim}_2(G + H) = n_1 + n_2$ if and only if one of the two following statements hold.

(a) Every vertex of G belongs to a non-singleton twin equivalence class of G and every vertex of H belongs to a non-singleton twin equivalence class of H.

(b) $\Delta(G) = n_1 - 1$, $\Delta(H) = n_2 - 1$, every vertex $u \in V(G)$ of degree $\delta(u) < n_1 - 1$ belongs to a non-singleton twin equivalence class of G and every vertex $v \in V(H)$ of degree $\delta(v) < n_2 - 1$ belongs to a non-singleton twin equivalence class of H.

Let G and H be two graphs of order $n_1 \geq 2$ and $n_2 \geq 2$, respectively. If $x, y \in V(G)$, then $\mathcal{C}_{G+H}(x, y) = \mathcal{C}_G(x, y)$. Analogously, if $x, y \in V(H)$, then $\mathcal{C}_{G+H}(x, y) = \mathcal{C}_H(x, y)$. Also, if $x \in V(G)$ and $y \in V(H)$, then $\mathcal{C}_{G+H}(x, y) = (V(G) - N_G(x)) \cup (V(H) - N_H(y))$. Therefore,

$$\mathcal{C}(G + H) = \min\{\mathcal{C}(G), \mathcal{C}(H), n_1 - \Delta(G) + n_2 - \Delta(H)\}.$$

Theorem 44. *Let G and H be two nontrivial graphs. Then the following assertions hold:*

(i) *For any $k \in \{1, \ldots, \mathcal{C}(G + H)\}$,*

$$\mathrm{adim}_k(G + H) \geq \mathrm{adim}_k(G) + \mathrm{adim}_k(H).$$

(ii) *For any $k \in \{1, \ldots, \min\{\mathcal{C}(H), \mathcal{C}(K_1 + G)\}\}$*

$$\mathrm{adim}_k(G + H) \leq \mathrm{adim}_k(K_1 + G) + \mathrm{adim}_k(H).$$

Proof. First we proceed to deduce the lower bound. Let A be a k-adjacency basis of $G + H$, $A_G = A \cap V(G)$, $A_H = A \cap V(H)$ and let $x, y \in V(G)$ be two different vertices. Notice that $A_G \neq \emptyset$ and $A_H \neq \emptyset$, as $n_1 \geq 2$ and $n_2 \geq 2$. Now, since $\mathcal{C}_{G+H}(x, y) = \mathcal{C}_G(x, y)$, it follows that $|A_G \cap \mathcal{C}_G(x, y)| = |A \cap \mathcal{C}_{G+H}(x, y)| \geq k$, and as a consequence, A_G is a k-adjacency generator for G. By analogy we deduce that A_H is a k-adjacency generator for H. Therefore, $\mathrm{adim}_k(G + H) = |A| = |A_G| + |A_H| \geq \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$.

To obtain the upper bound, first we suppose that there exists a k-adjacency basis U of $K_1 + G$ such that the vertex of K_1 does not belong to U. We claim that for any k-adjacency basis B of H the set $X = U \cup B$ is a k-adjacency generator for $G + H$. To see this we take two different vertices $a, b \in V(G + H)$. If $a, b \in V(G)$, then $|\mathcal{C}_{G+H}(a, b) \cap X| = |\mathcal{C}_{K_1+G}(a, b) \cap U| \geq k$. If $a, b \in V(H)$, then $|\mathcal{C}_{G+H}(a, b) \cap X| = |\mathcal{C}_H(a, b) \cap B| \geq k$. Now, assume that $a \in V(G)$ and $b \in V(H)$. Since U is a k-adjacency generator for $\langle b \rangle + G$, we have that $|\mathcal{C}_{\langle b \rangle+G}(a, b) \cap U| \geq k$. Hence, $|\mathcal{C}_{G+H}(a, b) \cap X| = |\mathcal{C}_{\langle b \rangle+G}(a, b) \cap U| \geq k$. Therefore, X is a k-adjacency generator for $G + H$ and, as a consequence, $\mathrm{adim}_k(G + H) \leq |X| = |U| + |B| = \mathrm{adim}_k(K_1 + G) + \mathrm{adim}_k(H)$.

Suppose from now on that the vertex u of K_1 belongs to any k-adjacency basis U of $K_1 + G$. We differentiate two cases:

Case 1. For any k-adjacency basis B of H, there exists a vertex x such that $B \subseteq N_H(x)$. We claim that $X = U' \cup (B \cup \{x\})$ is a k-adjacency generator for $G+H$, where $U' = U - \{u\}$. To see this we take two different vertices $a, b \in V(G + H)$.

Notice that since B is k-adjacency basis of H, there exists exactly one vertex $x \in V(H)$ such that $B \subseteq N_H(x)$ and for any $y \in V(H) - \{x\}$ it holds $|B - N_H(y)| \geq k$. If $a, b \in V(G)$, then $|\mathcal{C}_{G+H}(a,b) \cap X| = |\mathcal{C}_{K_1+G}(a,b) \cap U'| = |\mathcal{C}_{K_1+G}(a,b) \cap U| \geq k$. If $a, b \in V(H)$, then $|\mathcal{C}_{G+H}(a,b) \cap X| = |\mathcal{C}_H(a,b) \cap (B \cup \{x\})| \geq k$. Now, assume that $a \in V(G)$ and $b \in V(H)$. Since $U' \cup \{b\}$ is a k-adjacency basis of $\langle b \rangle + G$, we have that $|\mathcal{C}_{\langle b \rangle + G}(a,b) \cap U'| \geq k-1$. Furthermore, $|\mathcal{C}_{\langle a \rangle + H}(a,b) \cap (B \cup \{x\})| \geq 1$. Hence, $|\mathcal{C}_{G+H}(a,b) \cap X| = |\mathcal{C}_{\langle b \rangle + G}(a,b) \cap U'| + |\mathcal{C}_{\langle a \rangle + H}(a,b) \cap (B \cup \{x\})| \geq k$. Therefore, X is a k-adjacency generator for $G+H$ and, as a consequence, $\mathrm{adim}_k(G+H) \leq |X| = |U'| + |B \cup \{x\}| = (\mathrm{adim}_k(K_1+G)-1) + (\mathrm{adim}_k(H)+1) = \mathrm{adim}_k(K_1+G) + \mathrm{adim}_k(H)$.

Case 2. There exists a k-adjacency basis B' of H such that $|B' - N_H(h')| \geq 1$, for all $h' \in V(H)$. We take $X = U' \cup B'$ and we proceed as above to show that X is a k-adjacency generator for $G + H$. As above, for $a, b \in V(G)$ or $a, b \in V(H)$ we deduce that $|\mathcal{C}_{G+H}(a,b) \cap X| \geq k$. Now, for $a \in V(G)$ and $b \in V(H)$ we have $|\mathcal{C}_{\langle b \rangle + G}(a,b) \cap U'| \geq k-1$ and $|\mathcal{C}_{\langle a \rangle + H}(a,b) \cap B'| \geq 1$. Hence, $|\mathcal{C}_{G+H}(a,b) \cap X| = |\mathcal{C}_{\langle b \rangle + G}(a,b) \cap U'| + |\mathcal{C}_{\langle a \rangle + H}(a,b) \cap B| \geq k$ and, as a consequence, $\mathrm{adim}_k(G + H) \leq |X| = |U'| + |B'| = (\mathrm{adim}_k(K_1 + G) - 1) + \mathrm{adim}_k(H) \leq \mathrm{adim}_k(K_1 + G) + \mathrm{adim}_k(H)$. □

By Proposition 37 and Theorem 44 we obtain the following result.

Proposition 45. *Let G and H be two non-trivial graphs. If for any adjacency basis A of G, there exists $g \in V(G)$ such that $A \subseteq N_G(g)$ and for any adjacency basis B of H, there exists $h \in V(H)$ such that $B \subseteq N_H(h)$, then*

$$\mathrm{adim}_1(G+H) = \mathrm{adim}_1(G) + \mathrm{adim}_1(H) + 1$$

Otherwise,

$$\mathrm{adim}_1(G+H) = \mathrm{adim}_1(G) + \mathrm{adim}_1(H).$$

Corollary 46. *Let G and H be two nontrivial graphs and $k \in \{1, \ldots, \mathcal{C}(G+H)\}$. If $\mathrm{adim}_k(K_1+G) = \mathrm{adim}_k(G)$, then*

$$\mathrm{adim}_k(G+H) = \mathrm{adim}_k(G) + \mathrm{adim}_k(H).$$

In the previous section we showed that there are several classes of graphs where $\mathrm{adim}_k(K_1 + G) = \mathrm{adim}_k(G)$. This is the case, for instance, of graphs of diameter $D(G) \geq 6$, or $G \in \{P_n, C_n\}$, $n \geq 7$, or graphs of girth $\mathtt{g}(G) \geq 5$ and minimum degree $\delta(G) \geq 3$. Hence, for any of these graphs, any nontrivial graph H, and any $k \in \{1, \ldots, \min\{\mathcal{C}(H), \mathcal{C}(K_1 + G)\}\}$ we have that $\mathrm{adim}_k(G + H) = \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$.

Theorem 47. *Let G and H be two nontrivial graphs. Then the following assertions are equivalent:*

(i) *There exists a k-adjacency basis A_G of G and a k-adjacency basis A_H of H such that $|(A_G - N_G(x)) \cup (A_H - N_H(y))| \geq k$, for all $x \in V(G)$ and $y \in V(H)$.*

(ii) $\mathrm{adim}_k(G+H) = \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$.

Proof. Let A_G be a k-adjacency basis of G and and let A_H be a k-adjacency basis of H such that $|(A_G - N_G(x)) \cup (A_H - N_H(y))| \geq k$, for all $x \in V(G)$ and $y \in V(H)$. By Theorem 44, $\mathrm{adim}_k(G+H) \geq \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$. It remains to prove that $\mathrm{adim}_k(G+H) \leq \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$. We will prove that $A = A_G \cup A_H$ is a k-adjacency generator for $G+H$. We differentiate three cases for two vertices $x, y \in V(G+H)$. If $x, y \in V(G)$, then the fact that A_G is a k-adjacency basis of G leads to $k \leq |A_G \cap \mathcal{C}_G(x,y)| = |A \cap \mathcal{C}_{G+H}(x,y)|$. Analogously we deduce the case $x, y \in V(H)$. If $x \in V(G)$ and $y \in V(H)$, then the fact that $\mathcal{C}_{G+H}(x,y) = (V(G) - N_G(x)) \cup (V(H) - N_H(y))$ and $|(A_G - N_G(x)) \cup (A_H - N_H(y))| \geq k$ leads to $|A \cap \mathcal{C}_{G+H}(x,y)| \geq k$. Therefore, A is a k-adjacency generator for $G+H$, as a consequence, $|A| = |A_G| + |A_H| = \mathrm{adim}_k(G) + \mathrm{adim}_k(H) \geq \mathrm{adim}_k(G+H)$.

On the other hand, let B be a k-adjacency basis of $G+H$ such that $|B| = \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$ and let $B_G = B \cap V(G)$ and $B_H = B \cap V(H)$. Since for any $g_1, g_2 \in V(G)$ and $h \in V(H)$, $h \notin \mathcal{C}_{G+H}(g_1, g_2)$, we conclude that B_G is a k-adjacency generator for G and, by analogy, B_H is a k-adjacency generator for H. Thus, $\mathrm{adim}_k(G) \leq |B_G|$, $\mathrm{adim}_k(H) \leq |B_H|$ and $|B_G| + |B_H| = |B| = \mathrm{adim}_k(G) + \mathrm{adim}_k(H)$. Hence, $|B_G| = \mathrm{adim}_k(G)$, $|B_H| = \mathrm{adim}_k(H)$ and, as a consequence, B_G and B_H are k-adjacency bases of G and H, respectively. If there exists $g \in V(G)$ and $h \in V(H)$ such that $|(B_G - N_G(g)) \cup (B_H - N_H(h))| < k$, then $|B \cap \mathcal{C}_{G+H}(g,h)| = |(B_G - N_G(g)) \cup (B_H - N_H(h))| < k$, which is a contradiction. Therefore, the result follows. □

We would point out the following particular cases of the previous result.

Corollary 48. *Let C_n be a cycle graph of order $n \geq 5$ and $P_{n'}$ a path graph of order $n' \geq 4$. If $G \in \{K_t + C_n, N_t + C_n\}$, then*

$$\mathrm{adim}_1(G) = \left\lfloor \frac{2n+2}{5} \right\rfloor + t - 1 \textit{ and } \mathrm{adim}_2(G) = \left\lceil \frac{n}{2} \right\rceil + t.$$

If $G \in \{K_t + P_{n'}, N_t + P_{n'}\}$, then

$$\mathrm{adim}_1(G) = \left\lfloor \frac{2n'+2}{5} \right\rfloor + t - 1 \textit{ and } \mathrm{adim}_2(G) = \left\lceil \frac{n'+1}{2} \right\rceil + t.$$

Proof. Let $G_1 \in \{K_t, N_t\}$ and $G_2 \in \{P_n, C_n\}$. By Propositions 32 and 33 we deduce that $\mathrm{adim}_2(G_2) - \Delta(G_2) \geq 1$. On the other hand, for any 2-adjacency basis A of G_1 and $x \in V(G_1)$ we have $|B - N_{G_1}(y)| \in \{1, t\}$. Therefore, by Theorem 47 we obtain the result for $G = G_1 + G_2$. □

Notice that for $n \geq 7$ and $n' \geq 6$, the previous result can be derived from Corollary 46.

Corollary 49. *Let G be a graph of order $n \geq 7$ and maximum degree $\Delta(G) \leq 3$. Then for any integer $t \geq 2$ and $H \in \{K_t, N_t\}$,*

$$\mathrm{adim}_2(G+H) = \mathrm{adim}_2(G) + t.$$

Proof. By Theorem 18 we deduce that $\mathrm{adim}_2(G) \geq 4$, so for any 2-adjacency basis A of G and $x \in V(G)$ we have $|A - N_G(x)| \geq 1$. Moreover, for any 2-adjacency basis B of H and $y \in V(H)$ we have $|B - N_H(y)| \in \{1, t\}$. Therefore, by Theorem 47 we obtain the result. □

Corollary 50. *Let G and H be two graphs of order at least seven such that G is k_1-adjacency dimensional and H is k_2-adjacency dimensional. For any integer k such that $\Delta(G) + \Delta(H) - 4 \leq k \leq \min\{k_1, k_2\}$,*

$$\mathrm{adim}_k(G + H) = \mathrm{adim}_k(G) + \mathrm{adim}_k(H).$$

Proof. By Theorem 18, for any positive integer $k \leq \min\{k_1, k_2\}$, we have that $\mathrm{adim}_k(G) \geq k + 2$ and $\mathrm{adim}_k(H) \geq k + 2$. Thus, if $k \geq \Delta(G) + \Delta(H) - 4$, then $(\mathrm{adim}_k(G) - \Delta(G)) + (\mathrm{adim}_k(H) - \Delta(H)) \geq k$. Therefore, by Theorem 47 we conclude the proof. □

As a particular case of the result above we derive the following remark.

Remark 51. Let G and H be two 3-regular graphs of order at least seven. Then

$$\mathrm{adim}_2(G + H) = \mathrm{adim}_2(G) + \mathrm{adim}_2(H).$$

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Departament d'Enginyeria Informàtica i Matemàtiques,
Universitat Rovira i Virgili,
Av. Països Catalans 26, 43007 Tarragona
Spain
E-mails: alejandro.estrada@urv.cat
yunior.ramirez@urv.cat
juanalberto.rodriguez@urv.cat

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