Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.bg.ac.yu

Appl. Anal. Discrete Math. **4** (2010), 361–372. doi:10.2298/AADM100601026T

A NOTE ON BOUNDEDNESS OF SOLUTIONS TO A CLASS OF NON-AUTONOMOUS DIFFERENTIAL EQUATIONS OF SECOND ORDER

Cemil Tunç

By defining some appropriate Liapunov functions, we discuss boundedness of solutions to a class of non-autonomous and nonlinear differential equations of second order. In this work, we prove some results established in the literature by Liapunov's second method instead of the integral test. We give six examples to illustrate the theoretical analysis in this work and effectiveness of the method utilized here.

1. INTRODUCTION AND MAIN RESULTS

In 1972, Kroopnick [**3**] considered the following nonlinear differential equation of second order

$$x'' + a(t)b(x) = 0, \tag{1}$$

where a and b are continuous functions on $\Re^+ = [0, \infty)$ and $\Re = (-\infty, \infty)$, respectively. It is assumed that the derivative $a'(t)$ exists and is continuous. The author showed boundedness of solutions of Eq. (1) with appropriate conditions on $a(t)$ and $b(x)$. Namely, Kroopnick [**3**] proved the following theorem by the integral test:

Theorem A. (Kroopnick [**3**, Theorem I]) *If*

$$a(t) > \alpha > 0, a'(t) \le 0 \text{ on } [T, \infty), t \ge T, b(x) \text{ continuous,}$$

2000 Mathematics Subject Classification. 34C10, 34C11, 34D05.
Keywords and Phrases. Non-autonomous, differential equation, second order, boundedness.

and

$$\lim_{x\to\pm\infty} B(x) = \int^{x} b(u)\mathrm{d}u = \infty,$$

then all solutions of Eq. (1) *are bounded as* $t \to \infty$.

We write Eq. (1) in system form as

$$x' = y, \qquad (2)$$
$$y' = -\, a(t)b(x).$$

The first main problem of this paper is the following theorem.

Theorem 1. *In addition to the basic assumptions imposed upon the functions* $a(t)$ *and* $b(x)$, *we assume that there exists a positive constant* α *such that the following assumptions hold:*

$$a(t) > \alpha,\ \ a'(t) \le 0 \text{ for all } t \in \Re^{+},$$

$$B(x) = \int_0^x b(u)\mathrm{d}u \text{ is positive for all } x \neq 0 \text{ and } B(x) \to \infty \text{ as } |x| \to \infty.$$

Then every solution of Eq. (1), *together with its derivative, is bounded as* $t \to \infty$.

Proof. Define a Liapunov function as

$$V(t,x,y) = a(t)\int_0^x b(s)\mathrm{d}s + \frac{1}{2}\,y^2.$$

It follows that

$$V(t,0,0) = 0.$$

In view of the assumptions of Theorem 1, firstly, we find that

$$V(t,x,y) \ge \alpha\int_0^x b(s)\mathrm{d}s + \frac{1}{2}\,y^2 > 0,$$

for all $x \neq 0$ and $y \neq 0$.

Secondly, since $B(x) \to \infty$ as $|x| \to \infty$, $V(t,x,y) \le K$ implies $|x| \le K_1$ and $|y| \le K_2$, where the constants K_1 and K_2 depend on the constant K. Thus, we only need to show that $V(t,x,y)$ is bounded along every trajectory of (2) as $t \to \infty$.

Along a trajectory of (2) the time derivative of the Liapunov function $V(t,x,y)$ gives that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,V(t,x,y) = a'(t)\int_0^x b(s)\mathrm{d}s \le 0.$$

Integrating the last inequality on $[0,\ \infty]$ (for a positive constant K) we obtain

$$V(t,x,y) \le V(0,x(0),y(0)) = K,$$

where $(x(0),y(0))$ is the initial point through which the trajectory starts at $t \ge 0$. Thus, it follows from the above discussion that $V(t,x(t),y(t))$ is bounded for all $t \ge 0$. This shows that every solution of Eq. (1), together with its derivative, is bounded as $t \to \infty$. The proof of Theorem 1 is now completed. $\square$

Example 1. Consider the equation

$$x'' + \left(1 + \frac{1}{1+t^2}\right) x^5 = 0,$$

which is a special case of Eq. (1).

We write this equation in system form as

$$x' = y,$$
$$y' = -\left(1 + \frac{1}{1+t^2}\right) x^5.$$

Hence, it follows

$$a(t) = 1 + \frac{1}{1+t^2} \geq 1 > 0,\ a'(t) = -\frac{2t}{(1+t^2)^2} \leq 0, t \geq 0,$$

$$b(x) = x^5, x^6 = xb(x) > 0, (x \neq 0), \int_0^x b(s)\mathrm{d}s = \int_0^x s^5 \mathrm{d}s = \frac{x^6}{6} \to \infty \text{ as } |x| \to \infty.$$

The above discussion shows that all the assumptions of Theorem 1 hold. Thus, we conclude that all solutions of Eq. (1) are bounded as $t \to \infty$.

On the other hand, it is also seen that

$$V_1(t,x,y) = \left(1 + \frac{1}{1+t^2}\right) \frac{x^6}{6} + \frac{1}{2}$$

for all $x \neq 0$ and $y \neq 0$, $V_1(t,0,0) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} V_1(t,x,y) = -\frac{t}{3(1+t^2)^2} x^6 \leq 0.$$

The remainder of the proof can be completed by using the procedure in Theorem 1.

In 1981, Kroopnick [**4**] considered the equation

$$x'' + c(t)f(x)x' + a(t)b(x) = 0, \tag{3}$$

where a, c and f, b are continuous functions on $\Re^+ = [0,\infty)$ and $\Re = (-\infty,\infty)$, respectively. It is assumed that the derivative $a'(t)$ exists and is continuous. The author presented two theorems, which include some sufficient conditions for all solutions of Eq. (3) to be bounded as $t \to \infty$.

In [**4**], Kroopnick constructed the following theorems.

Theorem B. (Kroopnick [**4**, Theorem I]) *Suppose that $a(t)$ and $c(t)$ are continuous functions on $[0,\infty)$ and let $b(x)$ and $f(x)$ be continuous on $(-\infty,\infty)$. Furthermore, suppose that for some positive constant a_0, $a(t) \geq a_0$, $a'(t) \leq 0$, $c(t) \geq 0$ for $0 \leq t < \infty$, and $f(x) > 0$. Finally, if $B(x) = \int_0^x b(u)\mathrm{d}u \to \infty$ as $|x| \to \infty$, then every solution of Eq.* (3) *exists on $[0,\infty)$ and $|x(t)|$ and $|x'(t)|$ are bounded as $t \to \infty$.*

Theorem C. (Kroopnick [**4**, Theorem III]) *The hypotheses are the same as Theorem B except that* $a'(t) \geq 0$ *(* $0 \leq t < \infty$*). Furthermore, if* $xb(x) \geq 0$ $(x \in \Re)$*, then all solutions to Eq.* (3) *are bounded as* $t \to \infty$*.*

Kroopnick [**4**] proved the above theorems by using the integral test.

We write Eq. (3) in system form as

$$\begin{aligned} x' &= y, \\ y' &= -\, c(t)f(x)y - a(t)b(x). \end{aligned} \tag{4}$$

The second main problem of this paper is the following theorem.

Theorem 2. *In addition to the basic assumptions imposed upon the functions* $a(t)$*,* $c(t)$*,* $f(x)$ *and* $b(x)$*, we assume that there exists a positive constant* a_0 *such that the following assumptions hold:*

$$a(t) \geq a_0,\ \ a'(t) \leq 0, c(t) \geq 0 \ \textit{for all}\ t \in \Re^+,\ \ f(x) > 0 \ \textit{for all}\ x \in \Re$$

and

$$B(x) = \int_0^x b(u)\mathrm{d}u \ \textit{is positive for all}\ x \neq 0 \ \textit{and}\ B(x) \to \infty \ \textit{as}\ |x| \to \infty.$$

Then every solution of (3) *exists on* $[0, \infty)$ *and* $|x(t)|$ *and* $|x'(t)|$ *are bounded as* $t \to \infty$*.*

Proof. We employ the Liapunov function

$$V(t,x,y) = a(t) \int_0^x b(s)\mathrm{d}s + \frac{1}{2}\,y^2$$

to prove Theorem 2 as in the proof of Theorem 1. Clearly, in view of the assumptions of Theorem 2, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t,x,y) = a'(t) \int_0^x b(s)\mathrm{d}s - c(t)f(x)y^2 \leq 0.$$

The rest of the proof is similar to that of Theorem 1. Therefore, we omit the details. The proof of Theorem 2 is now completed. □

Example 2. Consider non-linear differential equation of second order:

$$x'' + t^2(1+e^x)\,x' + \left(1 + \frac{1}{1+t^2}\right)x^5 = 0.$$

The above equation is a special case of Eq (3) and can be stated as the system

$$\begin{aligned} x' &= y, \\ y' &= -t^2(1+e^x)y - \left(1 + \frac{1}{1+t^2}\right)x^5. \end{aligned}$$

Hence, we have

$$c(t) = t^2 \geq 0,\ t \geq 0,\ f(x) = 1 + e^x > 0.$$

By denoting Liapunov function as $V_1(t, x, y)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} V_1(t, x, y) = -\frac{t}{3(1+t^2)^2} x^6 - t^2(1+e^x)y^2 \leq 0$$

When we take into account the above discussion and Example 1, it follows that all the assumptions of Theorem 2 hold. Thus we conclude that all solutions of the above equation are bounded as $t \to \infty$.

REMARK 1. KROOPNICK [**3**, **4**] proved Theorem A and Theorem B using the integral test, without giving examples on the topic. Instead of this test, we use the Liapunov's second method to show boundedness of solutions of (1) and (2). Our conditions are the same as that in KROOPNICK [**3**, **4**, Theorem I], and we also give two examples to show effectiveness of the method used here. The procedure used in the proof of Theorem 1 and Theorem 2 is very clear and comprehensible, and the boundedness of solutions is obvious.

Theorem 3. *Together with all the assumptions of Theorem 2 except $a(t) \geq a_0$ and $a'(t) \leq 0$, we assume that*

$$a_0 \geq a(t) > 0 \text{ and } a'(t) \geq 0 \text{ for all } t \in \Re^+.$$

Then every solution of Eq. (3) exists on $[0, \infty)$ and $|x(t)|$ and $|x'(t)|$ are bounded as $t \to \infty$.

Proof. Define the Liapunov function

$$V_2(t, x, y) = \int_0^x b(s)\mathrm{d}s + \frac{1}{2a(t)} y^2$$

so that $V_2(t, 0, 0) = 0$ and

$$V_2(t, x, y) \geq \int_0^x b(s)\mathrm{d}s + \frac{1}{2a_0} y^2 > 0,$$

for all $x \neq 0$ and $y \neq 0$.

The time derivative of Liapunov function $V_2(t, x, y)$ along (4) gives that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_2(t, x, y) = -\frac{c(t)}{a(t)} f(x)y^2 - \frac{a'(t)}{2a^2(t)} y^2 \leq 0.$$

The rest of the proof is omitted. □

EXAMPLE 3. Consider the following second order non-linear differential equation

$$x'' + (1 - e^{-t})(1 + e^x)x' + \left(3 - \frac{1}{1+t^2}\right) x^5 = 0.$$

This equation can be stated as the system

$$\begin{aligned} x' &= y, \\ y' &= -(1-e^{-t})(1+e^{x})y - \left(3 - \frac{1}{1+t^2}\right)x^5. \end{aligned}$$

Hence we find the following

$$c(t) = 1 - e^{-t} \geq 0,\ a(t) = 3 - \frac{1}{1+t^2},$$

$$2 \geq 3 - \frac{1}{1+t^2} > 0,\ a_0 = 2,\ a'(t) = \frac{2t}{(1+t^2)^2} \geq 0,\ t \geq 0.$$

On the other hand, it follows that

$$V_3(t,x,y) = \frac{1}{6}\,x^6 + \frac{1+t^2}{4+6t^2}\,y^2 \geq \frac{1}{6}\,(x^6+y^2) > 0$$

for all $x \neq 0$ and $y \neq 0$, $V_3(t,0,0) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\,V_3(t,x,y) = -\frac{1+t^2}{2+3t^2}\,(1-e^{-t})(1+e^{x})y^2 - \frac{t}{(2+3t^2)^2}\,y^2 \leq 0.$$

In view of the discussion made above and that in Example 1 and Example 2, it follows that all the assumptions of Theorem 3 hold. Therefore, we conclude that all solutions of the above equation are bounded as $t \to \infty$.

Later, in 1987, Kroopnick [**5**] discussed under what conditions the solutions to

$$(m(t)x')' + a(t)b(x) = 0 \tag{5}$$

are bounded, where $a(t)$ and $m(t)$ are $\in C^1[0,\infty)$.

Kroopnick [**5**] established the following theorem and proved it by the integration test.

Theorem D. (Kroopnick [**5**, Theorem I]) *Suppose that $a(t)$ and $m(t)$ are $\in C^1[0,\infty)$ and, furthermore, suppose that $a(t) \geq a_0 > 0$ and $m(t) \geq m_0 > 0$ for some positive constants a_0 and m_0. Also, let $m'(t) \leq 0$, $a'(t) \leq 0$, and let $b(x) \in C(-\infty,\infty)$. Finally, assume that if*

$$\lim_{|x|\to\infty} B(x) = \int_0^x b(u)\mathrm{d}u = \infty,\ \text{then } |x| \text{ and } |x'| \text{ are bounded as } t \to \infty.$$

Then (5) is equivalent to

$$m(t)x'' + m'(t)x' + a(t)b(x) = 0,$$

and it can be written in a system form as

$$\begin{aligned} x' &= y, \\ y' &= -\frac{m'(t)}{m(t)}\,y - \frac{a(t)}{m(t)}\,b(x). \end{aligned} \tag{6}$$

The fourth main problem of this paper is the following theorem.

Theorem 4. *In addition to the basic assumptions imposed upon the functions $m(t)$, $a(t)$ and $b(x)$, we assume that there exist positive constants a_0 and m_0 such that the following assumptions hold:*

$$a(t) \geq a_0,\ a'(t) \leq 0,\ m_0 \geq m(t) > 0,\ m'(t) \geq 0 \text{ for all } t \in \Re^+,$$

$$B(x) = \int_0^x b(u)\mathrm{d}u \text{ is positive for all } x \neq 0 \text{ and } B(x) \to \infty \text{ as } |x| \to \infty.$$

Then every solution of Eq. (5), *together with its derivative, is bounded as $t \to \infty$.*

Proof. Define the Liapunov function

$$V_4(t,x,y) = \frac{a(t)}{m(t)} \int_0^x b(s)\mathrm{d}s + \frac{1}{2}\, y^2.$$

so that $V_4(t,0,0) = 0$ and

$$V_4(t,x,y) \geq \frac{a_0}{m_0} \int_0^x b(s)\mathrm{d}s + \frac{1}{2}\, y^2 > 0,$$

for all $x \neq 0$ and $y \neq 0$.

The time derivative of Liapunov function $V_4(t,x,y)$ along (6) gives that

$$\frac{\mathrm{d}}{\mathrm{d}t} V_4(t,x,y) = \frac{a'(t)m(t) - a(t)m'(t)}{m^2(t)} \int_0^x b(s)\mathrm{d}s - \frac{m'(t)}{m(t)}\, y^2 \leq 0.$$

The rest of the proof is similar to that of Theorem 1 and its details are ommited. □

Example 4. Consider the following second order non-linear differential equation

$$\left(3 - \frac{1}{1+t^2}\right) x'' + \frac{2t}{(1+t^2)^2} x' + \left(3 + \frac{1}{1+t^2}\right) x^5 = 0.$$

This equation can be stated as system

$$\begin{aligned} x' &= y, \\ y' &= -\frac{2t}{(1+t^2)(2+3t^2)}\, y - \frac{4+3t^2}{2+3t^2}\, x^5. \end{aligned}$$

Hence, it follows that

$$m(t) = 3 - \frac{1}{1+t^2},\ m_0 = 3 \geq 3 - \frac{1}{1+t^2} > 0,\ m'(t) = \frac{2t}{(1+t^2)^2} \geq 0,$$

$$a(t) = 3 + \frac{1}{1+t^2} \geq 3,\ a_0 = 3,\ a'(t) = -\frac{2t}{(1+t^2)^2} \leq 0, t \geq 0.$$

We also notice that

$$V_5(t,x,y)=\frac{4+3t^2}{6(2+3t^2)}\,x^6+\frac{1}{2}\,y^2\geq\frac{4+3t^2}{6(4+3t^2)}\,x^6+\frac{1}{2}\,y^2\geq\frac{1}{6}\,x^6+\frac{1}{2}\,y^2>0$$

for all $x\neq 0$ and $y\neq 0$, $V_5(t,0,0)=0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\,V_5(t,x,y)=-\frac{2t}{(2+3t^2)^2}\,x^6-\frac{2t}{(1+t^2)(2+3t^2)}\,y^2\leq 0.$$

Integrating the last inequality on $[0,\infty)$, one can conclude that $V_5(t,x(t),y(t))$ is bounded for all $t\geq 0$. This shows that every solution of the above equation, together with its derivative, is bounded as $t\to\infty$. In view of the discussion made above, it also follows that all the assumptions of Theorem 4 hold.

Remark 2. The assumptions of Theorem 3 and Theorem 4 are the same as that in Kroopnick [**4**, Theorem I] and Kroopnick [**5**, Theorem III] except $a_0\geq a(t)>0$ and $m_0\geq m(t)>0$ instead of $a(t)\geq a_0>0$ and $m(t)\geq m_0>0$, respectively.

In 1995, Kroopnick [**6**] first presented a boundedness theorem for the equation

$$x''+c(t,x,x')+a(t)b(x)=e(t), \tag{7}$$

where $c(t,x,x')$, $a(t)$, $b(x)$ and $e(t)$ are continuous on $\Re^+\times\Re\times\Re$, $\Re^+$, $\Re$ and $\Re^+$, respectively. It is also assumed that the derivative $a'(t)$ exists and is continuous.

Instead of Eq. (7), we consider it as a system

$$\begin{aligned}x'&=y,\\ y'&=-\,c(t,x,y)-a(t)b(x)+e(t).\end{aligned} \tag{8}$$

Kroopnick [**6**] proved the following theorem.

Theorem E. (Kroopnick [**6**, Theorem I]) *Given the differential equation* (7), *suppose that $c(t,x,y)$ is continuous on $\Re^+\times\Re\times\Re$, $c(t,x,y)y\geq 0$ and $e(t)$ is continuous on $\Re^+$ with $\int_0^\infty|e(t)|\,\mathrm{d}t<\infty$. Furthermore, if $a(t)>a_0>0$ for some constant a_0 and continuous on $\Re^+$, $a'(t)\leq 0$, $b(x)$ is continuous on $\Re$, and*

$$B(x)=\int_0^x b(u)\mathrm{d}u \textit{ approaches } \infty \textit{ as } |x|\to\infty,$$

then all solutions of Eq. (7) *as well as their derivatives are bounded as $t\to\infty$.*

Kroopnick [**6**] proved Theorem 5 by means of the integral test.

The fifth main problem of this paper is the following theorem.

Theorem 5. *In addition to the basic assumptions imposed upon the functions $a(t)$, $b(x)$, $c(t,x,y)$ and $e(t)$, we assume that there exists a positive constant a_0 such that the following assumptions hold:*

$$a(t)\geq a_0,\ a'(t)\leq 0,\ c(t,x,y)y\geq 0 \textit{ for all } t\in\Re^+, x,y\in\Re,$$

$$B(x)=\int_0^x b(u)\mathrm{d}u \textit{ is positive for all } x\neq 0 \textit{ and } B(x)\to\infty \textit{ as } |x|\to\infty$$

and

$$\int_0^\infty |e(t)|\,\mathrm{d}t<\infty.$$

Then all solutions of Eq. (7) *as well as their derivatives are bounded as* $t\to\infty$.

Proof. Define the Liapunov function

$$V(t,x,y)=a(t)\int_0^x b(s)\mathrm{d}s+\frac{1}{2}\,y^2.$$

In view of the fact that $a(t)\geq a_0>0$, we arrive at

$$V(t,x,y)\geq a_0\int_0^x b(s)\mathrm{d}s+\frac{1}{2}\,y^2>0,$$

for all $x\neq 0$ and $y\neq 0$.

The time derivative of $V(t,x,y)$ along a solution $(x,y)=(x(t),y(t))$ of (8) gives that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(t,x,y)&=a'(t)\int_0^x b(s)\mathrm{d}s-c(t,x,y)y+e(t)y\\&\leq|e(t)|\;\;|y|\leq|e(t)|\;\;+|e(t)|\;\;y^2\\&\leq|e(t)|\;\;+2\,|e(t)|\;\;V(t,x,y).\end{aligned}$$

Integrating the last inequality on $[0,\ \infty]$, for a positive constant K_3, we obtain

$$V(t,x(t),y(t))\leq V(0,x(0),y(0))+\int_0^t|e(s)|\,\mathrm{d}s+2\int_0^t|e(s)|\;\;V(s,x(s),y(s))\mathrm{d}s.$$

Using the convergence of the integral $\int_0^t|e(s)|\,\mathrm{d}s$ and the Gronwall-Reid-Bellman inequality (see Gronwall [**1**] and Mitrinovic [**2**]), we can conclude that $V(t,x(t),y(t))$ is bounded for all $t\geq 0$. This shows that every solution of Eq. (7), together with its derivative, is bounded as $t\to\infty$. The proof of Theorem 5 is now completed. □

Example 5. Consider equation

$$x''+(1+t^2+x^2+x'^2)x'+\left(1+\frac{1}{1+t^2}\right)x^5=\frac{1}{1+t^2}.$$

This equation can be written as the system

$$\begin{aligned}x'&=y,\\y'&=-(1+t^2+x^2+y^2)y-\left(1+\frac{1}{1+t^2}\right)x^5+\frac{1}{1+t^2}.\end{aligned}$$

Hence, it follows that

$$c(t,x,y)y = (1+t^2+x^2+y^2)y^2 \geq 0, t \geq 0,$$

$$e(t) = \frac{1}{1+t^2}, \quad \int_0^\infty |e(t)|\,\mathrm{d}t = \int_0^\infty \frac{1}{1+t^2}\,\mathrm{d}t = \frac{\pi}{2} < \infty.$$

Utilizing the function $V_1(t,x,y)$, it also follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_1(t,x,y) &= -\frac{t}{3(1+t^2)^2}\,x^6 - (1+t^2+x^2+y^2)y^2 + \frac{y}{1+t^2}\\ &\leq \frac{y}{1+t^2} \leq \frac{1}{1+t^2} + \frac{y^2}{1+t^2}\\ &\leq \frac{y}{1+t^2} \leq \frac{1}{1+t^2} + \frac{1}{1+t^2}\,V_1(t,x,y).\end{aligned}$$

Integrating the last inequality on $[0, \infty]$, using the Gronwall-Reid-Bellman inequality (see Gronwall [**1**] and Mitrinovic [**2**]) and taking into account the above discussion and that in Example 1, it follows that all the assumptions of Theorem 5 hold. Thus, we conclude that all solutions of the above equation are bounded as $t \to \infty$.

Kroopnick [**6**] presented a boundedness theorem for the equation

$$x'' + c(t,x,x') + a(t,x) = e(t), \tag{9}$$

where $c(t,x,x')$, $a(t,x)$ and $e(t)$ are continuous on $\Re^+ \times \Re \times \Re$, $\Re^+ \times \Re$ and $\Re^+$, respectively. It is also assumed that the derivative $x\,\dfrac{\partial}{\partial t}\,a(t,x)$ exists and is continuous.

We write Eq. (9) in system form as

$$\begin{aligned}x' &= y,\\ y' &= -\,c(t,x,y) - a(t,x) + e(t).\end{aligned} \tag{10}$$

Utilizing the integration test, Kroopnick [**6**] proved the following theorem.

Theorem F. (Kroopnick [**6**, Theorem II]) *Given equation in* (9) *and suppose that* $c(t,x,y)$ *is continuous on* $\Re^+ \times \Re \times \Re$, $c(t,x,y)y > 0$ *and* $a(t,x)$ *is continuous on* $\Re^+ \times \Re$ *with* $x\dfrac{\partial}{\partial t}a(t,x) \leq 0$. *Furthermore, if* $\int_0^{\pm\infty} a(t,u)\mathrm{d}u = \infty$ *uniformly in* t *and* $e(t)$ *is continuous on* $\Re^+$ *with* $\int_0^\infty |e(t)|\,\mathrm{d}t < \infty$, *then all solutions to equation (9) as well as their derivatives are bounded as* $t \to \infty$.

The last main problem of this paper is the following theorem.

Theorem 6. *In addition to the basic assumptions imposed upon the functions* $c(t,x,y)$ *and* $a(t,x)$, *we assume that the following assumptions hold:*

$$\int_0^x a(t,u)\mathrm{d}u \ \textit{is positive for all } t \textit{ and } x \neq 0,$$

and approaches ∞ *uniformly in* t *as* $|x| \to \infty$,

$$x\,\frac{\partial}{\partial t}\,a(t,x) \le 0 \text{ for all } t \in \Re^{+} \text{ and } x \in \Re,$$

$$c(t,x,y)y > 0 \text{ for all } t \in \Re^{+} \text{ and } x,\ y \in \Re, \text{ and } \int_0^{\infty} |e(t)|\,\mathrm{d}t < \infty.$$

Then, all solutions to equation (9) *as well as their derivatives are bounded as* $t \to \infty$.

Proof. Define the Liapunov function

$$V_6(t,x,y) = \int_0^x a(t,s)\mathrm{d}s + \frac{1}{2}\,y^2$$

so that $V_6(t,0,0) = 0$ and

$$V_6(t,x,y) = \int_0^x a(t,s)\mathrm{d}s + \frac{1}{2}\,y^2 > 0$$

for all $x \neq 0$ and $y \neq 0$.

Under the assumptions of Theorem 6, the time derivative of the Liapunov function $V_6(t,x,y)$ along (10) gives that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_6(t,x,y) = \int_0^x \frac{\partial}{\partial t}a(t,s)\mathrm{d}s - c(t,x,y)y + e(t)y \le |e(t)|\,|y|\,.$$

The rest of the proof is similar to that of Theorem 5. □

Example 6. Consider the equation

$$x'' + \frac{1+t^2+x^2+x'^2}{x'} + (1+e^{-t})x^5 = \frac{\cos t}{1+t^2}, \quad (x' \neq 0).$$

This equation can be written in system form as

$$\begin{aligned} x' &= y, \\ y' &= -\frac{1+t^2+x^2+y^2}{y} - (1+e^{-t})x^5 + \frac{\cos t}{1+t^2}, \quad (y \neq 0). \end{aligned}$$

Hence, we get

$$c(t,x,y)y = 1+t^2+x^2+y^2 > 0, \ \int_0^x a(t,s)\mathrm{d}s = \frac{1}{6}\,(1+e^{-t})x^6 \to \infty \text{ as } |x| \to \infty,$$

$$x\,\frac{\partial}{\partial t}\,a(t,x) = -e^{-t}x^6 \le 0,\ e(t) = \frac{\cos t}{1+t^2}\,,\ |e(t)| \le \frac{1}{1+t^2}\,,\ \int_0^{\infty} |e(t)|\,\mathrm{d}t \le \frac{\pi}{2} < \infty.$$

Subject to the above discussion, it follows that all the assumptions of Theorem 6 hold.

It also follows that

$$V_7(t,x,y) = (1+e^{-t})\frac{x^6}{6} + \frac{1}{2}\,y^2 > 0$$

for all $x \neq 0$ and $y \neq 0$, $V_7(t,0,0) = 0$ and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,V_7(t,x,y) &= -\frac{1}{6}\,e^{-t}x^6 - (1+t^2+x^2+y^2) + \frac{\cos t}{1+t^2}\,y\\ &\le \frac{1}{1+t^2} + \frac{y^2}{1+t^2} \le \frac{1}{1+t^2} + \frac{1}{1+t^2}\,V_7(t,x,y).\end{aligned}$$

Integrating the last inequality on $[0,\ \infty]$ and using the Gronwall-Reid-Bellman inequality (see Gronwall [**1**] and Mitrinović [**2**]), one can conclude that all solutions of the above equation are bounded as $t \to \infty$.

Remark 3. Theorem 5 and Theorem 6 was proved in [**6**] via the integral test. The conditions of Theorem 5 and Theorem 6 are the same as that in Kroopnick [**6**, Theorem I, Theorem II].

Acknowledgement. The author expresses his sincere thanks to the referee for many helpful comments, corrections and suggestions.

REFERENCES

1. T. H. Gronwall: *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations.* Ann. of Math., **20** (4) (1919), 292–296.
2. D. S. Mitrinović (in cooperation with P. M. Vasić): *Analytic inequalities.* Die Grundlehren der mathematischen Wissenschaften, Band 165 Springer-Verlag, New York-Berlin 1970.
3. A. Kroopnick: *Properties of solutions of differential equations of the form $x'' + a(t)b(x) = 0$.* Proc. Amer. Math. Soc. **34** (1972), 319–320.
4. A. Kroopnick: *Note on bounded L^p- solutions of a generalized Liénard equation.* Pacific J. Math., **94** (1) (1981), 171–175.
5. A. Kroopnick: *On the boundedness and oscillation of solutions to $(m(t)x')' + a(t)b(x) = 0$.* Internat. J. Math. Math. Sci., **10** (1) (1987), 47–50.
6. A. Kroopnick: *General boundedness theorems to some second order nonlinear differential equation with integrable forcing term.* Internat. J. Math. Math. Sci., **18** (4) (1995), 823–824.

Department of Mathematics,
Faculty of Arts and Sciences,
Yüzüncü Yıl University,
65080 Van
Turkey
E-mail: cemtunc@yahoo.com

(Received December 27, 2009)
(Revised June 26, 2010)