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TWIN DOMINATION AND TWIN IRREDUNDANCE IN DIGRAPHS

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Let $D = (V, A)$ be a digraph. A subset S of V is called a twin dominating set of D if for every vertex $v \in V - S$, there exist vertices $u_1, u_2 \in S$ (u_1 and u_2 may coincide) such that (v, u_1) and (u_2, v) are arcs in D. The minimum cardinality of a twin dominating set in D is called the twin domination number of D and is denoted by $\gamma^*(D)$. In this paper we present several basic results on these and other related parameters.

1. INTRODUCTION

Throughout this paper $D = (V, A)$ is a finite, directed graph with neither loops nor multiple arcs (but pairs of opposite arcs are allowed) and $G = (V, E)$ is a finite, undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ of G are denoted by n and m respectively. For basic terminology on graphs and digraphs, we refer to Chartrand and Lesniak [**3**].

Let $D = (V, A)$ be a digraph. For any vertex $u \in V$, the sets $O(u) = \{v : (u, v) \in A\}$ and $I(u) = \{v : (v, u) \in A\}$ are called the outset and inset of u. The indegree and outdegree of u are defined by $id(u) = |I(u)|$ and $od(u) = |O(u)|$. The minimum indegree, the minimum outdegree, the maximum indegree and the maximum outdegree of D are denoted by δ^-, δ^+, Δ^- and Δ^+ respectively.

Let $G = (V, E)$ be a graph. A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$ or simply γ.

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Although domination and other related concepts have been extensively studied for undirected graphs, the respective analogues on digraphs have not received much attention. Fu [**7**] studied the following natural analogue of domination in directed graphs. The out-domination number of a directed graph $D = (V, A)$ is the minimum cardinality of a subset S of V such that every vertex of $V - S$ is an out-neighbor of some vertex in S. The in-domination number is defined analogusly. Arumugam et al. [**1**] introduced the concepts of total and connected out-domination in digraphs.

A survey of results on domination in directed graphs by Ghoshal, Laskar and Pillone is given in Chapter 15 of Haynes et al. [**8**], but most of the results in this survey deal with the concepts of kernels and solutions in digraphs and on domination in tournaments. Chartrand et al. [**4**] introduced the concept of twin domination in digraphs.

Definition 1.1. [**4**] *Let $D = (V, A)$ be a digraph. A subset S of V is called a twin dominating set of D if for every vertex $v \in V - S$, there exist vertices $u_1, u_2 \in S$ (u_1 and u_2 may coincide) such that (v, u_1) and (u_2, v) are arcs in D. The minimum cardinality of a twin dominating set in D is called the twin domination number of D and is denoted by $\gamma^*(D)$.*

If $G = (V, E)$ is a graph and if G^* is the symmetric digraph obtained from G by replacing each edge $uv \in E$ by a pair of symmetric arcs (u, v) and (v, u), then a subset S of V is a dominating set of G if and only if S is a twin dominating set of the digraph G^*. Thus twin domination in digraphs includes domination in graphs as a special case.

In this paper we present in the second section some results on twin domination and in the third section some results on twin domination in oriented graphs.

2. MAIN RESULTS

We first give two sharp upper bounds for the twin domination number of a digraph.

We observe that for the directed path P_n with $n \geq 2$ vertices, $\gamma^*(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$ and for a directed cycle C_n with $n \geq 3$ vertices, $\gamma^*(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proposition 2.1. *Let $\ell(D)$ denote the length of a longest directed path in D. Then $\gamma^*(D) \leq n - \left\lfloor \frac{\ell(D)}{2} \right\rfloor$ and the bound is sharp.*

Proof. Let $P = (v_0, v_1, v_2, \ldots, v_k)$ be a longest path in D with $k = \ell(D)$. Let S be a minimum twin dominating set of P. Clearly $S_1 = S \cup (V(D) - V(P))$ is a twin dominating set of D and hence $\gamma^*(D) \leq |S_1| = n - \left\lfloor \frac{\ell(D)}{2} \right\rfloor$. The bound is attained trivially for directed paths, also for any digraph D obtained from a directed path P by attaching at least one in-edge at every vertex of D. $\square$

The proof of the following proposition is similar.

Proposition 2.2. *Let $c(D)$ denote the length of a longest directed cycle in D. Then $\gamma^*(D) \leq n - \left\lfloor \frac{c(D)}{2} \right\rfloor$ and the bound is sharp.*

Proposition 2.3. *Let T be a tournament of order $n \geq 2$. Then $\gamma^*(T) = 2$ if and only if there exist $u, v \in V$ such that $O(u) - \{v\} = I(v) - \{u\}$.*

Proof. Suppose $\gamma^*(T) = 2$ and let $S = \{u, v\}$ be a twin dominating set of T. Let $w \in V - \{u, v\}$. If w is out-dominated by u, then it is in-dominated by v and vice versa. Hence $O(u) - \{v\} = I(v) - \{u\}$. Conversely, if $O(u) - \{v\} = I(v) - \{u\}$, then $\{u, v\}$ is a twin dominating set of T and hence $\gamma^*(T) = 2$.

Corollary 2.4. *If T is a transitive tournament, then $\gamma^*(T) = 2$.*

Observation 2.5. *Since any tournament T of order n contains a directed hamiltonian path, it follows that $\gamma^*(T) \leq \left\lceil \frac{n+1}{2} \right\rceil$.*

Theorem 2.6. *Let T be a tournament of order $n \geq 3$. If there exists $u \in V(T)$ such that $od(u) = 0$ or $id(u) = 0$, then $\gamma^*(T) \leq \lceil \log_2 (n-1) \rceil + 1$.*

Proof. Without loss of generality we assume that $od(u) = 0$. Then u in-dominates the set $V(T) - \{u\}$ and u belongs to every twin dominating set of T. Let T_1 be the subtournament obtained by deleting u from T. Since $\sum_{v \in V(T_1)} od(v) = \frac{(n-1)(n-2)}{2}$, it follows that there exists a vertex u_1 in T_1 with $od(u_1) \geq \left\lceil \frac{n-2}{2} \right\rceil$. Now, let $T_2 = T_1 - O[u_1]$ and let u_2 be a vertex of T_2 which out-dominates at least $\left\lceil \frac{|V(T_2)|}{2} \right\rceil$ vertices of T_2.

By continuing this process we obtain an out-dominating set S of T_1 with $|S| \leq \lceil \log_2 (n-1) \rceil$. Now $S \cup \{u\}$ is a twin dominating set of T and hence $\gamma^*(T) \leq \lceil \log_2 (n-1) \rceil + 1$. □

We observe that if S is a twin dominating set of a digraph D, then any superset of S is also a twin dominating set, so that twin domination in digraphs is a superhereditary property. Hence a twin dominating set S is a minimal twin dominating set if and only if $S - \{u\}$ is not a twin dominating set for all $u \in S$. The following theorem gives a necessary and sufficient condition for a twin dominating set to be minimal.

Theorem 2.7. *A twin dominating set S of a digraph D is a minimal twin dominating set if and only if for each vertex $u \in S$, there exists $v \in V$ such that $O[v] \cap S = \{u\}$ or $I[v] \cap S = \{u\}$.*

Proof. Suppose S is a minimal twin dominating set of D and let $u \in S$. Then $S - \{u\}$ is not a twin dominating set of D and hence there exists $v \in V - (S - \{u\})$ such that v is not out-dominated or in-dominated by any vertex in $S - \{u\}$. Hence $O[v] \cap S = \{u\}$ or $I[v] \cap S = \{u\}$. The converse is obvious.

Definition 2.8. *Let $D = (V, A)$ be a directed graph, $S \subseteq V$ and $u \in S$. A vertex $v \in V$ is called an out-private neighbor of u with respect to S if $O[v] \cap S = \{u\}$ and v is called an in-private neighbor of u with respect to S if $I[v] \cap S = \{u\}$. The set of all out-private neighbors of u with respect to S is denoted by* $\mathrm{pn}^+[u, S]$ *and the set of all in-private neighbors of u with respect to S is denoted by* $\mathrm{pn}^-[u, S]$. *The set of all private neighbors of u with respect to S is denoted by* $\mathrm{pn}[u, S] = \mathrm{pn}^+[u, S] \cup \mathrm{pn}^-[u, S]$.

It follows from Theorem 2.7 that a twin dominating set S is a minimal twin dominating set if and only if $\mathrm{pn}[u, S] \neq \emptyset$ for all $u \in S$. This motivates the following definition.

Definition 2.9. *Let $D = (V, A)$ be a directed graph. A subset S of V is called a twin irredundant set if* $\mathrm{pn}[u, S] \neq \emptyset$ *for all $u \in S$.*

It follows from Theorem 2.7 that every minimal twin dominating set is a twin irredundant set. We observe that any subset of a twin irredundant set is a twin irredundant set and hence a twin irredundant set is maximal if and only if $S \cup \{u\}$ is not a twin irredundant set for all $u \in V - S$.

Definition 2.10. *The minimum cardinality of a maximal twin irredundant set in D is called the twin irredundance number of D and is denoted by $ir^*(D)$. The maximum cardinality of a twin irredundant set in D is called the upper twin irredundance number of D and is denoted $IR^*(D)$.*

Proposition 2.11. *A twin dominating set S is a minimal twin dominating set if and only if it is twin dominating and twin irredundant.*

Proof. It follows from the definition that a minimal twin dominating set is twin irredundant. Conversely, suppose a set S is both twin dominating and twin irredundant. Let $v \in S$. Since S is twin irredundant, $\mathrm{pn}^+[v, S] \neq \emptyset$ or $\mathrm{pn}^-[v, S] \neq \emptyset$. Now any vertex $w \in \mathrm{pn}^+[v, S] \cup \mathrm{pn}^-[v, S]$ is not twin dominated by any vertex in $S - \{v\}$. Hence $S - \{v\}$ is not a twin dominating set, so that S is a minimal twin dominating set.

Proposition 2.12. *Every minimal twin dominating set in a digraph D is a maximal twin irredundant set of D.*

Proof. It follows from Theorem 2.7 that S is twin irredundant. If S is not a maximal twin irredundant set, then there exists a vertex $u \in V - S$ such that $S \cup \{u\}$ is twin irredundant. Hence $\mathrm{pn}^+[u, S \cup \{u\}] \neq \emptyset$ or $\mathrm{pn}^-[u, S \cup \{u\}] \neq \emptyset$. Then any vertex $w \in \mathrm{pn}^+[u, S \cup \{u\}] \cup \mathrm{pn}^-[u, S \cup \{u\}]$ is not twin dominated by S, which is a contradiction.

Corollary 2.13. *For any digraph D, $ir^*(D) \leq \gamma^*(D) \leq \Gamma^*(D) \leq IR^*(D)$.*

In the study of domination in graphs, the concepts of maximal independent set, minimal dominating set and maximal irredundant set lead to the following inequality chain, known as the *domination chain* of parameters:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G).$$

This inequality chain was first established by COCKAYNE et al. [**6**]. This chain of inequalities has become one of the major focal points in domination theory. In the case of digraphs, the concept of twin domination and twin irredundance lead to an inequality chain of four parameters given in Corollary 2.13. In this context we pose the following problem.

Problem 2.14. *Define the concept of twin independence satisfying the condition that a twin independent set S is maximal twin independent if and only if S is twin-independent and twin dominating.*

The following result is the analogue of a theorem by BOLLOBÁS and COCKAYNE appearing in [**2**].

Theorem 2.15. *For any digraph D,*

$$\frac{\gamma^*(D)}{2} < ir^*(D) \leq \gamma^*(D) \leq 2ir^*(D) - 1.$$

Proof. Let $ir^*(D) = k$ and let $S = \{v_1, v_2, \dots, v_k\}$ be a twin irredundant set of D. Since S is twin irredundant, $\mathrm{pn}^+[v_i, S] \cup \mathrm{pn}^-[v_i, S] \neq \emptyset$, for $1 \leq i \leq k$. Let $S' = \{u_1, u_2, \dots, u_k\}$ where $u_i \in \mathrm{pn}^+[v_i, S] \cup \mathrm{pn}^-[v_i, S]$. The possibility of u_i being equal to v_i is not excluded. Clearly $|S \cup S'| \leq 2k = 2ir^*(D)$.

We claim that the set $S'' = S \cup S'$ is a twin dominating set of D. If not, there exists $w \in V - S''$ which is not twin dominated by S''. Hence $\mathrm{pn}^+[w, S \cup \{w\}] \cup \mathrm{pn}^-[w, S \cup \{w\}] \neq \emptyset$. Further since $u_i \in \mathrm{pn}^+[v_i, S] \cup \mathrm{pn}^-[v_i, S]$, we have $\mathrm{pn}^+[u_i, S \cup \{w\}] \cup \mathrm{pn}^-[u_i, S \cup \{w\}] \neq \emptyset$. Thus $S \cup \{w\}$ is a twin irredundant set, which is a contradiction. Therefore, S'' is a twin dominating set and since S is a maximal twin irredundant set, it follows that S'' is not a minimal twin dominating set. Therefore, $\gamma^*(D) \leq 2ir^*(D) - 1$ and $\dfrac{\gamma^*(D)}{2} < ir^*(D)$.

Theorem 2.16. *For any digraph D, $IR^*(D) \leq n - \delta$, where $\delta = min\{\delta^+, \delta^-\}$.*

Proof. The result is trivial if $\delta = 0$. Hence we assume that $\delta > 0$ and there exists $v \in V(D)$ with $od(v) = \delta$. Let S be a twin irredundant set in D. Suppose that v has k out-neighbors in S. Then v has $\delta - k$ out-neighbors in $V - S$. If $k = 0$, then $|V - S| \geq \delta$, so that $|S| \leq n - \delta$, as required. If $k > 0$, then each out-neighbor of v in S must have an out-private neighbor in $V - S$ and these k vertices must be distinct. Thus $|V - S| \geq (\delta - k) + k = \delta$.

3. TWIN DOMINATION IN ORIENTED GRAPHS

For the two orientations D_1 and D_2 of the cycle C_4 given in Figure 1, we have $\gamma^*(D_1) = 2$ and $\gamma^*(D_2) = 4$.

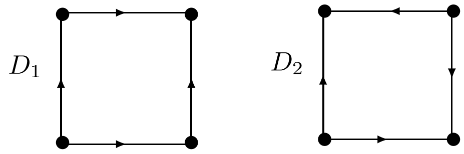

Figure 1. Two orientations of C_4 with twin domination numbers 2 and 4

Thus two distinct orientations of a graph can have distinct twin domination numbers. Motivated by this observation Chartrand et al. [**4**] introduced the following concept of orientable twin domination number $\mathrm{dom}^*(G)$ and upper orientable twin domination number $\mathrm{DOM}^*(G)$ of a graph G, which are defined by

$$\mathrm{dom}^*(G) = \min\{\gamma^*(D)\ :\ D \text{ is an orientation of } G\},$$

and

$$\mathrm{DOM}^*(G) = \max\{\gamma^*(D)\ :\ D \text{ is an orientation of } G\}.$$

Theorem 3.1. *Let G be a nontrivial connected graph of order n. Then $\mathrm{DOM}^*(G) = n$ if and only if G is bipartite.*

Proof. Let $\mathrm{DOM}^*(G) = n$. Then there exists an orientation D of G such that $\gamma^*(D) = n$. Clearly $od(v) = 0$ or $id(v) = 0$ for all $v \in V$. Now, let $C = (v_1, v_2, \dots, v_k, v_1)$ be a cycle of length k in D and let $od(v_1) = 0$. Then it follows that $od(v_i) = 0$ if i is odd and $id(v_i) = 0$ if i is even. Now, since $od(v_1) = 0$, we have $id(v_k) = 0$ and thus k is even. Hence G is bipartite. Conversely, assume that G is a bipartite graph with bipartition X and Y. Let D be the orientation of G obtained by directing all edges of G from X to Y. Then $X \cup Y$ is the unique minimum twin dominating set for D and hence $\mathrm{DOM}^*(G) = n$.

Corollary 3.2. *A nontrivial connected bipartite graph G admits an orientation D such that $\Gamma^*(D) = IR^*(D)$.*

Theorem 3.3. *For any two integers r, s with $r \leq s$,*

$$\mathrm{dom}^*(K_{r,s}) = \begin{cases} 2 & \textit{if } r = s = 1 \textit{ or } r = 2, \\ s & \textit{if } r = 1 \textit{ and } s \geq 2, \\ 3 & \textit{if } r = 3, \\ 4 & \textit{if } r \geq 4. \end{cases}$$

Proof. Let $V_1 = \{v_1, v_2, \dots, v_r\}$ and $V_2 = \{u_1, u_2, \dots, u_s\}$ be the bipartition of $G = K_{r,s}$. The result is obvious if $r = 1$ or 2. We consider two cases.

Case i. $r = 3$.

Let D_1 be an orientation of G such that $od(v_1) = 0$ and $id(v_2) = 0$. Then $V_1 = \{v_1, v_2, v_3\}$ is a twin dominating set of D_1, so that $\mathrm{dom}^*(G) \leq \gamma^*(D_1) \leq 3$.

Now, let D be any orientation of G and let S be a twin dominating of D. If $S \cap V_1 = \emptyset$, then $S = V_2$. If $S \cap V_2 = \emptyset$, then $S = V_1$. If $S \cap V_1 \neq \emptyset$ and $S \cap V_2 \neq \emptyset$, then $|S \cap V_1| \geq 2$ and $|S \cap V_2| \geq 2$. In all cases $|S| \geq 3$ and hence $\mathrm{dom}^*(G) \geq 3$. Thus $\mathrm{dom}^*(G) = 3$.

Case ii. $r \geq 4$.

Let D_1 be an orientation of G such that $od(v_1) = 0, id(v_2) = 0, u_1 \in O(v_i)$, $u_2 \in I(v_i), 3 \leq i \leq r$. Clearly $S = \{v_1, v_2, u_1, u_2\}$ is a twin dominating set of D_1, so that $\mathrm{dom}^*(G) \leq \gamma^*(D_1) = 4$. Further $\gamma^*(D) \geq 4$ for any orientation D of G and hence $\mathrm{dom}^*(G) = 4$. □

We now proceed to determine the upper and lower orientable twin domination numbers of several classes of graphs such as complete bipartite graphs, paths, cycles, wheels and complete graphs.

It follows from Theorem 3.1 that $\mathrm{DOM}^*(P_n) = n$, and $\mathrm{DOM}^*(C_n) = n$ if n is even. Also $\mathrm{DOM}^*(C_n) = n - 1$ if n is odd.

Theorem 3.4. *For the path $P_n = (v_1, v_2, v_3, \ldots, v_n)$, we have* $\mathrm{dom}^*(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Proof. If D_1 is the directed path on n vertices, then $\gamma^*(D_1) = \left\lceil \frac{n+1}{2} \right\rceil$ and hence $\mathrm{dom}^*(P_n) \leq \left\lceil \frac{n+1}{2} \right\rceil$. We now prove by induction on n, that $\gamma^*(D) \geq \left\lceil \frac{n+1}{2} \right\rceil$ for every orientation D of P_n. If $n = 2$, then $\gamma^*(D) = 2 \geq \left\lceil \frac{n+1}{2} \right\rceil$. We assume that the result is true for any path with less than n vertices and let D be an orientation of P_n. If $od(v_i) = 0$ or $id(v_i) = 0$ for all $v_i, 1 \leq i \leq n$ then the inequality is trivial. Suppose there exists v_i, $(1 < i < n)$ such that $id(v_i) = 1$ and $od(v_i) = 1$. Then $D - \{v_i\} = D_1 \cup D_2$ where $|D_1| = n_1 < n$ and $|D_2| = n_2 < n$ and $n_1 + n_2 = n - 1$. Let S_1 be a twin dominating set of D_1 and S_2 be a twin dominating set of D_2. Clearly $v_{i-1} \in S_1$ and $v_{i+1} \in S_2$ and hence $S = S_1 \cup S_2$ is a twin dominating set of D. Also by induction hypothesis, $|S_1| \geq \left\lceil \frac{n_1+1}{2} \right\rceil$ and $|S_2| \geq \left\lceil \frac{n_2+1}{2} \right\rceil$. Hence $|S| \geq \left\lceil \frac{n_1+n_2+2}{2} \right\rceil \geq \left\lceil \frac{n+1}{2} \right\rceil$. So $\gamma^*(D) \geq \left\lceil \frac{n+1}{2} \right\rceil$ for every orientation D of P_n and hence $\mathrm{dom}^*(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Theorem 3.5. *For every integer $n \geq 3$,* $\mathrm{dom}^*(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. If D_1 is the directed cycle on n vertices, then $\gamma^*(D_1) = \left\lceil \frac{n}{2} \right\rceil$ and hence $\mathrm{dom}^*(C_n) \leq \left\lceil \frac{n}{2} \right\rceil$. Further for any orientation D of C_n, any twin dominating set S of D contains either v_i or v_{i+1} for any i, and hence $|S| \geq \left\lceil \frac{n}{2} \right\rceil$. Thus $\mathrm{dom}^*(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Theorem 3.6. *For $n \geq 4$,*

$$\mathrm{dom}^*(W_n) = \begin{cases} \left\lceil \frac{n-1}{3} \right\rceil & \text{if } n = 5, \\ \left\lceil \frac{n-1}{3} \right\rceil + 1 & \text{otherwise.} \end{cases}$$

Proof. Let $W_n = w + C_{n-1}$ and $C_{n-1} = (v_1, v_2, \ldots, v_{n-1}, v_1)$. Let D_1 be an orientation of W_n such that $od(w) = 0$ and $id(v_i) = 0$ if $i \equiv 1 \pmod 3$. Clearly $\gamma^*(D_1) = \left\lceil \frac{n-1}{3} \right\rceil + 1$ and hence $\mathrm{dom}^*(W_n) \leq \left\lceil \frac{n-1}{3} \right\rceil + 1$. Now, let D be any orientation of W_n. Let S be a twin dominating set of D with $|S| = \gamma^*(D)$. If $w \notin S$, then $|S| \geq \left\lceil \frac{n-1}{2} \right\rceil \geq \left\lceil \frac{n-1}{3} \right\rceil + 1$. If $w \in S$, then $S - \{w\}$ is a dominating set of C_{n-1} and hence $|S - \{w\}| \geq \left\lceil \frac{n-1}{3} \right\rceil$. Thus $\mathrm{dom}^*(W_n) \geq |S| \geq \left\lceil \frac{n-1}{3} \right\rceil + 1$. Hence $\mathrm{dom}^*(W_n) = \left\lceil \frac{n-1}{3} \right\rceil + 1$.

Theorem 3.7. *For $n \geq 4$,*

$$\mathrm{DOM}^*(W_n) = \begin{cases} n-1 & \text{if } n \text{ is odd or } n = 4, \\ n-2 & \text{if } n \text{ is even and } n \geq 6. \end{cases}$$

Proof. If n is odd, let D_1 be an orientation of W_n such that $id(v_i) = 0$ if i is odd and $od(v_i) = 0$ if i is even. Clearly $\gamma^*(D_1) = n - 1$ and hence $\mathrm{DOM}^*(W_n) \geq n - 1$. Hence it follows from Theorem 3.1 that $\mathrm{DOM}^*(W_n) = n - 1$.

Now suppose n is even and $n \geq 6$, let D_1 be an orientation of W_n such that $id(v_i) = 0$ if i is odd, $id(v_i) = 0$ if i is even and $i \neq n-2$, and $id(v_{n-2}) = 1, od(v_{n-2}) = 2$. Clearly $\gamma^*(D_1) \geq n - 2$ and hence $\mathrm{DOM}^*(W_n) \geq n - 2$.

Now, let D be any orientation of W_n. We claim that $\gamma^*(D) \leq n - 2$. Since C_{n-1} is an odd cycle we may assume without loss of generality that v_1 has indegree and outdegree 1 in C_{n-1}. Now $\gamma^*(D - \{v_1, v_2, v_{n-1}\}) \leq n - 4$. If S_1 is a minimum twin dominating set of D_1, then $S = S_1 \cup \{v_2, v_{n-1}\}$ is a twin dominating set of D and $|S| \leq n - 2$. Hence $\mathrm{DOM}^*(W_n) = n - 2$.

Theorem 3.8. *For $r \geq 1$,*

$$\mathrm{dom}^*(K_{r,r,r}) = \begin{cases} 2 & \text{if } r = 1 \text{ or } 2, \\ 3 & \text{otherwise} \end{cases}$$

and $\mathrm{DOM}^*(K_{r,r,r}) = 2r$.

Proof. Let $V_1 = \{v_1, v_2, \ldots, v_r\}, V_2 = \{u_1, u_2, \ldots, u_r\}$ and $V_3 = \{w_1, w_2, \ldots, w_r\}$ be the partite sets of $G = K_{r,r,r}$. The result is obvious if $r = 1$ or 2. Suppose $r \geq 3$. Let D_1 be an orientation of G such that $I(v_1) = V_2 \cup V_3$, $O(u_1) = V_1 \cup V_3$, $O(w_1) = V_2 - \{u_1\}$ and $I(w_1) = (V_1 - \{v_1\}) \cup \{u_1\}$. Then $\{u_1, v_1, w_1\}$ is a twin dominating set of D_1 and hence $\mathrm{dom}^*(G) \leq \gamma^*(D_1) = 3$. Further $\gamma^*(D) \geq 3$ for any arbitrary orientation D of G and hence $\mathrm{dom}^*(G) = 3$.

We now proceed to prove that $\mathrm{DOM}^*(G) = 2r$. Let D_1 be an orientation of G such that $O(v_i) = V_2 \cup V_3$ and $O(u_i) = V_3$ for all $i, 1 \leq i \leq r$. Clearly $V_1 \cup V_3$ is the unique minimum twin dominating set of D_1, so that $\mathrm{DOM}^*(G) \geq \gamma^*(D_1) = 2r$. Now let D be an arbitrary orientation of G. Since each of the r triangles (u_i, v_i, w_i, u_i) contains at least one vertex with both outdegree and indegree at least 1, it follows that $\gamma^*(D) \leq 2r$ and hence $\mathrm{DOM}^*(G) = 2r$.

Observation 3.9. *It follows from Corollary* 2.4 *and Observation* 2.5 *that* $\mathrm{dom}^*(K_n) = 2$ *and* $\mathrm{DOM}^*(K_n) \leq \left\lceil \frac{n+1}{2} \right\rceil$.

Theorem 3.10. *For* $2 \leq n \leq 7$, *we have* $\mathrm{DOM}^*(K_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Proof. Let T_i be the orientation of K_i, $2 \leq i \leq 7$, given in Figure 2, where an arrow between two disjoint subsets A and B indicates that all edges with one end x in A and the other end y in B are given the orientation (x, y). We first prove the result for $n = 7$. It follows from Observation 2.5 that $\gamma^*(T_7) \leq 4$. Now let S be any twin dominating set of T_7. Let $C_1 = \{v_1, v_2, v_3\}$ and $C_2 = \{v_5, v_6, v_7\}$. Since no vertex of C_1 is out-dominated by a vertex of $V(T_7) - C_1$ and no vertex of C_2 is in-dominated by a vertex of $V(T_7) - C_2$, we have $S \cap V(C_1) \neq \emptyset$ and $S \cap V(C_2) \neq \emptyset$. Without loss of generality we assume that $v_1, v_5 \in S$. If $|S| = 3$, then S is of the form $\{v_1, v_2, v_5\}$ or $\{v_1, v_5, v_6\}$ or $\{v_1, v_4, v_5\}$ and it can be easily verified that none of these sets is a twin dominating set of T_7. Thus $|S| \geq 4$ and hence $\gamma^*(T_7) \geq 4$. Thus $\mathrm{DOM}^*(K_7) \geq 4$ and so $\mathrm{DOM}^*(K_7) = 4$. The proof is similar for $n = 2, 3, 4, 5$ *or* 6.

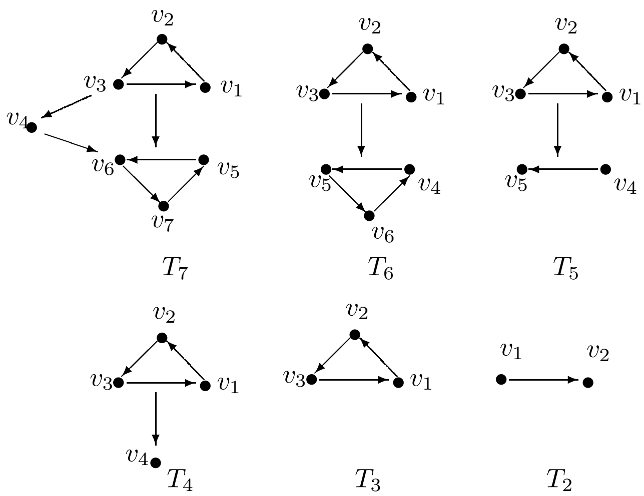

Figure 2. Tournaments with $\mathrm{DOM}^* = \left\lceil \frac{n+1}{2} \right\rceil, 2 \leq n \leq 7$

Conjecture 3.11. $\mathrm{DOM}^*(K_n) = \left\lceil \frac{n+1}{2} \right\rceil$ *for all* $n \geq 8$.

Since the n-dimensional hypercube Q_n is hamiltonian when $n \geq 2$, it follows that $\gamma^*(Q_n) \leq 2^{n-1}$. In this context we propose the following conjecture.

Conjecture 3.12. *For every positive integer* $n \geq 2$, $\mathrm{dom}^*(Q_n) = 2^{n-1}$.

Remark 3.13. For any digraph $D = (V, A)$, the vertex set V is trivially a twin dominating set and hence every digraph admits a partition of V into twin dominating sets. Hence as in graphs we can define the twin domatic number of a digraph D to be the maximum order of a partition of V into twin dominating sets. The study of this parameter is open.

Further one can investigate the effect of the removal of a vertex or an edge on the twin domination number.

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