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ON THE BOUNDEDNESS OF THE V-CONJUGATION OPERATOR ON HARDY SPACES

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Abstract. In this work we prove that the V-conjugation operator is of type (H^1, H^1). We provide an example that shows that the L^1-norm of the V-conjugate function is not equivalent to the H^1-norm of the original function on any Vilenkin group.

1. Introduction

The V-conjugate function was studied in the works of P. Simon [**1**], [**3**] and [**2**]. In [**1**, Theorem 4] it has been proved that the V-conjugation operator is of type (H^1, L^1). On the same paper it was speculated that the L^1-norm of the V-conjugate function is equivalent to the H^1-norm of the original function. In this paper we establish that these two norms are not equivalent on any Vilenkin group. Besides, we prove that the V-conjugate function of any element from H^1 also belongs to the space H^1, on which the boundedness remains valid.

Let $(m_0, m_1, \dots, m_n, \dots)$ be a bounded sequence of integers not less than 2. Let $G := \prod_{n=0}^{\infty} \mathbb{Z}_{m_n}$, where $\mathbb{Z}_{m_n}$ denotes the discrete group of order m_n, with addition *mod* m_n. Each element from G can be represented as a sequence $(x_n)_n$, where $x_n \in \{0, 1, \dots, m_n - 1\}$, for every integer $n \geq 0$. Addition in G is obtained coordinatewise.

The topology on G is generated by the subgroups:

$$I_n := \{x = (x_i)_i \in G, x_i = 0 \text{ for } i < n\},$$

and their translations

$$I_n(y) := \{x = (x_i)_i \in G, x_i = y_i \text{ for } i < n\}.$$

The basis $(e_n)_n$ is formed by elements $e_n = (\delta_{in})_i$.

Define the sequence $(M_n)_n$ as follows: $M_0 = 1$ and $M_{n+1} = m_n M_n$.

If $|I_n|$ denotes the normalized product measure of I_n then it can be easily seen that $|I_n| = M_n^{-1}$.

The generalized Rademacher functions are defined by

$$r_n(x) := e^{\frac{2\pi i x_n}{m_n}}, n \in \mathbb{N} \cup \{0\}, x \in G,$$

For every nonnegative integer n, there exists a unique sequence $(n_i)_i$ so that $n = \sum_{i=0}^{\infty} n_i M_i$.

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and the system of Vilenkin functions by

$$\psi_n(x) := \prod_{i=0}^{\infty} r^{n_i}(x), n \in \mathbb{N} \cup \{0\}, x \in G.$$

The Fourier coefficients, the partial sums of the Fourier series and the Dirichlet kernels are respectively defined as follows

$$\hat{f}(n) = \int f(x)\bar{\psi}_n(x)dx, \qquad S_n f = \sum_{k=0}^{n-1} \hat{f}(k)\psi_k, \qquad D_n = \sum_{k=0}^{n-1} \psi_k.$$

It can be easily seen that

$$S_n f(y) = \int D_n(y-x)f(x)dx,$$

and

$$D_{M_n}(x) = M_n 1_{I_n}(x).$$

The Hardy space H^1 (see [**1**]) consists of integrable functions f for which the two quadratic variations

$$q(f) = (|\hat{f}(0)|^2 + \sum_{n=0}^{\infty} |S_{M_{n+1}} f - S_{M_n} f|^2)^{\frac{1}{2}},$$

$$Q(f) = (|\hat{f}(0)|^2 + \sum_{n=0}^{\infty} \sum_{j=1}^{m_n-1} |S_{(j+1)M_n} f - S_{jM_n} f|^2)^{\frac{1}{2}}$$

belong to L^1.

Quadratic variation and conditional quadratic variation were also used in [**4**] and [**5**].

In this work we introduce a new quadratic variation

$$\tilde{q}(f)(x) = (|\hat{f}(0)|^2 + \sum_{n=0}^{\infty} \sup_{s\in\{0,\ldots,m_n-1\}} |S_{M_{n+1}} f(x+se_n) - S_{M_n} f(x)|^2)^{\frac{1}{2}}.$$

The V-conjugate function as defined in [**1**] and [**3**], has the form

$$\tilde{f} := \sum_{k=0}^{\infty} f * L_k D_{M_k},$$

where f is an integrable function and

$$L_k := -\sum_{j=1}^{\triangle_k} r_k^j + \sum_{j=\triangle_k+1}^{m_k-1} r_k^j,$$

$\triangle_k = [\frac{m_k-1}{2}]$, if $m_k > 2$, and $\triangle_k = 1$, if $m_k = 2$.

Using the expressions of L_k given in [**3**], we have

$$L_k(x) = 1 - \frac{1}{i}\frac{(-1)^k}{sin\dfrac{\pi x_k}{m_k}} + \frac{exp\left(-\dfrac{\pi x_k i}{m_k}\right)}{i}\left(sin\frac{x_k\pi}{m_k}\right)^{-1},$$

where $(x_k \neq 0, m_k \equiv 1(2))$.

For $m_k \equiv 0(2)$, $(m_k > 2)$, $(x_k \neq 0)$, we have

$$L_k(x) = 1 + \frac{1}{i} \frac{exp\left(-\dfrac{\pi x_k i}{m_k}\right)[1-(-1)^{x_k}]}{sin\dfrac{\pi x_k}{m_k}}.$$

When $m_k = 2$, we have $L_k(x) = -(-1)^{x_k}$, $(x \in G)$.

For $m_k \equiv 1(2))$, $s \in \{1, ..., m_k - 1\}$, we write

$$L_k(se_k) = 1 + \frac{i(-1)^s}{sin\dfrac{s\pi}{m_k}} - \frac{icos\dfrac{s\pi}{m_k} + sin\dfrac{s\pi}{m_k}}{sin\dfrac{s\pi}{m_k}} = \frac{i(-1)^s}{sin\dfrac{s\pi}{m_k}} - i\frac{cos\dfrac{s\pi}{m_k}}{sin\dfrac{s\pi}{m_k}}.$$

If $s \in \{2, ..., m_k - 1\}$ is even, then

$$L_k(se_k) = i\frac{1-cos\dfrac{s\pi}{m_k}}{sin\dfrac{s\pi}{m_k}} = 2i\frac{sin^2\dfrac{s\pi}{2m_k}}{2sin\dfrac{s\pi}{2m_k}cos\dfrac{s\pi}{2m_k}} = i\tan\frac{s\pi}{2m_k},$$

and clearly, $L_k(0) = 0$. If s is odd, we obtain

$$L_k(se_k) = -i\frac{1+cos\dfrac{s\pi}{m_k}}{sin\dfrac{s\pi}{m_k}} = -2i\frac{cos^2\dfrac{s\pi}{2m_k}}{2sin\dfrac{s\pi}{2m_k}cos\dfrac{s\pi}{2m_k}} = -i\cot\frac{s\pi}{2m_k}.$$

Notice that

$$\begin{aligned}
(f * L_k D_{M_k})(y) &= \int f(x)(L_k D_{M_k})(y-x)dx = M_k \int\limits_{I_k(y)} f(x)L_k(y-x)dx \\
&= M_k \sum_{j=0}^{m_{k-1}} \int\limits_{(y-x)\in(s\cdot e_k + I_{k+1})} f(x)L_k(y-x)dx = \\
&= M_k \sum_{j=0}^{m_{k-1}} L_k(se_k) \int\limits_{t\in(s\cdot e_k+I_{k+1})} f(y-t)dt = \sum_{s=0}^{m_{k-1}} L_k(se_k)a_s^k(y),
\end{aligned}$$

where $a_s^k(y) = M_k \int\limits_{t\in(s\cdot e_k+I_{k+1})} f(y-t)dt$.

Notice that if s is odd then

$$-L_k(se_k) = i\cot\frac{\pi s}{2m_k} = i\tan\left(\frac{\pi}{2} - \frac{\pi s}{2m_k}\right) = i\tan\left(\frac{m_k - s}{2m_k}\pi\right) = L_k((m_k - s)e_k).$$

This is clearly also valid when s is even. Therefore,

$$\sum_{s=0}^{m_k} L_k(se_k)a_s^k = \sum_{s=1}^{m_k} L_k(se_k)a_s^k = \sum_{s=1}^{m_k} L_k(se_k)(a_s^k - a_{m_k-s}^k) = \sum_{s=1}^{m_k} L_k(se_k)(a_s^k - a_{-s}^k).$$

Now, if m_k is even we have

$$\begin{aligned} L_k(se_k) &= 1 + \frac{1}{i}\frac{(\cos(\frac{\pi s}{m_k}) - i\sin(\frac{\pi s}{m_k}))[1-(-1)^s]}{sin\frac{\pi s}{m_k}} \\ &= 1 - \frac{(i\cos(\frac{\pi s}{m_k}) + \sin(\frac{\pi s}{m_k}))[1-(-1)^s]}{sin\frac{\pi s}{m_k}}. \end{aligned}$$

Then, for s even $L_k(se_k) = 1$, and for s odd we get

$$L_k(se_k) = 1 - 2\frac{i\cos\frac{\pi s}{m_k} + \sin\frac{\pi s}{m_k}}{sin\frac{\pi s}{m_k}} = -1 - 2i\cot\frac{\pi s}{m_k}.$$

Set $\gamma_s = 2\cot\frac{\pi s}{m_k}$.

Notice that

$$-\gamma_s = 2\cot(\pi - \frac{\pi s}{m_k}) = 2\cot\frac{\pi(m_k - s)}{m_k} = \gamma_{-s}.$$

The notation C will be used for an absolute positive constant that may vary in different contexts.

2. Main Results

Theorem 2.1. *Let G be a bounded Vilenkin group. There exists a sequence $(f_n)_n \in L^1(G)$ satisfying $\|\tilde{f}_n\|_1 = 1$ for every $n \in \mathbb{N}$, and $\|f_n\|_1 \to \infty$ as $n \to \infty$.*

Proof. Let

$$f_n(y) = \sum_{0\le k\le n, m_k odd} f^k(y)(1_{\{y_k=1\}} - 1_{\{y_k=0\}}) + \sum_{0\le k\le n, m_k even} f^k(y)\sum_{j=0}^{m_k-1}(-1)^j 1_{\{y_k=j\}},$$

where the functions $f^k(y)$ are recursively constructed from the formulae

$$f^k(y) = \frac{m_k}{L_k(e_k)}\tilde{f}_{k-1}(y),$$

if m_k is odd,

$$f^k(y) = \tilde{f}_{k-1}(y),$$

if m_k is even and $k \ge 2$,

$$f^0(y) = \frac{m_0}{L_0(y_0 - 1) - L_0(y_0)},$$

for m_0 odd, and $f^0(y) = (-1)^{y_0}$ if m_0 is even.

Let

$$\alpha_k(y_k) = 1 + \frac{L_k((y_k-1)e_k) - L_k(y_ke_k)}{L_k(e_k)},$$

if m_k is odd and $y_k \in \{0, \ldots, m_k - 1\}$, and

$$\alpha_k(y_k) = 1 + (-1)^{y_k},$$

for m_k even.

Since f_n is constant on every I_{n+1}-coset, it follows that $L_k D_{M_k} * f_n = 0$ if $k \geq n+1$ and that f^n are constant on I_n-cosets.

Moreover,

$$\begin{aligned} L_k D_{M_k} * f_k &= L_k D_{M_k} * f_{k-1} + L_k D_{M_k} * f^k(1_{\{y_k=1\}} - 1_{\{y_k=0\}}) \\ &= L_k D_{M_k} * f^k(1_{\{y_k=1\}} - 1_{\{y_k=0\}}). \end{aligned}$$

If m_k is an odd number then for $y \in G$,

$$\begin{aligned} L_k D_{M_k} * f_k(y) &= M_k \int_{I_k(y)} f^k(x) L_k(y-x)(1_{\{x_k=1\}} - 1_{\{x_k=0\}})dx \\ &= f^k(y) \frac{M_k}{M_{k+1}} (L_k((y_k-1)e_k) - L_k(y_k e_k)) \\ &= \tilde{f}_{k-1}(y) \frac{L_k((y_k-1)e_k) - L_k(y_k e_k)}{L_k(e_k)}. \end{aligned}$$

If m_k is even we get

$$\begin{aligned} L_k D_{M_k} * f_k(y) &= M_k \int_{I_k(y)} f^k(x) L_k(y-x)(\sum_{j=0}^{m_k-1} (-1)^j 1_{\{x_k=j\}}(x))dx \\ &= M_k f^k(y) \sum_{s=0}^{m_k-1} (-1)^{y_k+s} L_k(-se_k) \int_{I_{k+1}(y+se_k)} dx \\ &= \frac{(-1)^{y_k}}{m_k} f^k(y) \sum_{s=0}^{m_k-1} (-1)^s L_k(-se_k) \end{aligned}$$

$$= (-1)^{y_k} f^k(y) + \frac{(-1)^{y_k}}{m_k} f^k(y) \sum_{0 \leq s \leq m_k-1, s\ odd} 2i \cot \frac{-s\pi}{m_k} = (-1)^{y_k} \tilde{f}_{k-1}(y).$$

Using these facts it is easily seen that $\tilde{f}_0 = 1$. Besides, for arbitrary n we have

$$\tilde{f}_n(y) = \tilde{f}_{n-1}(y) + (L_n D_{M_n} * f_n)(y) = \alpha_n(y_n) \tilde{f}_{n-1}(y).$$

From $\tilde{f}_1(y) = \alpha_1(y_1)\tilde{f}_0(y) = \alpha_1(y_1)$, it follows that $\tilde{f}_n(y) = \prod_{k=1}^n \alpha_k(y_k)$.

The following step is to verify that $\alpha_k(y_k)$ is nonnegative for every $y_k \in \{0, \ldots, m_k - 1\}, k \geq 1$. This is obviously true when m_k is even. Now, if m_k is odd and y_k even but different from 0, we have

$$\begin{aligned} \alpha_k(y_k) &= 1 + \frac{L_k((y_k-1)e_k) - L_k(y_k e_k)}{L_k(e_k)} \\ &= 1 + \frac{\cot \frac{y_k-1}{2m_k}\pi + \tan \frac{y_k}{2m_k}\pi}{\cot \frac{\pi}{2m_k}} \geq 0. \end{aligned}$$

If m_k and y_k are both odd numbers with $y_k \neq 1$, then we have

$$\alpha_k(y_k) = 1 - \frac{\tan\frac{y_k-1}{2m_k}\pi + \cot\frac{y_k}{2m_k}\pi}{\cot\frac{\pi}{2m_k}} = 1 - \frac{\sin\frac{\pi}{2m_k}}{\cos(\frac{y_k-1}{2m_k}\pi)\sin(\frac{y_k}{2m_k}\pi)}.$$

But, from

$$\sin\frac{\pi}{2m_k} = \sin(\frac{y_k}{2m_k}\pi)\cos(\frac{y_k-1}{2m_k}\pi) - \sin(\frac{y_k-1}{2m_k}\pi)\cos(\frac{y_k}{2m_k}\pi),$$

it follows that

$$\sin\frac{\pi}{2m_k} \le \sin(\frac{y_k}{2m_k}\pi)\cos(\frac{y_k-1}{2m_k}\pi).$$

Now, if m_k is odd and $y_k = 0$ or $y_k = 1$, we get $\alpha_k(0) = \alpha_k(1) = 0$, which implies that $\alpha_k(y_k) \ge 0$.

The L^1-norm of the V-conjugate function is therefore given by

$$\|\tilde{f}_n\|_1 = \sum_{y_0=0}^{m_0-1}\sum_{y_1=0}^{m_1-1}\cdots\sum_{y_n=0}^{m_n-1}\int_{I_{n+1}(y_0,\dots,y_n)}|\tilde{f}_n(y)|dy$$

$$= \frac{1}{M_{n+1}}\sum_{y_0=0}^{m_0-1}\sum_{y_1=0}^{m_1-1}\cdots\sum_{y_n=0}^{m_n-1}\prod_{k=1}^{n}\alpha_k(y_k) = \frac{m_0}{M_{n+1}}\prod_{k=1}^{n}F_k,$$

where $F_k = \sum\limits_{y_k=0}^{m_k-1}\alpha_k(y_k)$.

For m_k odd we have

$$F_k = \sum_{y_k=0}^{m_k-1}(1 + \frac{L_k((y_k-1)e_k) - L_k(y_ke_k)}{L_k(e_k)})$$

$$= m_k + \frac{1}{L_k(e_k)}\sum_{y_k=0}^{m_k-1}(L_k((y_k-1)e_k) - L_k(y_ke_k))$$

$$= m_k + \frac{1}{L_k(e_k)}(L_k(-e_k) - L_k((m_k-1)e_k)) = m_k.$$

$F_k = m_k$ is also obviously true when m_k is even. Finally we get $\|\tilde{f}_n\|_1 = 1$.

Define the sets $E_k \subset G$ by $E_k = \{y_k = 0\} \cup \{y_k = 1\}$, if m_k is odd and $E_k = \{y : y_k \equiv 1, mod(2)\}$ if m_k is even.

Notice that $\alpha_k(y_k) = 0$ if and only if $y \in E_k$. Define $J_k \subset \{0, \dots, m_k - 1\}$ by $y \in E_k \Leftrightarrow y_k \in J_k$.

Suppose that $y \in E_k \setminus \bigcup\limits_{i=0}^{k-1} E_i$, for some $k \ge 1$. Then $\alpha_k(y_k) = 0$ and $\alpha_i(y_i) > 0$ for every $i \in \{0, \dots, k-1\}$. This means that whenever m_i is odd for $i \le k-1$, we get $f^i(y) = 0$, because $y_i \ne 0, 1$.

Also, if m_i is even then f^i is nonnegative by definition.

Notice that from $\alpha_k(y_k) = 0$, we get that $f^i(y) = 0$ whenever $i \ge k+1$.

Since $f^k(y)$ is either nonnegative or purely imaginary we conclude that

$$|f_n(y)| = |f^k(y) + \sum_{0\le i\le k-1, m_i\ even} f^i(y)| \ge |f^k(y)| \ge \tilde{f}_{k-1}(y) = \prod_{i=1}^{k-1}\alpha_i(y_i).$$

We obtain

$$\int_{E_k \setminus \bigcup_{i=0}^{k-1} E_i} |f_n(y)|dy \geq \int_{E_k \setminus \bigcup_{i=0}^{k-1} E_i} \prod_{i=1}^{k-1} \alpha_i(y_i)$$

$$= \sum_{y_0 \in J_0^c} \cdots \sum_{y_{k-1} \in J_{k-1}^c} \sum_{y_k \in J_k} \int_{I_{k+1}(y_0,\ldots,y_k)} \prod_{i=1}^{k-1} \alpha_i(y_i),$$

$$\geq \frac{2}{M_{k+1}} \sum_{y_0=0}^{m_0-1} \cdots \sum_{y_{k-1}=0}^{m_{k-1}-1} \prod_{i=1}^{k-1} \alpha_i(y_i) = \frac{2m_0}{M_{k+1}} \prod_{i=1}^{k-1} F_i = \frac{2m_0}{m_k},$$

where

$$J_i^c = \{0, \ldots, m_i - 1\} \setminus J_i, i \in \{0, \ldots, k-1\}.$$

From

$$\int |f_n(y)|dy = \sum_{k=1}^{n} \int_{E_k \setminus \bigcup_{i=0}^{k-1} E_i} |f_n(y)|dy \geq \sum_{k=1}^{n} \frac{2m_0}{m_k},$$

we get that $\|f_n\|_1 \to \infty$ as $n \to \infty$. □

Lemma 2.2. *There exists a constant $C > 0$ only depending on the sequence $(m_n)_n$ satisfying*

$$|L_n D_{M_n} * f(x)| \leq C \sup_{s \in \{0,\ldots,m_n-1\}} |S_{M_{n+1}} f(x + se_n) - S_{M_n} f(x)|,$$

for every $f \in L^1(G)$.

Proof. Suppose in the first case that m_n is odd.

$$|L_n D_{M_n} * f(x)| = |\sum_{s=0}^{m_n-1} L_k(se_k) a_s^n(x)| = |\sum_{s=1}^{\frac{m_n-1}{2}} L_k(se_k)(a_s^n(x) - a_{-s}^n(x))|$$

$$\leq C \sup_{s \in \{0,\ldots,m_n-1\}} |S_{M_{n+1}} f(x + se_n) - S_{M_n} f(x)| \sum_{s=1}^{\frac{m_n-1}{2}} |L_k(se_k)|$$

$$= C \sup_{s \in \{0,\ldots,m_n-1\}} |S_{M_{n+1}} f(x + se_n) - S_{M_n} f(x)|.$$

Suppose now that m_n is even. If f is a real function, the imaginary part of $L_n D_{M_n} * f(x)$ is given by

$$Im(L_n D_{M_n} * f)(x) = \sum_{1 \leq s \leq m_n-1, s \text{ odd}} \gamma_s a_s^n(x) = \sum_{1 \leq s \leq \frac{m_n}{2}-1, s \text{ odd}} \gamma_s (a_s^n(x) - a_{-s}^n(x)).$$

Hence,

$$|Im(L_n D_{M_n} * f)(x)| \leq \sup_{s \in \{0,\ldots,m_n-1\}} |S_{M_{n+1}} f(x + se_n) - S_{M_n} f(x)| \sum_{s=1}^{\frac{m_n}{2}-1} |\gamma_s|$$

$$= C \sup_{s \in \{0,\ldots,m_n-1\}} |S_{M_{n+1}} f(x + se_n) - S_{M_n} f(x)|.$$

On the other hand the real part consists of

$$Re(L_n D_{M_n} * f)(x) = \sum_{0\le s\le m_n-2, s\ even} a_s^n(x) - \sum_{1\le s\le m_n-1, s\ odd} a_s^n(x)$$

$$= m_n S_{M_n} - 2\sum_{1\le s\le m_n-1, s\ odd} a_s^n(x) = 2\sum_{1\le s\le m_n-1, s\ odd} (S_{M_n} f(x) - a_s^n(x)).$$

It follows

$$|Re(L_n D_{M_n} * f)(x)| \le C \sup_{s\in\{0,\dots,m_n-1\}} |S_{M_{n+1}} f(x+se_n) - S_{M_n} f(x)|.$$

Then, the result follows by an easy calculation. □

Lemma 2.3. *The norms $\|\tilde{q}(f)\|_1$ and $\|f\|_{H^1}$ are equivalent in H^1.*

Proof. Let $f \in H^1$. It is clear that $q(f) \le \tilde{q}(f)$. Let $x \in G$ be arbitrary.

$$\sup_{s\in\{0,\dots,m_n-1\}} |S_{M_{n+1}} f(x+se_n) - S_{M_n} f(x)|^2$$

$$= \sup_{s\in\{0,\dots,m_n-1\}} |S_{M_{n+1}} f(x+se_n) - S_{M_n} f(x+se_n)|^2$$

$$= \sup_{s\in\{0,\dots,m_n-1\}} |\sum_{j=0}^{m_n-1} (S_{(j+1)M_n} f(x+se_n) - S_{jM_n} f(x+se_n))|^2$$

$$\le C \sup_{s\in\{0,\dots,m_n-1\}} \sum_{j=0}^{m_n-1} |S_{(j+1)M_n} f(x+se_n) - S_{jM_n} f(x+se_n)|^2$$

$$= C \sup_{s\in\{0,\dots,m_n-1\}} \sum_{j=0}^{m_n-1} |r_n^j(x+se_n) M_n \int_{I_n(x+se_n)} f(t)\bar{r}_n^j(t)dt|^2$$

$$= C \sup_{s\in\{0,\dots,m_n-1\}} \sum_{j=0}^{m_n-1} |r_n^j(x) M_n \int_{I_n(x)} f(t)\bar{r}_n^j(t)dt|^2$$

$$= C \sup_{s\in\{0,\dots,m_n-1\}} \sum_{j=0}^{m_n-1} |S_{(j+1)M_n} f(x) - S_{jM_n} f(x)|^2.$$

It follows that $\tilde{q}(f) \le CQ(f)$, and the result follows by applying Theorem 3 from [**1**].

□

Theorem 2.4. *The V-conjugation operator is bounded on H^1. Namely, if $f \in H^1$, then $\tilde{f} \in H^1$, and there exists a constant $C > 0$ only depending on the sequence $(m_n)_n$ such that $\|\tilde{f}\|_{H^1} \le C\|f\|_{H^1}$.*

Proof. Let $f \in H^1$. Notice that $S_{M_n}\tilde{f} = \sum_{i=0}^{n-1} L_i D_{M_i} * f$. Therefore, by Lemma 2.2 we get

$$\begin{aligned}|S_{M_{n+1}}\tilde{f}(x) - S_{M_n}\tilde{f}(x)| &= |(L_n D_{M_n} * f)(x)| \\ &\le C \sup_{s\in\{0,\ldots,m_n-1\}} |S_{M_{n+1}} f(x+se_n) - S_{M_n} f(x)|,\end{aligned}$$

for some constant $C > 0$ only depending on the sequence $(m_n)_n$.

Then, Lemma 2.3 gives

$$\|\tilde{f}\|_{H^1} \le C\|\tilde{q}(f)\|_1 \sim \|f\|_{H^1}.$$

□

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