

## A PRODUCT KERNEL ASSOCIATED TO AXIALLY SYMMETRIC POTENTIALS

MARTHA GUZMÁN-PARTIDA\* AND CARLOS ROBLES-CORBALA

(Received 8 January, 2016)

**Abstract.** We characterize a family of weighted integrable distributions convolvable with a natural product version of the kernel  $\mathcal{K}_\alpha$  associated to generalized axially symmetric potentials.

### 1. Introduction

In this note, we consider a product version of the family of kernels  $\mathcal{K}_\alpha$ ,  $\alpha > -1$ , studied by J. Wittsten in [9], namely

$$\mathcal{K}_\alpha(x, t) := \frac{\Gamma((\alpha + n + 1)/2)}{\Gamma((\alpha + 1)/2) \pi^{n/2}} \frac{t^{\alpha+1}}{(|x|^2 + t^2)^{(\alpha+n+1)/2}}, \quad (1)$$

for  $(x, t) \in \mathbb{R}_+^{n+1}$ .

This kernel is related to the elliptic partial differential equation

$$D_\alpha u := t^{-\alpha} \left( \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_n^2} + \frac{\partial^2 u}{\partial t^2} - \frac{\alpha}{t} \frac{\partial u}{\partial t} \right) = 0, \quad (2)$$

with  $\alpha > -1$ . Solutions to (2) are called generalized axially symmetric potentials (see [8]). Notice that when  $\alpha = 0$  we recover the Laplace equation.

In the paper [9], the author proved that  $\|\mathcal{K}_{\alpha,t}\|_1 = 1$ , where  $\mathcal{K}_{\alpha,t}(x) := \mathcal{K}_\alpha(x, t)$ , that  $\mathcal{K}_{\alpha,t} \rightarrow \delta_0$  in  $\mathcal{S}'$  as  $t \rightarrow 0^+$  and,  $\mathcal{K}_\alpha$  is a solution to the equation (2) in  $\mathbb{R}_+^{n+1}$ . In fact, in that article, generalizing techniques developed in [1] and [2], is determined a certain class of weighted integrable distributions  $f$  convolvable in an appropriate sense with the kernel  $\mathcal{K}_\alpha$ , solving in this way the Dirichlet problem

$$\begin{cases} D_\alpha u = 0 & \text{in } \mathbb{R}_+^{n+1} \\ u = f & \text{in } \mathbb{R}^n. \end{cases}$$

Using some of the results obtained in [9] and [1] we obtain the main result of this article: to characterize the class of tempered distributions convolvable with a product domain version of  $\mathcal{K}_\alpha$  for the domain  $\mathbb{R}_+^2 \times \dots \times \mathbb{R}_+^2$  consisting of  $n$  copies of the upper half-plane  $\mathbb{R}_+^2$ . The notion of convolution that we appeal to is the so called  $\mathcal{S}'$ -convolution developed by Y. Hirata and H. Ogata in [4] and R. Shiraishi in [6] to extend to appropriate pairs of tempered distributions the classical definition of convolution. The reason for this choice of convolution is the same that in the

---

2010 *Mathematics Subject Classification* 46F20, 35J25.

*Key words and phrases:* Generalized axially symmetric potentials, weighted spaces of distributions.

\* Corresponding author.

case of the classical Poisson kernel for the upper-half space, which is the fact that the Fourier transform of  $\mathcal{K}_{\alpha,t}$  is given by (see [9])

$$\widehat{\mathcal{K}_{\alpha,t}}(\xi) = \frac{t|\xi|^{\alpha+1}}{\Gamma(\alpha+1)} \int_1^\infty e^{-t|\xi|s} (s^2 - 1)^{\alpha/2} ds,$$

which fails to be smooth at  $\xi = 0$ .

The paper is organized as follows: in the second section we remind and state some definitions and results needed for our exposition. In the third section we obtain necessary and sufficient conditions on the class of tempered distributions convolvable with our product domain version of the kernel  $\mathcal{K}_\alpha$ . Finally, in the fourth section we obtain distributional boundary values for generalized axially symmetric potentials for the product domain case.

We use standard notation for the spaces of functions and distributions to be used. As usual, we will denote by the same letter  $C$  a constant that could be changing line by line.

## 2. $\mathcal{S}'$ -convolution and the Product Kernel

To introduce the notion of  $\mathcal{S}'$ -convolution that we will use, we give a brief account of the spaces of functions and distributions required. All of this can be found in [5] and [3].

The space of integrable distributions  $\mathcal{D}'_{L^1}$  is, by definition, the strong dual of the space  $\dot{B}$  of smooth functions  $\varphi : \mathbb{R}^n \rightarrow \mathbb{C}$  for which  $\partial^\alpha \varphi \rightarrow 0$  as  $|x| \rightarrow \infty$ , for each multi-index  $\alpha$ . The space  $\dot{B}$  is a closed subspace of  $B$ , the set of smooth functions  $\varphi$  such that  $\partial^\alpha \varphi$  is bounded for every multi-index  $\alpha$ , endowed with the topology of the uniform convergence on  $\mathbb{R}^n$  of each derivative. The space  $C_0^\infty$  of compactly supported smooth functions defined on  $\mathbb{R}^n$ , is dense in  $\dot{B}$  but not in  $B$ . Thus,  $\mathcal{D}'_{L^1}$  is a subspace of  $\mathcal{D}'$ , the space of distributions. Moreover, each integrable distribution is uniquely determined by its restriction to  $C_0^\infty$ . The space  $\mathcal{E}'$  of compactly supported distributions is a subspace of  $\mathcal{D}'_{L^1}$ . Furthermore, according to [5], each  $T \in \mathcal{D}'_{L^1}$  can be represented as

$$T = \sum_{finite} \partial^\alpha f_\alpha, \quad (3)$$

where  $f_\alpha \in L^1$ . Consequently, we have the inclusions  $\mathcal{E}' \subset \mathcal{D}'_{L^1} \subset \mathcal{S}'$ . The space  $\mathcal{D}'_{L^1}$  is closed under multiplication by functions in  $B$ , a fact that we will use repeatedly.

It is possible to consider  $\mathcal{D}'_{L^1}$  as the strong dual of the space  $B$ , provided that we endow  $B$  with a topology that gives rise to the following notion of sequence convergence: A sequence  $\{\varphi_j\}$  converges to  $\varphi$  if, for each multi-index  $\alpha$ ,  $\sup_j \|\partial^\alpha \varphi_j\|_\infty < \infty$  and the sequence  $\{\partial^\alpha \varphi_j\}$  converges to  $\partial^\alpha \varphi$  uniformly on compact sets. If we denote as  $B_c$  the resulting topological space, it can be proved that  $C_0^\infty$ , and so  $\dot{B}$ , is dense in  $B_c$  ([5]). Since every distribution  $T \in \mathcal{D}'_{L^1}$  is well defined on  $C_0^\infty$  and it is continuous on  $C_0^\infty$  with respect to the topology of  $B_c$ , it turns out that  $\mathcal{D}'_{L^1}$  is also the dual of  $B_c$  ([5]).

Y. Hirata and H. Ogata ([4]) introduced the  $\mathcal{S}'$ -convolution to extend the validity of the Fourier exchange formula  $\widehat{T * S} = \widehat{T} \widehat{S}$ , originally proved by L. Schwartz ([5]) for pairs of distributions in the Cartesian product  $\mathcal{O}'_c \times \mathcal{S}'$ . Later, R. Shiraishi

proved in [6] an equivalent definition of  $\mathcal{S}'$ -convolution, which is the one we shall use.

**Definition 2.1.** ([6]) *Given two tempered distributions  $T$  and  $S$ , it is said that the  $\mathcal{S}'$ -convolution of  $T$  and  $S$  exists if  $T(\check{S} * \varphi) \in \mathcal{D}'_{L^1}$  for every  $\varphi \in \mathcal{S}$ . When the  $\mathcal{S}'$ -convolution exists, the map*

$$\begin{aligned} \mathcal{S} &\rightarrow \mathbb{C} \\ \varphi &\rightarrow (T(\check{S} * \varphi), 1)_{\mathcal{D}'_{L^1}, B_c} \end{aligned}$$

*is linear and continuous. Thus, it defines a tempered distribution which will be denoted by  $T * S$ .*

Here,  $T(\check{S} * \varphi)$  is the multiplicative product of the distribution  $T$  with the convolution  $\check{S} * \varphi$ . This convolution is a smooth function and furthermore, the function and each derivative has at most polynomial growth at infinity. R. Shiraishi proved in [6] that  $T * S$  exists if and only if  $S * T$  exists, and they coincide. It is important to remark the fact that Definition 2.1 coincides with the classical definition in all the cases in which the latter makes sense.

Associated to the kernel  $\mathcal{K}_{\alpha,t}(x)$  given by (1) there is a class of weighted distributions optimal for  $\mathcal{S}'$ -convolution. We denote for  $x \in \mathbb{R}^n$ ,  $w(x) = (1 + |x|^2)^{1/2}$ , and  $\mu \in \mathbb{R}$

$$w^\mu \mathcal{D}'_{L^1} = \{T \in \mathcal{S}' : w^{-\mu} T \in \mathcal{D}'_{L^1}\}$$

with the topology induced by the map

$$\begin{aligned} w^\mu \mathcal{D}'_{L^1} &\rightarrow \mathcal{D}'_{L^1} \\ T &\rightarrow w^{-\mu} T. \end{aligned}$$

J. Wittsten proved the following result.

**Theorem 2.2.** ([9]) *Let  $\alpha > -1$  and  $T \in \mathcal{S}'$ . The following statements are equivalent:*

- (a):  $T \in w^{n+\alpha+1} \mathcal{D}'_{L^1}$ .
- (b):  $T$  is  $\mathcal{S}'$ -convolvable with the kernel  $\mathcal{K}_{\alpha,t}$  for each  $t > 0$ .

The product domain version of the kernel  $\mathcal{K}_\alpha$  will be defined as follows:

$$\bar{\mathcal{K}}_\alpha(x, t) = \prod_{j=1}^n \mathcal{K}_{\alpha_j}(x_j, t_j) \quad (4)$$

where  $(x, t) = ((x_1, t_1), \dots, (x_n, t_n)) \in \mathbb{R}_+^2 \times \dots \times \mathbb{R}_+^2$  the cartesian product of  $n$  copies of the upper half-plane and  $\alpha = (\alpha_1, \dots, \alpha_n)$  with  $\alpha_j > -1$ , for  $j = 1, \dots, n$ .

The kernel given by (4) is a generalization of the product domain version of the Poisson kernel studied in [1]. This product domain version of the Poisson kernel is given by

$$\mathcal{P}_{(t)}(x) = \prod_{j=1}^n P_{t_j}(x_j), \quad (5)$$

where  $(x, t)$  has the same meaning as above and  $P_{t_j}(x_j)$  is the classical Poisson kernel for the upper half-plane. The kernel (5) is closely related to families of  $n$ -harmonic functions on the product domain  $\mathbb{R}_+^2 \times \dots \times \mathbb{R}_+^2$ .

In the same paper [1], the authors consider the following spaces of weighted distributions.

**Definition 2.3.** ([1]) Let  $w_j = (1 + x_j^2)^{1/2}$ ,  $j = 1, \dots, n$  and let  $\mu_1, \dots, \mu_n \in \mathbb{R}$ . Then

$$w_1^{\mu_1} \dots w_n^{\mu_n} \mathcal{D}'_{L^1} = \{T \in \mathcal{S}' : w_1^{-\mu_1} \dots w_n^{-\mu_n} T \in \mathcal{D}'_{L^1}\}$$

with the topology induced by the map

$$\begin{aligned} w_1^{\mu_1} \dots w_n^{\mu_n} \mathcal{D}'_{L^1} &\rightarrow \mathcal{D}'_{L^1} \\ T &\rightarrow w_1^{-\mu_1} \dots w_n^{-\mu_n} T. \end{aligned}$$

Moreover, they also prove:

**Theorem 2.4.** ([1]) Given  $T \in \mathcal{S}'$ , the following statements are equivalent:

- (a):  $T \in w_1^2 \dots w_n^2 \mathcal{D}'_{L^1}$ .
- (b):  $T$  is  $\mathcal{S}'$ -convolvable with  $\mathcal{P}_{(t)}$  for each  $(t) > (0)$ .

Here,  $(t) > (0)$  means that  $t_1 > 0, \dots, t_n > 0$ .

In the next section we will determine necessary and sufficient conditions on tempered distributions in order to be  $\mathcal{S}'$ -convolvable with (4).

### 3. Optimal Spaces for $\mathcal{S}'$ -convolution with $\bar{\mathcal{K}}_\alpha$

We will denote by  $\bar{\mathcal{K}}_{\alpha,t}(x) := \bar{\mathcal{K}}_\alpha(x, t)$ . It is reasonable to expect that the appropriate space of weighted integrable distributions  $\mathcal{S}'$ -convolvable with the kernel  $\bar{\mathcal{K}}_{\alpha,t}$  involves products of weights  $w_j^{\alpha_j+2}$ . As we will show, this space will be  $w_1^{2+\alpha_1} \dots w_n^{2+\alpha_n} \mathcal{D}'_{L^1}$ . The first task to do is to give a characterization of this space.

**Proposition 3.1.** Given  $T \in \mathcal{S}'$ , and  $\mu_1, \dots, \mu_n \in \mathbb{R}$ , the following statements are equivalent:

- (a):  $T \in w_1^{\mu_1} \dots w_n^{\mu_n} \mathcal{D}'_{L^1}$ .
- (b):  $T = T_0 + \sum x_{j_1}^{\mu_{j_1}} \dots x_{j_k}^{\mu_{j_k}} T_{j_1 \dots j_k}$ , where  $T_0 \in \mathcal{E}'$ ,  $T_{j_1 \dots j_k} \in \mathcal{D}'_{L^1}$ , and the sum is taken over all the different  $k$ -tuples  $(j_1, \dots, j_k)$  with  $1 \leq j_1 < \dots < j_k \leq n$ ,  $1 \leq k \leq n$ .

**Proof.** It is clear that (b) implies (a).

For the converse, we employ the same proof given in [1], Proposition 14, which we write for the sake of completeness. Let  $\theta \in C_0^\infty(\mathbb{R})$ ,  $0 \leq \theta \leq 1$ ,  $\theta(x) = 1$  for  $|x| < 1$  and  $\theta(x) = 0$  for  $|x| > 2$ . Writing  $\theta_j = \theta(x_j)$  we have that

$$\begin{aligned} 1 &= (\theta_1 + (1 - \theta_1)) \dots (\theta_n + (1 - \theta_n)) \\ &= \theta_1 \dots \theta_n + \sum (1 - \theta_{i_1}) \dots (1 - \theta_{i_k}) \theta_{j_1} \dots \theta_{j_{n-k}}, \end{aligned}$$

where the sum collects all the possible cross-products containing at least one factor of the form  $1 - \theta_{i_l}$  without repetition. Hence, we can write

$$T = \theta_1 \dots \theta_n T + \sum x_{i_1}^{\mu_{i_1}} \dots x_{i_k}^{\mu_{i_k}} \frac{1 - \theta_{i_1}}{x_{i_1}^{\mu_{i_1}}} \dots \frac{1 - \theta_{i_k}}{x_{i_k}^{\mu_{i_k}}} w_1^{\mu_1} \dots w_n^{\mu_n} \theta_{j_1} \dots \theta_{j_{n-k}} w_1^{-\mu_1} \dots w_n^{-\mu_n} T.$$

Now, we notice that  $\theta_1 \dots \theta_n T \in \mathcal{E}'$ ,  $w_1^{-\mu_1} \dots w_n^{-\mu_n} T \in \mathcal{D}'_{L^1}$  and the functions

$$\frac{1 - \theta_{i_1}}{x_{i_1}^{\mu_{i_1}}} \dots \frac{1 - \theta_{i_k}}{x_{i_k}^{\mu_{i_k}}} w_{i_1}^{\mu_{i_1}} \dots w_{i_k}^{\mu_{i_k}}, w_{j_1}^{\mu_{j_1}} \dots w_{j_k}^{\mu_{j_k}} \theta_{j_1} \dots \theta_{j_{n-k}}$$

belong to  $B$ . □

Using this proposition we can obtain the following characterization.

**Theorem 3.2.** *Given  $T \in \mathcal{S}'$ , and  $\alpha_1, \dots, \alpha_n > -1$ , the following statements are equivalent:*

- (a):  $T \in w_1^{\alpha_1+2} \dots w_n^{\alpha_n+2} \mathcal{D}'_{L^1}$ .
- (b):  $T$  is  $\mathcal{S}'$ -convolvable with the kernel  $\bar{\mathcal{K}}_{\alpha,t}$  for every  $(t) > 0$ .

**Proof.** Suppose that  $w_1^{-(\alpha_1+2)} \dots w_n^{-(\alpha_n+2)} T \in \mathcal{D}'_{L^1}$  and let  $\varphi \in \mathcal{S}$ . According to Definition 2.1 it suffices to show that  $w_1^{\alpha_1+2} \dots w_n^{\alpha_n+2} (\bar{\mathcal{K}}_{\alpha,t} * \varphi) \in B$ . By Proposition 7 in [1] and Theorem 3.3 in [9] we can obtain

$$\begin{aligned} |\partial^\beta (\bar{\mathcal{K}}_{\alpha,t} * \varphi)(x)| &\leq \frac{C_{\alpha,n}}{t_1 \dots t_n} \prod_{j=1}^n \left(1 + \frac{x_j^2}{t_j^2}\right)^{-(\alpha_j+2)/2} \\ &\quad \times \left( \|\partial^\beta \varphi\|_{L^1} + \sum \frac{1}{t_{i_1}^{\alpha_{i_1}+2} \dots t_{i_k}^{\alpha_{i_k}+2}} \|y_{i_1}^{\alpha_{i_1}+2} \dots y_{i_k}^{\alpha_{i_k}+2} \varphi\|_{L^1} \right), \end{aligned} \quad (6)$$

where the sum is taken over all the possible  $k$ -tuples  $(i_1, \dots, i_k)$  with  $1 \leq i_1 < \dots < i_k \leq n$ ,  $1 \leq k \leq n$ .

The estimate (6) implies that  $w_1^{\alpha_1+2} \dots w_n^{\alpha_n+2} (\bar{\mathcal{K}}_{\alpha,t} * \varphi) \in B$ .

Conversely, by assumption,  $T(\bar{\mathcal{K}}_{\alpha,t} * \varphi) \in \mathcal{D}'_{L^1}$  for each  $\varphi \in \mathcal{S}$ . Now, if we choose  $\varphi \in \mathcal{S}$  such that  $\varphi(x) = \prod_{j=1}^n \varphi_j(x_j)$ , where  $\varphi_j \in \mathcal{S}(\mathbb{R})$ ,  $j = 1, \dots, n$ , hence

$$\bar{\mathcal{K}}_{\alpha,t} * \varphi = \prod_{j=1}^n \mathcal{K}_{\alpha_j, t_j} * \varphi_j.$$

With the notation employed in Proposition 3.1 we can write

$$T = \theta_1 \dots \theta_n T + \sum (1 - \theta_{i_1}) \dots (1 - \theta_{i_k}) \theta_{j_1} \dots \theta_{j_{n-k}} T$$

and  $\theta_1 \dots \theta_n T \in \mathcal{E}' \subset \mathcal{D}'_{L^1}$ . Now, notice that

$$\begin{aligned} &(1 - \theta_{i_1}) \dots (1 - \theta_{i_k}) \theta_{j_1} \dots \theta_{j_{n-k}} T \\ &= x_{i_1}^{\alpha_{i_1}+2} \dots x_{i_k}^{\alpha_{i_k}+2} \frac{1 - \theta_{i_1}}{x_{i_1}^{\alpha_{i_1}+2} (\mathcal{K}_{\alpha_{i_1}, t_{i_1}} * \psi_{i_1})} \dots \frac{1 - \theta_{i_k}}{x_{i_k}^{\alpha_{i_k}+2} (\mathcal{K}_{\alpha_{i_k}, t_{i_k}} * \psi_{i_k})} \\ &\quad \times \frac{\theta_{j_1}}{\mathcal{K}_{\alpha_{j_1}, t_{j_1}} * \psi_{j_1}} \dots \frac{\theta_{j_{n-k}}}{\mathcal{K}_{\alpha_{j_{n-k}}, t_{j_{n-k}}} * \psi_{j_{n-k}}} T (\bar{\mathcal{K}}_{\alpha,t} * \varphi), \end{aligned} \quad (7)$$

where  $\psi_s \in \mathcal{S}(\mathbb{R})$  for  $s = 1, \dots, n$  will be selected later and  $\varphi = \psi_1 \dots \psi_n$ .

As in Theorem 15 in [1] we can pick  $\psi = \psi_1 = \dots = \psi_n \in C_0^\infty(\mathbb{R})$  such that  $\psi = 0$  for  $|u| \geq 1/3$ ,  $\psi > 0$  for  $|u| < 1/3$  and therefore

$$(\mathcal{K}_{\alpha_i, t_i} * \psi)(x_i) \geq C_{\alpha_i} \frac{t_i^{\alpha_i+1}}{(x_i^2 + t_i^2)^{(\alpha_i+2)/2}} \|\psi\|_{L^1} \quad \text{for } |x_i| > 1/3 \quad (8)$$

and

$$(\mathcal{K}_{\alpha_i, t_i} * \psi)(x_i) \geq C \frac{t_i^{\alpha_i+1}}{(1 + t_i^2)^{(\alpha_i+2)/2}} \|\psi\|_{L^1} \quad \text{for } |x_i| < 1. \quad (9)$$

Now, (8) and (9) show that each quotient in (7) belongs to  $B$ , thus the whole expression in (7) belongs to  $\mathcal{D}'_{L^1}$ . An appealing to Proposition 3.1 concludes the proof.  $\square$

#### 4. Distributional Boundary Values

For future reference, we state a couple of remarks that will be very useful later. The proofs of them can be obtained using similar estimates to those done above, as well as the analogous results obtained in [1] and [9].

**Remark 4.1.** For  $T \in w_1^{\alpha_1+2} \dots w_n^{\alpha_n+2} \mathcal{D}'_{L^1}$  the  $\mathcal{S}'$ -convolution with the kernel  $\bar{\mathcal{K}}_{\alpha,t}$  is the function whose value at  $x$  is given by

$$\left( \left( \prod_{i=1}^n w_i^{-(\alpha_i+2)}(y_i) \right) T_y, \prod_{j=1}^n w_j^{\alpha_j+2}(y_j) \mathcal{K}_{\alpha_j,t_j}(x_j - y_j) \right)_{\mathcal{D}'_{L^1}, B_c}.$$

**Remark 4.2.** We have the representation

$$w_1^{\alpha_1+2} \dots w_n^{\alpha_n+2} \mathcal{D}'_{L^1} = \left\{ T \in \mathcal{S}' : T = \sum_{finite} \partial^\beta f_\beta, f_\beta \in L^1 \left( w_1^{-(\alpha_1+2)} \dots w_n^{-(\alpha_n+2)} \right) \right\}.$$

As usual, we denote by  $w_1^{\alpha_1+2} \dots w_n^{\alpha_n+2} \mathcal{D}_{L^1}$  the space of smooth functions  $f$  such that  $w_1^{-(\alpha_1+2)} \dots w_n^{-(\alpha_n+2)} f$  and all of its derivatives belong to  $L^1(\mathbb{R}^n)$ . We are ready to prove the following Proposition.

**Proposition 4.3.** Let  $T \in \mathcal{S}'$  and  $\alpha = (\alpha_1, \dots, \alpha_n)$  with  $\alpha_i > -1$  for  $i = 1, \dots, n$ .

- (a): The space  $w_1^{\alpha_1+2} \dots w_n^{\alpha_n+2} \mathcal{D}'_{L^1}$  is closed under differentiation. Furthermore, if  $T \in w_1^{\alpha_1+2} \dots w_n^{\alpha_n+2} \mathcal{D}'_{L^1}$  then  $\partial^\beta (T * \bar{\mathcal{K}}_{\alpha,t}) = (\partial^\beta T) * \bar{\mathcal{K}}_{\alpha,t}$  for each multi-index  $\beta$ .
- (b): For  $T \in w_1^{\alpha_1+2} \dots w_n^{\alpha_n+2} \mathcal{D}'_{L^1}$  the  $\mathcal{S}'$ -convolution  $T * \bar{\mathcal{K}}_{\alpha,t}$  belongs to the space  $w_1^{\alpha_1+2} \dots w_n^{\alpha_n+2} \mathcal{D}_{L^1}$ .

**Proof.** (a) The first assertion is obtained immediatly from Remark 4.2. The second one can be proved using general properties of  $\mathcal{S}'$ -convolution as in [9], Proposition 3.4.

(b) By assertion (a), it suffices to prove that  $T * \bar{\mathcal{K}}_{\alpha,t} \in L^1 \left( w_1^{-(\alpha_1+2)} \dots w_n^{-(\alpha_n+2)} \right)$ .

We may assume also that  $w_1^{-(\alpha_1+2)} \dots w_n^{-(\alpha_n+2)} T = \partial^\beta f$  for some  $f \in L^1(\mathbb{R}^n)$ . By Remark 4.1 we have

$$\begin{aligned} T * \bar{\mathcal{K}}_{\alpha,t}(x) &= \left( \left( \prod_{i=1}^n w_i^{-(\alpha_i+2)}(y_i) \right) T_y, \prod_{j=1}^n w_j^{\alpha_j+2}(y_j) \mathcal{K}_{\alpha_j,t_j}(x_j - y_j) \right)_{\mathcal{D}'_{L^1}, B_c} \\ &= \sum_{\gamma \leq \beta} (-1)^{|\beta|} C_{\beta,\gamma} \int_{\mathbb{R}^n} f(y) \prod_{j=1}^n \partial_{y_j}^{\gamma_j} w_j^{\alpha_j+2}(y_j) \partial_{y_j}^{\beta_j - \gamma_j} \mathcal{K}_{\alpha_j,t_j}(x_j - y_j) dy. \end{aligned}$$

Now, notice that for  $j = 1, \dots, n$

$$\left| \partial^{\gamma_j} w_j^{(\alpha_j+2)}(y_j) \right| \left| \partial_{y_j}^{\beta_j-\gamma_j} \mathcal{K}_{\alpha_j, t_j}(x_j - y_j) \right| \leq \frac{C_{\gamma_j, \alpha_j} (1 + y_j^2)^{(\alpha_j+2-\gamma_j)/2}}{t_j^{1+\beta_j-\gamma_j} \left( 1 + \frac{(x_j - y_j)^2}{t_j^2} \right)^{(\alpha_j+2+(\beta_j-\gamma_j))/2}}.$$

Collecting previous information we get

$$\begin{aligned} & \int_{\mathbb{R}^n} |T * \bar{\mathcal{K}}_{\alpha, t}(x)| w_1^{-(\alpha_1+2)}(x_1) \dots w_n^{-(\alpha_n+2)}(x_n) dx \\ & \leq \sum_{\gamma \leq \beta} \frac{C_{\beta, \gamma, \alpha}}{\prod_{1 \leq j \leq n} t_j^{1+\beta_j-\gamma_j}} I_{\beta \gamma \alpha}, \end{aligned}$$

with

$$\begin{aligned} I_{\beta \gamma \alpha} &= \int_{\mathbb{R}^n} |f(y)| \left[ \prod_{j=1}^n \int_{-\infty}^{\infty} w_j^{-(\alpha_j+2)}(x_j) \left( 1 + \frac{(x_j - y_j)^2}{t_j^2} \right)^{-(\alpha_j+2+(\beta_j-\gamma_j))/2} dx_j \right] \\ & \times \prod_{j=1}^n (1 + y_j^2)^{(\alpha_j+2-\gamma_j)/2} dy. \end{aligned}$$

Applying Lema 2.8 in [2] we obtain that

$$\begin{aligned} I_{\beta \gamma \alpha} &\leq C_{\beta, \gamma, \alpha, y} \int_{\mathbb{R}^n} |f(y)| \prod_{j=1}^n (1 + |y_j|)^{-(\alpha_j+2)} \left( (1 + |y_j|)^{\alpha_j+2-\gamma_j} \right) dy \\ &< \infty \end{aligned}$$

since  $f \in L^1(\mathbb{R}^n)$ . This conclude the proof.  $\square$

**Remark 4.4.** The proof of Proposition 4.3 shows that the  $\mathcal{S}'$ -convolution with the kernel  $\bar{\mathcal{K}}_{\alpha, t}$  preserves the space  $L^1(w_1^{-(\alpha_1+2)} \dots w_n^{-(\alpha_n+2)})$ .

For  $(t) > (0)$  we will write  $(t) \rightarrow (0)^+$  if  $t_j \rightarrow 0^+$  for each  $j = 1, \dots, n$ . Finally, we prove the following result which guarantees that boundary values are obtained in the desired topology.

**Theorem 4.5.** Given  $T \in w_1^{\alpha_1+2} \dots w_n^{\alpha_n+2} \mathcal{D}'_{L^1}$ , the  $\mathcal{S}'$ -convolution  $T * \bar{\mathcal{K}}_{\alpha, t}$  converges to  $T$  in  $w_1^{\alpha_1+2} \dots w_n^{\alpha_n+2} \mathcal{D}'_{L^1}$  as  $(t) \rightarrow (0)^+$ .

**Proof.** According to [2], Theorem 3.6 or [9], Theorem 4.3, it suffices to show that if  $T = f \in L^1(w_1^{-(\alpha_1+2)} \dots w_n^{-(\alpha_n+2)})$  then

$$f * \bar{\mathcal{K}}_{\alpha, t} \rightarrow f \text{ in } L^1(w_1^{-(\alpha_1+2)} \dots w_n^{-(\alpha_n+2)}) \text{ as } (t) \rightarrow (0)^+.$$

We can suppose without loss of generality that  $f(x) = f_1(x_1) \dots f_n(x_n)$  with  $f_j \in L(w_j^{-(\alpha_j+2)})$ ,  $j = 1, \dots, n$ . This can be done because  $L^1(w_1^{-(\alpha_1+2)}) \otimes \dots \otimes L^1(w_n^{-(\alpha_n+2)})$  is dense in  $L^1(w_1^{-(\alpha_1+2)} \dots w_n^{-(\alpha_n+2)})$  (see [7], p. 476).

Applying Theorem 4.3 in [9] for  $n = 1$  we obtain for  $j = 1, \dots, n$

$$f_j * \mathcal{K}_{\alpha_j, t_j} \rightarrow f_j \text{ in } L^1(w_j^{-(\alpha_j+2)})$$

when  $t_j \rightarrow 0^+$  and therefore

$$f * \overline{\mathcal{K}}_{\alpha,t} \rightarrow f$$

in  $L^1 \left( w_1^{-(\alpha_1+2)} \dots w_n^{-(\alpha_n+2)} \right)$  as  $(t) \rightarrow (0)^+$ . □

### References

- [1] J. Alvarez, M. Guzmán-Partida and U. Skórnik,  *$\mathcal{S}'$ -convolvability with the Poisson kernel in the Euclidean case and the product domain case*, Studia Math., **156** (2003), pp. 143-163.
- [2] J. Alvarez, M. Guzmán-Partida and S. Pérez-Esteva, *Harmonic extensions of distributions*, Math. Nachr., **280** (2007), pp. 1443-1466.
- [3] J. Barros-Neto, *An Introduction to the Theory of Distributions*, Marcel Dekker, 1973.
- [4] Y. Hirata and H. Ogata, *On the exchange formula for distributions*, J. Sci. Hiroshima Univ. Ser. A, **22** (1958), pp. 147-152.
- [5] L. Schwartz, *Théorie des Distributions*, Hermann, 1966.
- [6] R. Shiraishi, *On the definition of convolutions for distributions*, J. Sci. Hiroshima Univ. Ser. A, **23** (1959), pp. 19-32.
- [7] F. Trèves, *Topological Vector Spaces, Distributions and Kernels*, Dover, 2006.
- [8] A. Weinstein, *Generalized axially symmetric potential theory*, Bull. Amer. Math. Soc., **59** (1953), pp. 20-38.
- [9] J. Wittsten, *Generalized axially symmetric potentials with distributional boundary values*, Bull. Sci. Math., **139** (2015), pp. 892-922.

Martha Guzmán-Partida  
Departamento de Matemáticas  
Universidad de Sonora  
Hermosillo, Sonora, 83000, México  
martha@mat.uson.mx

Carlos Robles-Corbala  
Departamento de Matemáticas  
Universidad de Sonora  
Hermosillo, Sonora, 83000, México  
crobles@mat.uson.mx