

## THE TWO VERSIONS OF THE DIRICHLET PROBLEM FOR THE HEAT EQUATION

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**Abstract.** There are two versions of the Dirichlet problem for the heat equation on an arbitrary open set in Euclidean space. For one of them, there is already a characterization of resolitivity in terms of caloric measure. We prove that there is a similar characterization for the other, that the measure involved is essentially the same caloric measure, and that a boundary function is resolutive with respect to one version of the problem if and only if it is resolutive with respect to the other. We also prove that, for any boundary function, the upper solutions for the two versions coincide.

### 1. Introduction

Let  $E$  be an arbitrary open subset of  $\mathbb{R}^{n+1}$ . We take the boundary  $\partial E$  of  $E$  relative to the one-point compactification of  $\mathbb{R}^{n+1}$ . Thus  $\partial E$  contains the point at infinity if and only if  $E$  is unbounded. There are two distinct versions of the Dirichlet problem for the heat equation on  $E$ . In the first, the problem is formulated in exactly the same way as for Laplace's equation. That is, a continuous, real-valued function  $f$  on  $\partial E$  is given, and a temperature  $u$  on  $E$  such that  $\lim_{p \rightarrow q} u(p) = f(q)$  for all  $q \in \partial E$  is sought. This version has the advantage that it can be treated in an axiomatic setting that includes both parabolic and elliptic equations, as in [2, 4]. On the other hand, it takes no account of the fact that the temporal variable behaves differently to the spatial variables, and is thus out of line with earlier works that considered only particular types of open set. Moreover, the treatments in [2, 4] are bound up in their axiomatic systems, and so are unnecessarily tortuous for the heat equation. In [5], Doob asserted that a direct approach, more in line with the traditional approach to the Dirichlet problem for Laplace's equation, was possible for the heat equation, but gave few details. Sections 2-4 of this paper are devoted to providing such details, but guided by the treatment of the second form of the problem in [9] rather than by [5].

The second version of the Dirichlet problem for the heat equation does take into account the special nature of the temporal variable. This version was begun in [7], and carried through to [9] with different notations. It requires a classification of the boundary points of  $E$ , in which we use the following notations for the upper and lower half-balls. Given  $p_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$  and  $r > 0$ , we denote by  $H(p_0, r)$  the open lower half-ball  $\{(x, t) : |x - x_0|^2 + (t - t_0)^2 < r^2, t < t_0\}$ , and by  $H^*(p_0, r)$

the open upper half-ball  $\{(x, t) : |x - x_0|^2 + (t - t_0)^2 < r^2, t > t_0\}$ .

**Definitions.** Let  $q$  be a boundary point of the open set  $E$ . We call  $q$  a *normal* boundary point if either

- (a)  $q$  is the point at infinity, or
- (b)  $q \in \mathbb{R}^{n+1}$  and for every  $r > 0$ ,  $H(q, r) \setminus E \neq \emptyset$ .

Otherwise, we call  $q$  an *abnormal* boundary point; in this case, there is some  $r_0 > 0$  such that  $H(q, r_0) \subseteq E$ . The abnormal boundary points are of two kinds, according to whether they can be approached from above by points in  $E$ . If there is some  $r_1 < r_0$  such that  $H^*(q, r_1) \cap E = \emptyset$ , then  $q$  is called a *singular* boundary point. In this case,  $H(q, r_1) = B(q, r_1) \cap E$ . On the other hand if, for every  $r < r_0$ , we have  $H^*(q, r) \cap E \neq \emptyset$ , then  $q$  is called a *semi-singular* boundary point.

The set of all normal boundary points of  $E$  is denoted by  $\partial_n E$ , that of all abnormal points by  $\partial_a E$ , that of all singular points by  $\partial_s E$ , and that of all semi-singular points by  $\partial_{ss} E$ . Thus  $\partial E = \partial_n E \cup \partial_a E$  and  $\partial_a E = \partial_s E \cup \partial_{ss} E$ . The *essential boundary*  $\partial_e E$  is defined by

$$\partial_e E = \partial_n E \cup \partial_{ss} E = \partial E \setminus \partial_s E.$$

The second version of the Dirichlet problem for the heat equation is formulated as follows. In this, we use the notation  $\lim_{p \rightarrow q+} u(p)$  as an abbreviation for  $\lim_{(x,t) \rightarrow (y,s+)} u(p)$ , where  $p = (x, t)$  and  $q = (y, s)$ . A continuous, real-valued function  $f$  on the essential boundary  $\partial_e E$  is given, and a temperature  $u$  on  $E$  such that  $\lim_{p \rightarrow q} u(p) = f(q)$  for all  $q \in \partial_n E$ , and  $\lim_{p \rightarrow q+} u(p) = f(q)$  for all  $q \in \partial_{ss} E$ , is sought.

We deal with the generalised forms of the Dirichlet problem, in which the boundary function  $f$  is not required to be continuous and can take the values  $-\infty$  and  $+\infty$ . For each version of the problem, we try to associate with  $f$  a temperature on  $E$ , using the PWB method. We shall not discuss the boundary behaviour of such temperatures here. We now establish different notations for the PWB method relative to the two versions of the problem. Our terminology will follow [9], where further details can be found. For proofs that the corresponding concepts in [2, 4, 5] are equivalent, see [3] or [8].

Let  $f$  be an extended real-valued function defined on  $\partial E$ . For any lower bounded hypertemperature  $v$  on  $E$ , we put  $v$  in the class  $\mathfrak{U}_f^E$  if and only if both

$$\liminf_{p \rightarrow q} v(p) \geq f(q) \quad \text{for all } q \in \partial_n E,$$

and

$$\liminf_{p \rightarrow q+} v(p) \geq f(q) \quad \text{for all } q \in \partial_{ss} E.$$

This is the same as in [7, 9]. We put  $v$  in the class  $\overline{\mathfrak{H}}_f^E$  if and only if

$$\liminf_{p \rightarrow q} v(p) \geq f(q) \quad \text{for all } q \in \partial E.$$

This is similar to [2, 4, 5].

Clearly  $\overline{\mathfrak{H}}_f^E \subseteq \mathfrak{U}_f^E$ , so that if  $\overline{H}_f^E = \inf\{v : v \in \overline{\mathfrak{H}}_f^E\}$  is the *upper H-solution*, and  $U_f^E = \inf\{v : v \in \mathfrak{U}_f^E\}$  is the *upper S-solution*, then  $U_f^E \leq \overline{H}_f^E$  on  $E$ .

Dually, for any upper bounded hypotemperature  $w$  on  $E$ , we put  $w$  in the class  $\mathfrak{L}_f^E$  if and only if both

$$\limsup_{p \rightarrow q} w(p) \leq f(q) \quad \text{for all } q \in \partial_n E,$$

and

$$\limsup_{p \rightarrow q+} w(p) \leq f(q) \quad \text{for all } q \in \partial_{ss} E.$$

We put  $w$  in the class  $\underline{\mathfrak{H}}_f^E$  if and only if

$$\limsup_{p \rightarrow q} w(p) \leq f(q) \quad \text{for all } q \in \partial E.$$

Clearly  $\underline{\mathfrak{H}}_f^E \subseteq \mathfrak{L}_f^E$ , so that if  $\underline{H}_f^E = \sup\{w : w \in \underline{\mathfrak{H}}_f^E\}$  is the *lower H-solution*, and  $L_f^E = \sup\{w : w \in \mathfrak{L}_f^E\}$  is the *lower S-solution*, then  $L_f^E \geq \underline{H}_f^E$  on  $E$ .

If  $\underline{H}_f^E = \overline{H}_f^E$  and is a temperature on  $E$ , we denote it by  $H_f^E$  and say that  $f$  is *H-resolutive* for  $E$ . We also call  $H_f^E$  the *PWB solution to the H-Dirichlet problem* for  $f$  on  $E$ . Similarly, if  $L_f^E = U_f^E$  and is a temperature on  $E$ , we denote it by  $S_f^E$  and say that  $f$  is *S-resolutive* for  $E$ . We also call  $S_f^E$  the *PWB solution to the S-Dirichlet problem* for  $f$  on  $E$ .

Since  $\underline{H}_f^E \leq L_f^E \leq U_f^E \leq \overline{H}_f^E$  on  $E$ , the function  $f$  is H-resolutive only if it is S-resolutive, and then  $H_f^E = S_f^E$ . The main purpose of this paper is to establish that  $f$  is S-resolutive only if it is H-resolutive. In [10] (or [9]), there is a characterization of S-resolutivity in terms of caloric measure. In [5, p.332], Doob claims that there is a similar characterization of H-resolutivity in terms of parabolic measure, but gives no details to support that claim. Guided by the treatment in [10], we prove in Section 4 that there is such a characterization. In Section 5, we show that the parabolic measure coincides with the caloric measure, and deduce the equivalence of the two notions of resolutive. In Section 6, we prove a property of caloric measure that requires us to consider boundary functions that are not necessarily resolutive. This leads us to show, in Section 7, that the upper H-solution and the upper S-solution of any boundary function coincide. Thus the PWB method does not distinguish between the two forms of the Dirichlet problem.

## 2. Upper and Lower PWB Solutions of the H-Dirichlet Problem

In this section, we begin the systematic treatment of the H-Dirichlet problem. Lemmas 1 and 2, and Theorem 1, are well-known but are included for completeness. Lemmas 3 and 4 are new, as is the precise form of Lemma 5.

**Lemma 1.** *Let  $E$  be an open set, and let  $f$  be an extended real-valued function defined on the boundary  $\partial E$ . If  $u \in \underline{\mathfrak{H}}_f^E$  and  $v \in \overline{\mathfrak{H}}_f^E$ , then  $u \leq v$  on  $E$ . Consequently  $\underline{H}_f^E \leq \overline{H}_f^E$ .*

**Proof.** Since  $u$  is a hypotemperature and  $v$  is a hypertemperature on  $E$ ,  $v - u$  is a hypertemperature, by [9, Corollaries 3.55 and 3.57]. Furthermore, if  $q \in \partial E$  (and is possibly the point at infinity) and  $f(q)$  is finite, then

$$\liminf_{p \rightarrow q} (v - u)(p) \geq \liminf_{p \rightarrow q} v(p) - \limsup_{p \rightarrow q} u(p) \geq f(q) - f(q) = 0.$$

On the other hand, if  $f(q) = +\infty$  then  $\lim_{p \rightarrow q} v(p) = +\infty$  and  $\limsup_{p \rightarrow q} u(p) < +\infty$  because  $u$  is upper bounded, so that  $\liminf_{p \rightarrow q} (v - u)(p) \geq 0$ ; and if  $f(q) = -\infty$  then  $\lim_{p \rightarrow q} u(p) = -\infty$  and  $\liminf_{p \rightarrow q} v(p) > -\infty$ , so that  $\liminf_{p \rightarrow q} (v - u)(p) \geq 0$ . Therefore, by [9, Theorem 3.13],  $v \geq u$  on  $E$ . It follows that  $\underline{H}_f^E \leq \overline{H}_f^E$ .  $\square$

**Lemma 2.** *Let  $E$  be an open set, let  $f$  and  $g$  be extended real-valued functions on  $\partial E$ , and let  $\alpha \in \mathbb{R}$ .*

- (a) *Without further conditions,  $\overline{H}_{-f}^E = -\underline{H}_f^E$ .*
- (b) *If  $\alpha > 0$ , then  $\overline{H}_{\alpha f}^E = \alpha \overline{H}_f^E$  and  $\underline{H}_{\alpha f}^E = \alpha \underline{H}_f^E$ .*
- (c) *If  $f \leq g$ , then  $\overline{H}_f^E \leq \overline{H}_g^E$  and  $\underline{H}_f^E \leq \underline{H}_g^E$ .*
- (d) *Let  $(f + g)(q)$  be defined arbitrarily at each point  $q \in \partial E$  where  $f(q) + g(q)$  is undefined. Then for each point  $p \in E$ ,*

$$\overline{H}_{f+g}^E(p) \leq \overline{H}_f^E(p) + \overline{H}_g^E(p)$$

*provided that the sum on the right-hand side is defined, and*

$$\underline{H}_{f+g}^E(p) \geq \underline{H}_f^E(p) + \underline{H}_g^E(p)$$

*with the same proviso.*

**Proof.** (a) Since  $w \in \overline{\mathfrak{H}}_{-f}^E$  if and only if  $-w \in \underline{\mathfrak{H}}_f^E$ , we have

$$\overline{H}_{-f}^E = \inf\{w : -w \in \underline{\mathfrak{H}}_f^E\} = -\sup\{v : v \in \underline{\mathfrak{H}}_f^E\} = -\underline{H}_f^E.$$

(b) If  $\alpha > 0$ , then  $w \in \overline{\mathfrak{H}}_f^E$  if and only if  $\alpha w \in \overline{\mathfrak{H}}_{\alpha f}^E$ . Therefore

$$\overline{H}_{\alpha f}^E = \inf\{\alpha w : w \in \overline{\mathfrak{H}}_f^E\} = \alpha \overline{H}_f^E.$$

Similarly  $\underline{H}_{\alpha f}^E = \alpha \underline{H}_f^E$ .

(c) If  $f \leq g$ , then  $\overline{\mathfrak{H}}_g^E \subseteq \overline{\mathfrak{H}}_f^E$  and  $\underline{\mathfrak{H}}_f^E \subseteq \underline{\mathfrak{H}}_g^E$ , so that  $\overline{H}_f^E \leq \overline{H}_g^E$  and  $\underline{H}_f^E \leq \underline{H}_g^E$ .

(d) Let  $v \in \overline{\mathfrak{H}}_f^E$  and  $w \in \overline{\mathfrak{H}}_g^E$ . Then  $v + w$  is a lower bounded hypertemperature on  $E$ , and at all points  $q \in \partial E$  where  $f(q) + g(q)$  is well-defined, we have

$$\liminf_{p \rightarrow q} (v + w)(p) \geq \liminf_{p \rightarrow q} v(p) + \liminf_{p \rightarrow q} w(p) \geq f(q) + g(q).$$

At any point  $q \in \partial E$  where  $f(q) + g(q)$  is undefined, then without loss of generality we take  $f(q) = +\infty$  and  $g(q) = -\infty$ . This implies that  $\lim_{p \rightarrow q} v(p) = +\infty$ , and therefore that  $\lim_{p \rightarrow q} (v + w)(p) = +\infty$  because  $w$  is lower bounded. Thus, regardless of the value we assign to  $(f + g)(q)$ , we have  $\lim_{p \rightarrow q} (v + w)(p) \geq (f + g)(q)$ . Hence  $v + w \in \overline{\mathfrak{H}}_{f+g}^E$ . Now let  $p \in E$ , so that  $v(p) + w(p) \geq \overline{H}_{f+g}^E(p)$ . Clearly  $\overline{H}_{f+g}^E(p) \leq \overline{H}_f^E(p) + \overline{H}_g^E(p)$  if the sum on the right-hand side is defined and either term is  $+\infty$ . Since  $\overline{H}_f^E(p) = +\infty$  if and only if  $v(p) = +\infty$  for all  $v \in \overline{\mathfrak{H}}_f^E$ , it only remains to consider the case where there exist  $v \in \overline{\mathfrak{H}}_f^E$  and  $w \in \overline{\mathfrak{H}}_g^E$  such that

$v(p) < +\infty$  and  $w(p) < +\infty$ . In this case  $\overline{H}_f^E(p) + \overline{H}_g^E(p)$  is defined, and since  $\overline{H}_{f+g}^E(p) \leq v(p) + w(p)$  we have  $\overline{H}_{f+g}^E(p) \leq \overline{H}_f^E(p) + w(p)$ , and hence the first result.

The proof for the lower solutions now follows easily from (a).  $\square$

**Definition.** Let  $E$  be an open set, and let  $f \in C(\partial E)$ . We say that a temperature  $u$  on  $E$  is a *classical solution of the H-Dirichlet problem for  $f$*  if

$$\lim_{p \rightarrow q} u(p) = f(q) \quad \text{for all } q \in \partial E.$$

It is an important fact that, if there is a classical solution of the H-Dirichlet problem for  $f$ , then the PWB solution for  $f$  exists and coincides with the classical solution.

**Theorem 1.** *Let  $E$  be an open set, and let  $f \in C(\partial E)$ . If there is a classical solution  $u$  of the H-Dirichlet problem for  $f$  on  $E$ , then  $f$  is H-resolutive and  $H_f = u$  on  $E$ .*

**Proof.** Since  $f \in C(\partial E)$  it is bounded, and therefore  $u$  is bounded, in view of the boundary point maximum principle. Therefore, because of its boundary limits,  $u$  belongs to both  $\overline{\mathfrak{H}}_f^E$  and  $\underline{\mathfrak{H}}_f^E$ . Hence  $u \geq \overline{H}_f^E$  and  $u \leq \underline{H}_f^E$ , and so it follows from Lemma 1 that  $u = \overline{H}_f^E = \underline{H}_f^E$ . Since  $u$  is a temperature on  $E$ , this implies that  $f$  is H-resolutive and  $H_f^E = u$  on  $E$ .  $\square$

It follows easily from Theorem 1 that, if  $f(q) = \alpha \in \mathbb{R}$  for all  $q \in \partial E$ , then  $f$  is H-resolutive and  $H_f^E = \alpha$  on  $E$ . Furthermore, in view of Lemma 2(c) and Lemma 1, if  $g : \partial E \rightarrow [\alpha, \beta]$ , then  $\alpha \leq \underline{H}_g^E \leq \overline{H}_g^E \leq \beta$  on  $E$ .

Given an open subset  $E$  of  $\mathbb{R}^{n+1}$  and a point  $p_0 \in E$ , we denote by  $\Lambda(p_0; E)$  the set of points  $q \in E$  that are lower than  $p_0$  relative to  $E$ , in the sense that there is a polygonal path  $\gamma \subseteq E$  joining  $p_0$  to  $q$  along which the temporal variable  $t$  is strictly decreasing.

**Lemma 3.** *Let  $E$  be an open set, let  $p_0 \in E$ , and put  $\Lambda = \Lambda(p_0; E)$ . Let  $f$  be a function defined on  $\partial E$ , and define a function  $g$  on  $\partial \Lambda$  by*

$$g(p) = \begin{cases} f(p) & \text{if } p \in \partial \Lambda \cap \partial E, \\ -\infty & \text{if } p \in \partial \Lambda \setminus \partial E. \end{cases}$$

Then  $\overline{\mathfrak{H}}_g^\Lambda$  is precisely the class of restrictions to  $\Lambda$  of the members of  $\overline{\mathfrak{H}}_f^E$ , so that  $\overline{H}_g^\Lambda$  is the restriction to  $\Lambda$  of  $\overline{H}_f^E$ .

**Proof.** We first show that, given any hypertemperature  $v \in \overline{\mathfrak{H}}_f^E$ , its restriction to  $\Lambda$  belongs to  $\overline{\mathfrak{H}}_g^\Lambda$ . Obviously the restriction is a lower bounded hypertemperature on  $\Lambda$ . Let  $q \in \partial \Lambda$ . If  $q \in \partial E$  also, then

$$\liminf_{p \rightarrow q, p \in \Lambda} v(p) \geq \liminf_{p \rightarrow q, p \in E} v(p) \geq f(q) = g(q).$$

On the other hand, if  $q \notin \partial E$  then

$$\liminf_{p \rightarrow q, p \in \Lambda} v(p) \geq -\infty = g(q).$$

Hence the restriction of  $v$  to  $\Lambda$  belongs to  $\overline{\mathfrak{H}}_g^\Lambda$ .

In the opposite direction, given any hypertemperature  $w \in \overline{\mathfrak{H}}_g^\Lambda$ , we define a function  $\bar{w}$  on  $E$  by putting

$$\bar{w}(p) = \begin{cases} w(p) & \text{if } p \in \Lambda, \\ +\infty & \text{if } p \in E \setminus \overline{\Lambda}, \\ \liminf_{q \rightarrow p, q \in \Lambda} w(q) & \text{if } p \in \partial \Lambda \cap E. \end{cases}$$

We claim that  $\bar{w} \in \overline{\mathfrak{H}}_f^E$ . We show that  $\bar{w}$  is a hypertemperature on  $E$  using [9, Theorem 3.51]. Clearly  $\bar{w}$  is lower semicontinuous on  $E$ , and is also lower bounded on  $E$  because  $w$  is lower bounded on  $\Lambda$ . It remains to show that, given any point  $p \in E$  and any  $\epsilon > 0$ , we can find a positive number  $c < \epsilon$  such that the inequality  $\bar{w}(p) \geq \mathcal{V}(\bar{w}; p; c)$  holds. Clearly we can do this if  $p \in E \setminus \partial \Lambda$ , so suppose that  $p \in E \cap \partial \Lambda$ . Since  $\partial_e \Lambda \subseteq \partial_e E$  by [9, Lemma 8.4],  $p \in \partial_s \Lambda$ . Therefore we can find  $r_0 > 0$  such that  $H(p, 2r_0) = B(p, 2r_0) \cap \Lambda$ . We now choose  $c_0 > 0$  such that  $\overline{\Omega}(q; c) \subseteq \Lambda$  whenever  $q \in H(p, r_0)$  and  $c \leq c_0$ . Then, for any  $c \leq c_0$ , we have

$$\bar{w}(p) = \liminf_{q \rightarrow p, q \in \Lambda} w(q) \geq \liminf_{q \rightarrow p, q \in \Lambda} \mathcal{V}(w; q; c) = \liminf_{q \rightarrow p, q \in \Lambda} \mathcal{V}(\bar{w}; q; c) \geq \mathcal{V}(\bar{w}; p; c),$$

by Fatou's lemma. Hence  $\bar{w}$  is a hypertemperature on  $E$ .

We now take any point  $q \in \partial E$ . If  $q \notin \partial \Lambda$ , then

$$\liminf_{p \rightarrow q} \bar{w}(p) = +\infty \geq f(q).$$

On the other hand, if  $q \in \partial \Lambda$  then

$$\liminf_{p \rightarrow q} \bar{w}(p) = \liminf_{p \rightarrow q, p \in \Lambda} w(p) \geq g(q) = f(q).$$

Hence  $\bar{w} \in \overline{\mathfrak{H}}_f^E$ . Thus  $w$  is the restriction to  $\Lambda$  of a function in  $\overline{\mathfrak{H}}_f^E$ .  $\square$

**Lemma 4.** *Let  $E$  be an open set, and let  $f$  be an extended real-valued function defined on  $\partial E$ . If there are points  $p_0, q_0 \in E$  such that  $q_0 \in \Lambda(p_0; E)$ ,  $\overline{H}_f^E(p_0) < +\infty$ , and  $\overline{H}_f^E(q_0) > -\infty$ , then  $\overline{H}_f^E$  is a temperature on  $\Lambda(q_0; E)$ .*

**Proof.** We put  $\Lambda = \Lambda(p_0; E)$ . Let  $g$  be defined on  $\partial \Lambda$  as in Lemma 3, so that  $\overline{H}_f^E = \overline{H}_g^\Lambda$  on  $\Lambda$ . Since  $\overline{H}_f^E(p_0) < +\infty$ , we can find a hypertemperature  $w_0 \in \overline{\mathfrak{H}}_f^E$  such that  $w_0(p_0) < +\infty$ . By [9, Corollary 3.55],  $w_0$  is a supertemperature on  $\Lambda$ . By Lemma 3, the restriction of  $w_0$  to  $\Lambda$  belongs to  $\overline{\mathfrak{H}}_g^\Lambda$ , and so we can write  $\overline{H}_g^\Lambda = \inf \mathcal{F}$ , where  $\mathcal{F}$  is the class of all supertemperatures that belong to  $\overline{\mathfrak{H}}_g^\Lambda$ .

We show that  $\mathcal{F}$  is a saturated family of supertemperatures on  $\Lambda$ , with a view to applying [9, Theorem 3.26]. Let  $u, v \in \mathcal{F}$ . Then  $u \wedge v$  is a lower bounded

supertemperature on  $\Lambda$ . Moreover, whenever  $q \in \partial\Lambda$ , we have

$$\liminf_{p \rightarrow q} (u \wedge v)(p) = \left( \liminf_{p \rightarrow q} u(p) \right) \wedge \left( \liminf_{p \rightarrow q} v(p) \right) \geq g(q).$$

Hence  $u \wedge v \in \mathcal{F}$ . We now take any function  $w \in \mathcal{F}$ , and any circular cylinder  $D$  such that  $\bar{D} \subseteq \Lambda$ . By [9, Theorem 3.21], the Poisson integral of the restriction of  $w$  to  $\partial_n D$  exists, and if  $\pi_D w$  is defined on  $\Lambda$  to be equal to that Poisson integral on  $\bar{D} \setminus \partial_n D$ , and equal to  $w$  elsewhere on  $\Lambda$ , then  $\pi_D w$  is a supertemperature on  $\Lambda$  which is lower bounded on  $\Lambda$  by the same lower bound as  $w$ . Furthermore, since the compact set  $\bar{D} \subseteq \Lambda$  and  $\pi_D w = w$  on  $\Lambda \setminus \bar{D}$ , the boundary behaviour of  $\pi_D w$  is the same as that of  $w$ . Therefore  $\pi_D w \in \mathcal{F}$ , and so  $\mathcal{F}$  is a saturated family of supertemperatures on  $\Lambda$ .

Since  $q_0 \in \Lambda$  and, by Lemma 3,  $\bar{H}_g^\Lambda(q_0) = \bar{H}_f^E(q_0) > -\infty$ , it follows from [9, Theorem 3.26] that  $\bar{H}_g^\Lambda$  is a temperature on  $\Lambda(q_0; \Lambda) = \Lambda(q_0; E)$ , as required.  $\square$

**Corollary.** *Let  $E$  be an open set, and let  $f$  be an extended real-valued function defined on  $\partial E$ . If there is a point  $p_0 \in E$  such that  $\underline{H}_f^E(p_0)$  and  $\bar{H}_f^E(p_0)$  are both finite, then  $\underline{H}_f^E$  and  $\bar{H}_f^E$  are temperatures on  $\Lambda(p_0; E)$ .*

**Proof.** Since  $\underline{H}_f^E(p_0) > -\infty$ , we can find a hypotemperature  $u \in \mathfrak{H}_f^E$  such that  $u(p_0) > -\infty$ . By [9, Corollary 3.55],  $u$  is a subtemperature on  $\Lambda(p_0; E)$ , and in particular is finite on a dense subset  $F$  of  $\Lambda(p_0; E)$ . Therefore

$$-\infty < u(q) \leq \underline{H}_f^E(q) \leq \bar{H}_f^E(q)$$

for all  $q \in F$ . Since  $\bar{H}_f^E(p_0) < +\infty$ , it follows from Lemma 4 that  $\bar{H}_f^E$  is a temperature on the set

$$\bigcup_{q \in F} \Lambda(q; E) = \Lambda(p_0; E).$$

Applying this result to  $-f$ , and using Lemma 2(a), we obtain the result for  $\underline{H}_f^E$ .  $\square$

**Lemma 5.** *Let  $E$  be an open set, and let  $f$  be the limit of an increasing sequence  $\{f_j\}$  of extended real-valued functions on  $\partial E$  such that  $\bar{H}_{f_m}^E > -\infty$  on  $E$  for some  $m$ . If  $p_0$  is a point in  $E$  such that  $\bar{H}_{f_j}^E(p_0) < +\infty$  for all  $j$ , then*

$$\bar{H}_f^E = \lim_{j \rightarrow \infty} \bar{H}_{f_j}^E$$

on  $\Lambda(p_0; E)$ .

**Proof.** By Lemma 2(c), the sequence  $\{\bar{H}_{f_j}^E\}$  is increasing on  $E$ , and  $\bar{H}_{f_j}^E \leq \bar{H}_f^E$  on  $E$  for all  $j$ . Therefore  $\lim_{j \rightarrow \infty} \bar{H}_{f_j}^E \leq \bar{H}_f^E$  on  $E$ , and we may suppose that  $\bar{H}_{f_j}^E > -\infty$  on  $E$  for all  $j$ .

For each  $j$ , Lemma 4 and our hypothesis that  $\bar{H}_{f_j}^E(p_0) < +\infty$  now imply that  $\bar{H}_{f_j}^E$  is a temperature on  $\Lambda(p; E)$  for all  $p \in \Lambda(p_0; E)$ , and thus on  $\Lambda(p_0; E)$  itself.

We put  $\Lambda = \Lambda(p_0; E)$ , and define a function  $g_j$  on  $\partial\Lambda$  by

$$g_j(p) = \begin{cases} f_j(p) & \text{if } p \in \partial\Lambda \cap \partial E, \\ -\infty & \text{if } p \in \partial\Lambda \setminus \partial E. \end{cases}$$

Then Lemma 3 shows that  $\overline{H}_{f_j}^E = \overline{H}_{g_j}^\Lambda$  on  $\Lambda$ . Given any positive number  $\epsilon$  and any point  $p_1 \in \Lambda$ , we can find a hypertemperature  $w_j \in \overline{\mathfrak{H}}_{g_j}^\Lambda$  such that

$$w_j(p_1) - \overline{H}_{g_j}^\Lambda(p_1) < 2^{-j}\epsilon.$$

Since  $\overline{H}_{g_j}^\Lambda$  is a temperature on  $\Lambda$ , [9, Theorem 3.60] shows that  $\lim_{j \rightarrow \infty} \overline{H}_{g_j}^\Lambda$  is a hypertemperature on  $\Lambda$ . Moreover, since  $w_j - \overline{H}_{g_j}^\Lambda$  is a nonnegative hypertemperature on  $\Lambda$ , the same is true of  $\sum_{j=1}^\infty (w_j - \overline{H}_{g_j}^\Lambda)$ , and hence of the function

$$v = \lim_{j \rightarrow \infty} \overline{H}_{g_j}^\Lambda + \sum_{j=1}^\infty (w_j - \overline{H}_{g_j}^\Lambda).$$

For each  $k$ , we have

$$v \geq \overline{H}_{g_k}^\Lambda + (w_k - \overline{H}_{g_k}^\Lambda) = w_k,$$

so that  $v$  is lower bounded on  $\Lambda$  and

$$\liminf_{p \rightarrow q} v(p) \geq g_k(q)$$

for all  $q \in \partial\Lambda$ . Therefore, if  $g = \lim_{k \rightarrow \infty} g_k$  on  $\partial\Lambda$ , we have  $v \in \overline{\mathfrak{H}}_g^\Lambda$  and hence  $v \geq \overline{H}_g^\Lambda$ . In particular,

$$\overline{H}_g^\Lambda(p_1) \leq v(p_1) \leq \lim_{j \rightarrow \infty} \overline{H}_{g_j}^\Lambda(p_1) + \sum_{j=1}^\infty 2^{-j}\epsilon = \lim_{j \rightarrow \infty} \overline{H}_{g_j}^\Lambda(p_1) + \epsilon.$$

This holds for all  $\epsilon > 0$ , so that

$$\overline{H}_g^\Lambda(p_1) \leq \lim_{j \rightarrow \infty} \overline{H}_{g_j}^\Lambda(p_1) \leq \overline{H}_g^\Lambda(p_1).$$

Therefore, by Lemma 3,

$$\overline{H}_f^E(p_1) \leq \lim_{j \rightarrow \infty} \overline{H}_{f_j}^E(p_1) \leq \overline{H}_f^E(p_1).$$

Since  $p_1$  is an arbitrary point of  $\Lambda$ , the result is established.  $\square$

An earlier version of Lemma 5, with extra hypotheses and less precision, is given in [2, Lemma 4.1.6].

### 3. H-Resolutivity and PWB Solutions

Apart from Lemma 6, all the results in this section are standard but included for completeness.

**Lemma 6.** *Let  $E$  be an open set, and let  $f$  be an extended real-valued function on  $\partial E$ . If, for each point  $q_0 \in E$ , we can find a point  $p_0 \in E$  such that  $q_0 \in \Lambda(p_0; E)$  and  $\underline{H}_f^E(p_0) = \overline{H}_f^E(p_0) \in \mathbb{R}$ , then  $f$  is H-resolutive for  $E$ .*

**Proof.** Let  $q_0 \in E$ . By hypothesis, there is a point  $p_0 \in E$  such that  $q_0 \in \Lambda(p_0; E)$  and  $\underline{H}_f^E(p_0)$  and  $\overline{H}_f^E(p_0)$  are both finite. Therefore, by Lemma 4 Corollary, the functions  $\underline{H}_f^E$  and  $\overline{H}_f^E$  are temperatures on the neighbourhood  $\Lambda(p_0; E)$  of  $q_0$ . Thus  $\underline{H}_f^E$  and  $\overline{H}_f^E$  are temperatures on the whole of  $E$ , so that the function  $v = \underline{H}_f^E - \overline{H}_f^E$  is a nonpositive temperature on  $E$ , in view of Lemma 1. For any point  $q \in E$ , our hypothesis shows that there is a point  $p$  such that  $q \in \Lambda(p; E)$  and  $v(p) = 0$ , so that  $v = 0$  on  $\Lambda(p; E)$  by the strong maximum principle. In particular,  $v(q) = 0$ . Hence  $\underline{H}_f^E = \overline{H}_f^E$  on  $E$ , and  $f$  is H-resolutive for  $E$ .  $\square$

**Theorem 2.** Let  $E$  be an open set, let  $f$  and  $g$  be extended real-valued functions on  $\partial E$ , and let  $\alpha \in \mathbb{R}$ .

- (a) If  $f$  is H-resolutive, then  $\alpha f$  is H-resolutive and  $H_{\alpha f}^E = \alpha H_f^E$  on  $E$ .
- (b) If  $f$  and  $g$  are both H-resolutive, and  $(f + g)(q)$  is defined arbitrarily at each point  $q \in \partial E$  where  $f(q) + g(q)$  is undefined, then  $f + g$  is H-resolutive and  $H_{f+g}^E = H_f^E + H_g^E$  on  $E$ .

**Proof.** (a) If  $\alpha = 0$ , the result is trivial. If  $\alpha > 0$ , then Lemma 2(b) and the H-resolutivity of  $f$  show that  $\underline{H}_{\alpha f}^E = \alpha \underline{H}_f^E = \alpha \overline{H}_f^E = \overline{H}_{\alpha f}^E$ . The result follows. If  $\alpha < 0$ , then  $-\alpha f$  is H-resolutive; this, together with Lemma 2(a), shows that  $\underline{H}_{\alpha f}^E = -\overline{H}_{-\alpha f}^E = -\underline{H}_{-\alpha f}^E = \overline{H}_{\alpha f}^E$ . The result follows.

- (b) If  $f$  and  $g$  are both H-resolutive, then Lemma 1 and Lemma 2(d) show that

$$H_f^E + H_g^E = \underline{H}_f^E + \underline{H}_g^E \leq \underline{H}_{f+g}^E \leq \overline{H}_{f+g}^E \leq \overline{H}_f^E + \overline{H}_g^E = H_f^E + H_g^E,$$

which implies the result.  $\square$

**Theorem 3.** Let  $E$  be an open set, and let  $\{f_j\}$  be a sequence of real-valued, H-resolutive functions on  $\partial E$ . If  $\{f_j\}$  converges uniformly on  $\partial E$  to a function  $f$ , then  $f$  is H-resolutive and  $H_{f_j}^E \rightarrow H_f^E$  uniformly on  $E$ .

**Proof.** Given any  $\epsilon > 0$ , we choose a number  $k$  such that  $|f_j - f| < \epsilon$  on  $\partial E$  for all  $j > k$ . For such  $j$ , if  $w \in \overline{\mathfrak{H}}_{f_j}^E$  then  $w + \epsilon \in \overline{\mathfrak{H}}_f^E$ . Therefore  $\overline{H}_f^E \leq w + \epsilon$ , and it follows that  $\overline{H}_f^E \leq \overline{H}_{f_j}^E + \epsilon$ . Similarly  $\underline{H}_f^E \geq \underline{H}_{f_j}^E - \epsilon$  for all  $j > k$ . It now follows from Lemma 1 and the H-resolutivity of the functions  $f_j$  that

$$H_{f_j}^E - \epsilon \leq \underline{H}_f^E \leq \overline{H}_f^E \leq H_{f_j}^E + \epsilon.$$

These inequalities show that  $|H_{f_j}^E - \underline{H}_f^E| < \epsilon$  and  $|H_{f_j}^E - \overline{H}_f^E| < \epsilon$  for all  $j > k$ , so that the sequence  $\{H_{f_j}^E\}$  converges uniformly on  $E$  to both  $\underline{H}_f^E$  and  $\overline{H}_f^E$ . Therefore  $\underline{H}_f^E = \overline{H}_f^E \in C(E)$ , so that  $f$  is H-resolutive for  $E$ , by Lemma 6.  $\square$

**Lemma 7.** Let  $E$  be an open set, let  $K$  be a compact subset of  $E$ , and let  $w$  be a function on  $E \cup \partial E$  that is both a subtemperature on  $E$  and an element of  $C((E \cup \partial E) \setminus K)$ . Then the restriction of  $w$  to  $\partial E$  is H-resolutive for  $E$ .

**Proof.** We denote by  $f$  the restriction of  $w$  to  $\partial E$ . Since  $f \in C(\partial E)$  it is bounded, and so we can find real numbers  $\alpha$  and  $\beta$  such that  $\alpha \leq f \leq \beta$  on  $\partial E$ . Then  $\alpha \leq \underline{H}_f^E \leq \beta$  on  $E$ , so that  $\underline{H}_f^E$  is a temperature on  $E$ , by Lemma 4. For every

point  $q \in \partial E$  we have  $\lim_{p \rightarrow q} w(p) = f(q) \leq \beta$ , and so  $w$  is upper bounded, by [9, Theorem 8.2]. It follows that  $w \in \mathfrak{H}_f^E$ , and hence  $w \leq \underline{H}_f^E$  on  $E$ . Therefore

$$\liminf_{p \rightarrow q} \underline{H}_f^E(p) \geq \lim_{p \rightarrow q} w(p) = f(q)$$

for all  $q \in \partial E$ , so that  $\underline{H}_f^E \in \overline{\mathfrak{H}}_f^E$ , and hence  $\underline{H}_f^E \geq \overline{H}_f^E$  on  $E$ . Since  $\underline{H}_f^E \leq \overline{H}_f^E$  by Lemma 1, equality holds and, because  $\underline{H}_f^E$  is a temperature on  $E$ ,  $f$  is H-resolutive.  $\square$

**Theorem 4.** *If  $E$  is an open set and  $f \in C(\partial E)$ , then  $f$  is H-resolutive for  $E$ .*

**Proof.** Let  $\mathcal{G}$  denote the class of real-valued functions on  $E \cup \partial E$  that are both supertemperatures on  $E$  and continuous on  $(E \cup \partial E) \setminus K$  for some compact subset  $K$  of  $E$ . Let  $\mathcal{D}$  denote the class of differences  $u - v$  of functions in  $\mathcal{G}$ , and let  $\mathcal{F}$  denote the class of restrictions to  $\partial E$  of the functions in  $\mathcal{D}$ . Then  $\mathcal{F}$  is a linear subspace of  $C(\partial E)$  that contains the constant functions. By Lemma 7, the restrictions to  $\partial E$  of the functions in  $\mathcal{G}$  are H-resolutive, and so Theorem 2 shows that the functions in  $\mathcal{F}$  are all H-resolutive. Furthermore, for any point  $q_0 \notin \partial E$ , the class  $\mathcal{D}$  contains the function  $G(\cdot; q_0) \wedge \alpha$  for every positive number  $\alpha$ , and so  $\mathcal{F}$  separates points of  $\partial E$ . Finally, if  $u, v \in \mathcal{G}$  then [9, Corollaries 3.18 and 3.19] imply that  $u \wedge v, u + v \in \mathcal{G}$ , so that if  $u_1, u_2, v_1, v_2 \in \mathcal{G}$  the function

$$(u_1 - v_1) \vee (u_2 - v_2) = u_1 + u_2 - (u_2 + v_1) \wedge (u_1 + v_2) \in \mathcal{D}.$$

Thus  $f \vee g \in \mathcal{F}$  whenever  $f, g \in \mathcal{F}$ . It now follows from the Stone-Weierstrass theorem for the one-point compactification of  $\mathbb{R}^{n+1}$  that  $\mathcal{F}$  is dense in  $C(\partial E)$  with respect to the supremum norm. So every function in  $C(\partial E)$  can be expressed as the uniform limit of a sequence in  $\mathcal{F}$ . Since every function in  $\mathcal{F}$  is H-resolutive, it follows from Theorem 3 that every every function in  $C(\partial E)$  is H-resolutive.  $\square$

*Remark.* For earlier versions of Theorem 4 in the context of harmonic spaces, see [2, Satz 4.1.5] (for bounded open sets) and [4, Theorem 1.2.2].

#### 4. The Parabolic Measure on the Boundary

In this section, we develop the analogue for the heat equation of the notion of harmonic measure. Since we are giving details for Doob's program, and want a different terminology from that used for the S-Dirichlet problem in [10], we shall adopt his terminology and call the measure *parabolic*.

**Theorem 5.** *Let  $E$  be an open set, and let  $p \in E$ . Then there is a unique nonnegative Borel measure  $\omega_p^E$  on  $\partial E$  such that the equality*

$$H_f^E(p) = \int_{\partial E} f d\omega_p^E$$

*holds for every  $f \in C(\partial E)$ . Moreover  $\omega_p^E(\partial E) = 1$ .*

**Proof.** Any function  $f \in C(\partial E)$  has a PWB solution  $H_f^E$  on  $E$ , by Theorem 4. We show that the mapping  $f \mapsto H_f^E(p)$  is a positive linear functional on the

Banach space  $C(\partial E)$  with the supremum norm. By Theorem 2, if  $f, g \in C(\partial E)$  and  $\alpha, \beta \in \mathbb{R}$ , then

$$H_{\alpha f + \beta g}^E = H_{\alpha f}^E + H_{\beta g}^E = \alpha H_f^E + \beta H_g^E,$$

so that the mapping in question is a linear functional on  $C(\partial E)$ . Furthermore, if  $f \geq 0$  then  $0 \leq H_f^E$  on  $E$ , by Lemma 2. Hence the linear functional  $f \mapsto H_f^E(p)$  is positive. It now follows from the Riesz Representation Theorem that there is a unique nonnegative Borel measure  $\omega_p^E$  on  $\partial E$  such that  $H_f^E(p) = \int_{\partial E} f d\omega_p^E$  for every  $f \in C(\partial E)$ . In particular, if  $f(q) = 1$  for all  $q \in \partial E$ , then  $H_f^E = 1$  on  $E$  by Theorem 1, so that  $1 = H_f^E(p) = \int_{\partial E} d\omega_p^E = \omega_p^E(\partial E)$ .  $\square$

*Remark.* Given Theorem 4, Theorem 5 is a standard deduction. In the context of harmonic spaces, see [4, p.19]. In the same context, Bauer used a different definition of harmonic measure; see [2, Satz 4.1.5] (for bounded open sets and without the uniqueness assertion).

**Definition.** Let  $E$  be an open set, and let  $p \in E$ . Then the completion of the measure  $\omega_p^E$  of Theorem 5, is called the *parabolic measure* relative to  $E$  and  $p$ . It will also be denoted by  $\omega_p^E$ . A function on  $\partial E$  will be called  $\omega_p^E$ -*measurable* if it is measurable with respect to the completed measure.

**Lemma 8.** *Let  $E$  be an open set, and let  $f$  be a lower finite, lower semicontinuous function on  $\partial E$ . Then*

$$\overline{H}_f^E(p) = \underline{H}_f^E(p) = \int_{\partial E} f d\omega_p^E$$

for all  $p \in E$ , and if  $\overline{H}_f^E < +\infty$  on a dense subset of  $E$ , then  $f$  is  $H$ -resolutive for  $E$ .

**Proof.** There is an increasing sequence  $\{f_j\}$  of functions in  $C(\partial E)$  that converges pointwise to  $f$  on  $\partial E$ . By Theorem 4, each function  $f_j$  is  $H$ -resolutive for  $E$  so that, in particular, each  $H_{f_j}^E$  is finite-valued on  $E$ . Therefore, by Lemma 5,

$$\overline{H}_f^E = \lim_{j \rightarrow \infty} H_{f_j}^E$$

on  $E$ . Furthermore, Lemma 2(c) shows that  $H_{f_j}^E \leq \underline{H}_{f_j}^E$  on  $E$  for all  $j$ , so it follows that  $\overline{H}_f^E \leq \underline{H}_f^E$  on  $E$ . Since Lemma 1 shows that  $\underline{H}_f^E \leq \overline{H}_f^E$  on  $E$ , equality holds. Therefore, for all  $p \in E$  we have

$$\underline{H}_f^E(p) = \overline{H}_f^E(p) = \lim_{j \rightarrow \infty} H_{f_j}^E(p) = \lim_{j \rightarrow \infty} \int_{\partial E} f_j d\omega_p^E = \int_{\partial E} f d\omega_p^E,$$

by Theorem 5 and the Lebesgue monotone convergence theorem. Finally, since  $\overline{H}_f^E \geq H_{f_1}^E > -\infty$  on  $E$ , it follows from Lemma 4 that  $\overline{H}_f^E$  is a temperature on  $E$  if it is upper finite on a dense subset of  $E$ , so that  $f$  is  $H$ -resolutive for  $E$  in this case.  $\square$

**Corollary.** *Let  $E$  be an open set, let  $D$  be an open subset of  $E$ , and let  $u$  be a nonnegative supertemperature on  $E$ . Let  $f$  be defined on  $\partial D$  by*

$$f(q) = \begin{cases} u(q) & \text{if } q \in \partial D \cap E, \\ 0 & \text{if } q \in \partial D \setminus E. \end{cases}$$

*Then  $f$  is  $H$ -resolutive for  $D$ , and*

$$H_f^D(p) = \int_{\partial D} f d\omega_p^D$$

*for all  $p \in D$ .*

**Proof.** The function  $f$  is lower bounded and lower semicontinuous on  $\partial D$ , so that

$$\overline{H}_f^D(p) = \underline{H}_f^D(p) = \int_{\partial D} f d\omega_p^D$$

for all  $p \in D$ , by Lemma 8. Furthermore  $u \in \overline{\mathfrak{H}}_f^D$ , so that  $u \geq \overline{H}_f^D$  on  $D$ , and hence  $\overline{H}_f^D < +\infty$  on a dense subset of  $D$ . Therefore Lemma 8 shows that  $f$  is  $H$ -resolutive for  $D$ , and

$$H_f^D(p) = \int_{\partial D} f d\omega_p^D$$

for all  $p \in D$ . □

**Lemma 9.** *Let  $E$  be an open set, let  $p \in E$ , and let  $f$  be an extended real-valued function on  $\partial E$ . Given any number  $A > \overline{H}_f^E(p)$ , we can find a lower finite, lower semicontinuous function  $g$  on  $\partial E$ , such that  $f \leq g$  on  $\partial E$  and  $\overline{H}_g^E(p) < A$ . Given any number  $B < \underline{H}_f^E(p)$ , we can find an upper finite, upper semicontinuous function  $h$  on  $\partial E$ , such that  $h \leq f$  on  $\partial E$  and  $\underline{H}_h^E(p) > B$ .*

**Proof.** Since  $\overline{H}_f^E(p) < A$ , we can find a function  $w \in \overline{\mathfrak{H}}_f^E$  such that  $w(p) < A$ . We define a function  $g$  on  $\partial E$  by putting  $g(q) = \liminf_{p \rightarrow q} w(p)$  for all  $q \in \partial E$ . Then  $g$  is lower bounded and lower semicontinuous on  $\partial E$ . Since  $w \in \overline{\mathfrak{H}}_f^E$ , we also have  $g \geq f$  on  $\partial E$ . Finally, we note that  $w \in \overline{\mathfrak{H}}_g^E$ , which implies that  $\overline{H}_g^E(p) \leq w(p) < A$ .

Given  $B < \underline{H}_f^E(p)$ , we have  $-B > -\underline{H}_f^E(p) = \overline{H}_{-f}^E(p)$ , by Lemma 2(a). Therefore, by the part just proved, we can find a lower finite, lower semicontinuous function  $-h$  on  $\partial E$ , such that  $-f \leq -h$  on  $\partial E$  and  $\overline{H}_{-h}^E(p) < -B$ . So  $h$  is an upper finite, upper semicontinuous function on  $\partial E$ , such that  $h \leq f$  on  $\partial E$  and  $\underline{H}_h^E(p) = -\overline{H}_{-h}^E(p) > B$ . □

**Theorem 6.** *Let  $E$  be an open set, let  $p \in E$ , and let  $f$  be an extended real-valued function on  $\partial E$ .*

(a) *If  $\int_{\partial E} f d\omega_p^E$  exists, then*

$$\overline{H}_f^E(p) = \underline{H}_f^E(p) = \int_{\partial E} f d\omega_p^E. \quad (4.1)$$

(b) *Conversely, if  $\overline{H}_f^E(p) = \underline{H}_f^E(p)$  and is finite, then  $f$  is  $\omega_p^E$ -integrable (and (4.1) holds).*

**Proof.** (a) We prove that (4.1) holds for increasingly general classes of functions.

If  $f$  is the characteristic function  $\chi_A$  of a relatively open subset  $A$  of  $\partial E$ , then  $f$  is finite and lower semicontinuous on  $\partial E$ , so that (4.1) follows from Lemma 8.

We denote by  $\mathcal{B}$  the  $\sigma$ -algebra of all Borel subsets of  $\partial E$ , and by  $\mathcal{F}$  the class of all sets  $A \in \mathcal{B}$  for which (4.1) holds when  $f = \chi_A$ . We prove that  $\mathcal{F} = \mathcal{B}$ . We know that  $\mathcal{F}$  contains all the relatively open subsets of  $\partial E$ , so we can prove that  $\mathcal{F} = \mathcal{B}$  by showing that  $\mathcal{F}$  is a  $\sigma$ -algebra. Clearly  $\partial E \in \mathcal{F}$ . Suppose that  $A \in \mathcal{F}$ , so that

$$\omega_p^E(A) = \int_{\partial E} \chi_A d\omega_p^E = \overline{H}_{\chi_A}^E(p) = \underline{H}_{\chi_A}^E(p).$$

We denote by  $A^c$  the complement of  $A$  in  $\partial E$ . Then, using Theorem 5 and Lemma 2, we have

$$\begin{aligned} \omega_p^E(A^c) &= 1 - \omega_p^E(A) = 1 - \overline{H}_{\chi_A}^E(p) = \underline{H}_1^E(p) + \underline{H}_{-\chi_A}^E(p) \leq \underline{H}_{\chi_{A^c}}^E(p) \leq \overline{H}_{\chi_{A^c}}^E(p) \\ &\leq \overline{H}_1^E(p) + \overline{H}_{-\chi_A}^E(p) = 1 - \underline{H}_{\chi_A}^E(p) = 1 - \omega_p^E(A) = \omega_p^E(A^c). \end{aligned}$$

Therefore equality holds throughout, and hence

$$\underline{H}_{\chi_{A^c}}^E(p) = \overline{H}_{\chi_{A^c}}^E(p) = \omega_p^E(A^c) = \int_{\partial E} \chi_{A^c} d\omega_p^E.$$

Thus  $A^c \in \mathcal{F}$ . We now let  $\{F_j\}$  be an expanding sequence of sets in  $\mathcal{F}$ , and put  $F = \bigcup_{j=1}^{\infty} F_j$ . By Lemma 2, we have  $1 \geq \underline{H}_{\chi_F}^E \geq \underline{H}_{\chi_{F_{j+1}}}^E \geq \underline{H}_{\chi_{F_j}}^E \geq 0$  for all  $j$ . It therefore follows from Lemma 5 that

$$\underline{H}_{\chi_F}^E(p) \geq \lim_{j \rightarrow \infty} \underline{H}_{\chi_{F_j}}^E(p) = \lim_{j \rightarrow \infty} \overline{H}_{\chi_{F_j}}^E(p) = \overline{H}_{\chi_F}^E(p) \geq \underline{H}_{\chi_F}^E(p).$$

Hence

$$\underline{H}_{\chi_F}^E(p) = \overline{H}_{\chi_F}^E(p) = \lim_{j \rightarrow \infty} \overline{H}_{\chi_{F_j}}^E(p) = \lim_{j \rightarrow \infty} \omega_p^E(F_j) = \omega_p^E(F) = \int_{\partial E} \chi_F d\omega_p^E,$$

so that  $F \in \mathcal{F}$ . It follows that  $\mathcal{F}$  is a  $\sigma$ -algebra, and hence  $\mathcal{F} = \mathcal{B}$ .

Now we extend (4.1) to the characteristic functions of all  $\omega_p^E$ -measurable sets. Let  $A$  be such a set. Then we can write  $A = F \cup Y$  for some Borel set  $F$  and some subset  $Y$  of a Borel set  $Z$  with  $\omega_p^E(Z) = 0$ . Then  $\omega_p^E(A) = \omega_p^E(F)$ , and

$$\underline{H}_{\chi_A}^E(p) \leq \underline{H}_{\chi_A}^E(p) \leq \overline{H}_{\chi_A}^E(p) \leq \overline{H}_{\chi_{F \cup Z}}^E(p) \leq \overline{H}_{\chi_F}^E(p) + \overline{H}_{\chi_Z}^E(p),$$

by Lemmas 1 and 2. Since  $Z, F \in \mathcal{B}$ , we have

$$\overline{H}_{\chi_Z}^E(p) = \int_{\partial E} \chi_Z d\omega_p^E = 0,$$

and (4.1) with  $f = \chi_F$ . Hence

$$\underline{H}_{\chi_A}^E(p) \leq \underline{H}_{\chi_A}^E(p) \leq \overline{H}_{\chi_A}^E(p) \leq \overline{H}_{\chi_F}^E(p) = \int_{\partial E} \chi_F d\omega_p^E = \underline{H}_{\chi_F}^E(p).$$

Therefore equality holds throughout, and so

$$\underline{H}_{\chi_A}^E(p) = \overline{H}_{\chi_A}^E(p) = \overline{H}_{\chi_F}^E(p) = \int_{\partial E} \chi_F d\omega_p^E = \omega_p^E(F) = \omega_p^E(A) = \int_{\partial E} \chi_A d\omega_p^E.$$

Thus (4.1) holds with  $f = \chi_A$ .

Our next step is to extend (4.1) to all nonnegative,  $\omega_p^E$ -measurable, simple functions on  $\partial E$ . Suppose that  $f$  can be written in the form  $f = \sum_{i=1}^k \alpha_i \chi_{A_i}$ , for some positive numbers  $\alpha_1, \dots, \alpha_k$  and  $\omega_p^E$ -measurable sets  $A_1, \dots, A_k$ . Then (4.1) holds for each function  $\chi_{A_i}$ , and therefore Lemmas 1 and 2 can be used to show that

$$\sum_{i=1}^k \alpha_i \omega_p^E(A_i) = \sum_{i=1}^k \alpha_i \underline{H}_{\chi_{A_i}}^E(p) \leq \underline{H}_f^E(p) \leq \overline{H}_f^E(p) \leq \sum_{i=1}^k \alpha_i \overline{H}_{\chi_{A_i}}^E(p) \leq \sum_{i=1}^k \alpha_i \omega_p^E(A_i).$$

Hence

$$\underline{H}_f^E(p) = \overline{H}_f^E(p) = \sum_{i=1}^k \alpha_i \omega_p^E(A_i) = \sum_{i=1}^k \alpha_i \int_{\partial E} \chi_{A_i} d\omega_p^E = \int_{\partial E} f d\omega_p^E,$$

so that (4.1) holds for  $f$ .

We now consider the case where  $f$  is an arbitrary nonnegative,  $\omega_p^E$ -measurable function on  $\partial E$ . We write  $f$  as the limit of an increasing sequence  $\{g_j\}$  of nonnegative,  $\omega_p^E$ -measurable, simple functions on  $\partial E$ . Since (4.1) holds for each function  $g_j$ , the Lebesgue monotone convergence theorem gives

$$\underline{H}_{g_j}^E(p) = \overline{H}_{g_j}^E(p) = \int_{\partial E} g_j d\omega_p^E \rightarrow \int_{\partial E} f d\omega_p^E.$$

Moreover, using Lemma 2 we obtain

$$\underline{H}_f^E(p) \geq \lim_{j \rightarrow \infty} \underline{H}_{g_j}^E(p) = \lim_{j \rightarrow \infty} \overline{H}_{g_j}^E(p).$$

Each function  $g_j$  is bounded, so that each  $\overline{H}_{g_j}^E$  is also bounded, and hence Lemma 5 can be used to show that  $\lim_{j \rightarrow \infty} \overline{H}_{g_j}^E(p) = \overline{H}_f^E(p)$ . Since  $\overline{H}_f^E \geq \underline{H}_f^E$ , it follows that

$$\underline{H}_f^E(p) = \overline{H}_f^E(p) = \int_{\partial E} f d\omega_p^E,$$

as required.

Finally, we let  $f$  be an arbitrary  $\omega_p^E$ -measurable function for which  $\int_{\partial E} f d\omega_p^E$  exists. Then (4.1) holds for the positive and negative parts of  $f$ , so that Lemma 2 gives

$$\int_{\partial E} f d\omega_p^E = \overline{H}_{f^+}^E(p) - \underline{H}_{f^-}^E(p) = \overline{H}_{f^+}^E(p) + \overline{H}_{-f^-}^E(p) \geq \overline{H}_f^E(p),$$

and also

$$\int_{\partial E} f d\omega_p^E = \underline{H}_{f^+}^E(p) - \overline{H}_{f^-}^E(p) = \underline{H}_{f^+}^E(p) + \underline{H}_{-f^-}^E(p) \leq \underline{H}_f^E(p) \leq \overline{H}_f^E(p),$$

with the help of Lemma 1. Now (4.1) follows.

(b) Since  $\overline{H}_f^E(p)$  is finite, it follows from Lemma 9 that, given any positive integer  $j$ , we can find a lower finite, lower semicontinuous function  $g_j$  on  $\partial E$  such that  $f \leq g_j$  on  $\partial E$  and

$$\overline{H}_{g_j}^E(p) < \overline{H}_f^E(p) + \frac{1}{j}.$$

Furthermore, because  $\underline{H}_f^E(p)$  is finite, Lemma 9 also shows that we can find an upper finite, upper semicontinuous function  $h_j$  on  $\partial E$  such that  $h_j \leq f$  on  $\partial E$  and

$$\underline{H}_{h_j}^E(p) > \underline{H}_f^E(p) - \frac{1}{j}.$$

We put

$$g = \inf_j g_j, \quad h = \sup_j h_j,$$

and note that  $g, h$  are Borel measurable and satisfy  $h \leq f \leq g$  on  $\partial E$ . By Lemma 8,

$$\overline{H}_f^E(p) = \inf_j \overline{H}_{g_j}^E(p) = \inf_j \int_{\partial E} g_j d\omega_p^E \geq \int_{\partial E} g d\omega_p^E.$$

Moreover, by Lemmas 2 and 8 we have

$$\begin{aligned} \underline{H}_f^E(p) &= -\overline{H}_{-f}^E(p) = -\inf_j \overline{H}_{-h_j}^E(p) = -\inf_j \int_{\partial E} (-h_j) d\omega_p^E \\ &= \sup_j \int_{\partial E} h_j d\omega_p^E \leq \int_{\partial E} h d\omega_p^E. \end{aligned}$$

Hence

$$\underline{H}_f^E(p) \leq \int_{\partial E} h d\omega_p^E \leq \int_{\partial E} g d\omega_p^E \leq \overline{H}_f^E(p) = \underline{H}_f^E(p) \in \mathbb{R},$$

so that  $h = g$   $\omega_p^E$ -almost everywhere on  $\partial E$ . Since  $g$  and  $h$  are Borel measurable, it follows that there is a Borel set  $Z$  such that  $\omega_p^E(Z) = 0$  and  $h = f = g$  on  $(\partial E) \setminus Z$ . All subsets of  $Z$  are  $\omega_p^E$ -measurable, so that  $f$  is an  $\omega_p^E$ -measurable function and

$$\underline{H}_f^E(p) \leq \int_{\partial E} f d\omega_p^E \leq \overline{H}_f^E(p) = \underline{H}_f^E(p) \in \mathbb{R}.$$

Thus  $f$  is  $\omega_p^E$ -integrable (and (4.1) holds). □

**Corollary 1.** *Let  $E$  be an open set, and let  $f$  be an extended real-valued function on  $\partial E$ . If, for each point  $q_0 \in E$ , we can find a point  $p_0 \in E$  such that  $q_0 \in \Lambda(p_0; E)$ ,  $f$  is  $\omega_{p_0}^E$ -measurable, and both  $\overline{H}_f^E(p_0)$  and  $\underline{H}_f^E(p_0)$  are finite, then  $f$  is  $H$ -resolutive for  $E$  with*

$$H_f^E(p) = \int_{\partial E} f d\omega_p^E$$

for all  $p \in E$ .

**Proof.** Let  $q_0 \in E$ , and let  $p_0$  be a point as described in the statement of the theorem. Since  $f$  is  $\omega_{p_0}^E$ -measurable, so is  $f^+$ . Therefore Theorem 6(a) gives

$$\overline{H}_{f^+}^E(p_0) = \underline{H}_{f^+}^E(p_0) = \int_{\partial E} f^+ d\omega_{p_0}^E.$$

Since  $\overline{H}_f^E(p_0) < +\infty$ , there is a hypertemperature  $w \in \overline{\mathfrak{H}}_f^E$  such that  $w(p_0) < +\infty$ , and since  $w$  is lower bounded on  $E$ , there is a number  $\alpha$  such that  $w + \alpha \in \overline{\mathfrak{H}}_{f^+}^E$ . Therefore  $\overline{H}_{f^+}^E(p_0) < +\infty$ , and obviously  $\overline{H}_{f^+}^E(p_0) > -\infty$ . Lemma 6 now shows that  $f^+$  is H-resolutive for  $E$ . In particular, for all  $p \in E$  we have  $\overline{H}_{f^+}^E(p) = \underline{H}_{f^+}^E(p) \in \mathbb{R}$ , so that  $f^+$  is  $\omega_p^E$ -integrable and

$$H_{f^+}^E(p) = \int_{\partial E} f^+ d\omega_p^E$$

by Theorem 6(b). This result holds if  $f$  is replaced by  $-f$  because, by Lemma 2,  $\overline{H}_{-f}^E(p_0) = -\underline{H}_f^E(p_0)$  and  $\underline{H}_{-f}^E(p_0) = -\overline{H}_f^E(p_0)$ , which are finite; so  $f^+$  can be replaced by  $(-f)^+ = f^-$ . Therefore, by Theorem 2, the function  $f = f^+ - f^-$  is H-resolutive and

$$H_f^E(p) = H_{f^+}^E(p) - H_{f^-}^E(p) = \int_{\partial E} f^+ d\omega_p^E - \int_{\partial E} f^- d\omega_p^E = \int_{\partial E} f d\omega_p^E.$$

□

**Corollary 2.** *Let  $E$  be an open set, and let  $f$  be an extended real-valued function on  $\partial E$ . Then the following statements are equivalent:*

- (a)  $f$  is H-resolutive for  $E$ ;
- (b) for each point  $q_0 \in E$ , we can find a point  $p_0 \in E$  such that  $q_0 \in \Lambda(p_0; E)$  and  $f$  is  $\omega_{p_0}^E$ -integrable;
- (c)  $f$  is  $\omega_p^E$ -integrable for all  $p \in E$ .

If these statements hold, then

$$H_f^E(p) = \int_{\partial E} f d\omega_p^E$$

for all  $p \in E$ .

**Proof.** If statement (a) holds, then Theorem 6(b) shows that statement (c) holds also. If (c) holds, then obviously (b) holds too. Now suppose that (b) holds, and let  $q_0 \in E$ . Then we can find a point  $p_0 \in E$  such that  $q_0 \in \Lambda(p_0; E)$  and  $f$  is  $\omega_{p_0}^E$ -integrable, so that

$$\underline{H}_f^E(p_0) = \overline{H}_f^E(p_0) = \int_{\partial E} f d\omega_{p_0}^E$$

by Theorem 6(a), and the integral is finite. It now follows from Lemma 6 that (a) holds, and so the equivalence of the three statements is established.

Finally, if statement (a) holds, then

$$H_f^E(p) = \int_{\partial E} f d\omega_p^E$$

for all  $p \in E$ , by Theorem 6(b). □

It follows from Corollary 2 that, if  $A$  is a subset of  $\partial E$  which is  $\omega_p^E$ -measurable for all  $p \in E$ , then its characteristic function  $\chi_A$  is H-resolutive and  $H_{\chi_A}^E(p) = \omega_p^E(A)$  for all  $p \in E$ . Therefore, if  $\omega_{p_0}^E(A) = 0$  for some point  $p_0 \in E$ , then  $\omega_p^E(A) = 0$  for all  $p \in \Lambda(p_0; E)$ , by the minimum principle.

**5. The Equivalence of H-Resolutivity and S-Resolutivity**

Let  $f$  be an extended real-valued function on the boundary of an open set  $E$ . We have already observed that  $\underline{H}_f^E \leq L_f^E \leq U_f^E \leq \overline{H}_f^E$  on  $E$ , so that if  $f$  is H-resolutive for  $E$  then it is S-resolutive for  $E$ , with  $H_f^E = S_f^E$  on  $E$ . As a necessary step towards proving the converse, we now show that the caloric and parabolic measures coincide.

**Theorem 7.** *Let  $E$  be an open set, and let  $p \in E$ . We extend the caloric measure  $\mu_p^E$  on  $\partial_e E$ , to a measure  $\bar{\mu}_p^E$  on  $\partial E$ , by putting  $\bar{\mu}_p^E(\partial_s E) = 0$ . Then the parabolic measure  $\omega_p^E$  is equal to  $\bar{\mu}_p^E$  on  $\partial E$ .*

**Proof.** By [9, Theorem 8.27] or [10, Theorem 4.1], we have

$$S_f^E(p) = \int_{\partial_e E} f d\mu_p^E = \int_{\partial E} f d\bar{\mu}_p^E$$

for all  $f \in C(\partial E)$ . By Theorem 5, there is a unique nonnegative Borel measure  $\omega_p^E$  on  $\partial E$  such that

$$H_f^E(p) = \int_{\partial E} f d\omega_p^E$$

for all  $f \in C(\partial E)$ . Since  $S_f^E = H_f^E$  on  $E$  whenever the latter function exists, we also have the representation

$$H_f^E(p) = \int_{\partial E} f d\bar{\mu}_p^E$$

for all  $f \in C(\partial E)$ . Therefore the uniqueness assertion in Theorem 5 shows that  $\omega_p^E = \bar{\mu}_p^E$  on  $\partial E$ . □

*Remark.* Theorem 7 shows that the parabolic measure  $\omega_p^E$  is supported in  $\partial_e E$ , which was proved earlier by Suzuki [6].

**Corollary.** *Let  $E$  be an open set, let  $p_0 \in E$ , and put  $\Lambda = \Lambda(p_0; E)$ . Then for any point  $p \in \Lambda$ , the parabolic measure  $\omega_p^E$  is supported in  $\partial_e \Lambda$ , and  $\omega_p^\Lambda$  is the restriction to  $\partial_e \Lambda$  of  $\omega_p^E$ .*

**Proof.** Since  $\omega_p^E = \bar{\mu}_p^E$  on  $\partial E$ , the result follows from [9, Lemma 8.29] or [10, Lemma 4.3]. □

We can now combine Theorem 6 Corollary 2 with [9, Corollary 8.34] or [10, Corollary 4.8] to obtain the equivalence of H-resolutivity and S-resolutivity.

**Theorem 8.** *Let  $E$  be an open set, and let  $f$  be an extended real-valued function on  $\partial E$ . Then  $f$  is H-resolutive for  $E$  if and only if  $f$  is S-resolutive for  $E$ .*

**Proof.** If  $f$  is S-resolutive for  $E$  then, by [9, Corollary 8.34] or [10, Corollary 4.8],  $f$  is  $\bar{\mu}_p^E$ -integrable for all  $p \in E$ . Therefore  $f$  is  $\omega_p^E$ -integrable for all  $p \in E$ , by Theorem 7, and hence  $f$  is H-resolutive for  $E$ , by Theorem 6 Corollary 2.

The converse has already been demonstrated. □

Despite Theorem 8, consideration of the S-Dirichlet problem often gives sharper, more precise results, than does consideration of the H-Dirichlet problem. However, the relation between reductions and Dirichlet solutions is much easier to establish using H-Dirichlet solutions. The proof of the following theorem incorporates both of these facts.

**Theorem 9.** *Let  $E$  be an open set, let  $D$  be an open subset of  $E$ , and let  $u$  be a nonnegative supertemperature on  $E$ . Let  $f$  be defined on  $\partial D$  by*

$$f(q) = \begin{cases} u(q) & \text{if } q \in \partial D \cap E, \\ 0 & \text{if } q \in \partial D \setminus E. \end{cases}$$

Then

$$H_f^D = S_f^D = R_u^{E \setminus (D \cup \partial_s D)} = R_u^{E \setminus D}$$

on  $D$ .

**Proof.** We first prove that  $H_f^D = R_u^{E \setminus D}$  on  $D$ . Let  $v$  be a supertemperature in  $\overline{\mathfrak{H}}_f^D$ . Then  $\liminf_{p \rightarrow q} v(p) \geq f(q) \geq 0$  for all points  $q \in \partial D$ , so that  $v \geq 0$  on  $D$  by the boundary minimum principle. We put

$$w = \begin{cases} u \wedge v & \text{on } D, \\ u & \text{on } E \setminus D, \end{cases}$$

and note that  $w \geq 0$  on  $E$ . Moreover,  $w$  is a supertemperature on  $E$ , by [2, Satz 1.3.10] or [9, Lemma 7.20]. Since  $w = u$  on  $E \setminus D$ , we have  $w \geq R_u^{E \setminus D}$  on  $E$ . Therefore  $v \geq R_u^{E \setminus D}$  on  $D$ , and it follows that  $\overline{H}_f^D \geq R_u^{E \setminus D}$  on  $D$ .

Now suppose, instead, that  $v$  is a nonnegative supertemperature on  $E$  such that  $v \geq u$  on  $E \setminus D$ . Then for all points  $q \in \partial D \cap E$ , we have

$$\liminf_{p \rightarrow q, p \in D} v(p) \geq \liminf_{p \rightarrow q, p \in E} v(p) = v(q) \geq u(q) = f(q) \geq 0.$$

Moreover, for all points  $q \in \partial D \setminus E$ , we obviously have

$$\liminf_{p \rightarrow q, p \in D} v(p) \geq 0 = f(q).$$

Hence the restriction of  $v$  to  $D$  belongs to  $\overline{\mathfrak{H}}_f^D$ , so that  $v \geq \overline{H}_f^D$  on  $D$ . It follows that  $R_u^{E \setminus D} \geq \overline{H}_f^D$  on  $D$ , and hence that equality holds. Since  $f$  is H-resolutive, by Lemma 8 Corollary, we have established that  $H_f^D = R_u^{E \setminus D}$  on  $D$ .

By [11, Theorem 2.5], we have  $S_f^D \leq R_u^{E \setminus (D \cup \partial_s D)}$  on  $D$ . Hence

$$H_f^D = S_f^D \leq R_u^{E \setminus (D \cup \partial_s D)} \leq R_u^{E \setminus D} = H_f^D$$

on  $D$ , and the result follows.  $\square$

*Remark.* The proof that  $\overline{H}_f^D = R_u^{E \setminus D}$  goes back to [2, Satz 4.1.4], for the case where  $\overline{D} \subseteq E$ .

## 6. A Property of Caloric Measure

In order to prove our next theorem, we need the following lemma. The lemma was given in [2, Lemma 4.2.4] in the context of harmonic spaces, but the proof contained a significant error, which we correct in the present context.

**Lemma 10.** *Let  $E$  be an open set, let  $D$  be an open subset of  $E$ , and let  $f$  be an extended real-valued function defined on  $\partial E$  for which  $\overline{H}_f^E$  is a temperature. If  $g$  is defined on  $\partial D$  by*

$$g(q) = \begin{cases} f(q) & \text{if } q \in \partial D \cap \partial E, \\ \overline{H}_f^E(q) & \text{if } q \in \partial D \cap E, \end{cases}$$

then  $\overline{H}_g^D = \overline{H}_f^E$  on  $D$ .

**Proof.** Let  $u \in \overline{\mathfrak{H}}_f^E$ , and let  $v$  be its restriction to  $D$ . Clearly  $v$  is a lower bounded hypertemperature on  $D$ . Moreover, for any point  $q \in \partial D \cap \partial E$ , we have

$$\liminf_{p \rightarrow q, p \in D} v(p) \geq \liminf_{p \rightarrow q, p \in E} u(p) \geq f(q) = g(q);$$

and for any  $q \in \partial D \cap E$ , we have

$$\liminf_{p \rightarrow q, p \in D} v(p) \geq \liminf_{p \rightarrow q, p \in E} u(p) = u(q) \geq \overline{H}_f^E(q) = g(q).$$

Consequently  $v \in \overline{\mathfrak{H}}_g^D$ , and so  $u = v \geq \overline{H}_g^D$  on  $D$ . Since  $u$  is arbitrary, it follows that  $\overline{H}_f^E \geq \overline{H}_g^D$  on  $D$ .

We now take any function  $v \in \overline{\mathfrak{H}}_g^D$ , and define a function  $u$  on  $E$  by putting

$$u(p) = \begin{cases} \overline{H}_f^E(p) & \text{if } p \in E \setminus D, \\ v(p) \wedge \overline{H}_f^E(p) & \text{if } p \in D. \end{cases}$$

For any point  $q \in \partial D \cap E$ , we have

$$\liminf_{p \rightarrow q} v(p) \geq g(q) = \overline{H}_f^E(q),$$

and so it follows from [2, Satz 1.3.10] (or the proof of [9, Lemma 7.20]) that  $u$  is a hypertemperature on  $E$ . Given any function  $w \in \overline{\mathfrak{H}}_f^E$ , we consider the function

$$w_1 = w + u - \overline{H}_f^E,$$

which is a hypertemperature on  $E$  because  $\overline{H}_f^E$  is a temperature. We show that  $w_1 \in \overline{\mathfrak{H}}_f^E$ . We put

$$A = \{p \in E : u(p) = \overline{H}_f^E(p)\} \quad \text{and} \quad B = \{p \in D : u(p) = v(p)\},$$

and note that  $E = A \cup B$ . For each point  $q \in \partial E \cap \partial A$ , we have

$$\liminf_{p \rightarrow q, p \in A} w_1(p) = \liminf_{p \rightarrow q, p \in A} w(p) \geq \liminf_{p \rightarrow q, p \in E} w(p) \geq f(q);$$

and for each point  $q \in \partial E \cap \partial B$ , we have

$$\liminf_{p \rightarrow q, p \in B} w_1(p) \geq \liminf_{p \rightarrow q, p \in B} u(p) = \liminf_{p \rightarrow q, p \in B} v(p) \geq \liminf_{p \rightarrow q, p \in D} v(p) \geq g(q) = f(q);$$

thus  $\liminf_{p \rightarrow q} w_1(p) \geq f(q)$  for all  $q \in \partial E$ . Now  $w_1 = w$  on  $E \setminus D$ , and at those points of  $D$  where  $\overline{H}_f^E \leq v$ . At other points of  $D$  we have  $u = v$ , so that  $w_1 = w + v - \overline{H}_f^E \geq v$ . Since both  $w$  and  $v$  are lower bounded,  $w_1$  is too. Thus  $w_1 \in \overline{\mathfrak{H}}_f^E$ , and so  $w_1 \geq \overline{H}_f^E$ . Given any point  $p_0 \in D$  and any positive number  $\epsilon$ , we can choose  $w$  such that  $w(p_0) \leq \overline{H}_f^E(p_0) + \epsilon$ . It then follows that

$$\overline{H}_f^E(p_0) \leq w_1(p_0) = w(p_0) + u(p_0) - \overline{H}_f^E(p_0) \leq u(p_0) + \epsilon \leq v(p_0) + \epsilon,$$

and therefore that  $\overline{H}_f^E(p_0) \leq v(p_0)$ . This holds for all  $p_0 \in D$  and every  $v \in \overline{\mathfrak{H}}_g^D$ . Therefore  $\overline{H}_f^E \leq \overline{H}_g^D$  on  $D$ , and hence equality holds.  $\square$

The next theorem and its proof were suggested by [1, Theorem 6.4.8].

**Theorem 10.** *Let  $E$  be an open set, let  $D$  be an open subset of  $E$ , let  $q \in D$ , and let  $A \subseteq \partial_e E \cap \partial_e D$ . If  $A$  is  $\mu_q^E$ -measurable, then it is also  $\mu_q^D$ -measurable with  $\mu_q^D(A) \leq \mu_q^E(A)$ , and equality holds if  $\partial_e D \subseteq \partial E$ .*

**Proof.** It follows from Theorem 7 that  $A$  is  $\omega_q^E$ -measurable, and that we need to prove that  $A$  is  $\omega_q^D$ -measurable with  $\omega_q^D(A) \leq \omega_q^E(A)$ , and equality holds if  $\partial_e D \subseteq \partial E$ . We denote by  $\chi_A$  the characteristic function of  $A$  on  $\mathbb{R}^{n+1}$ . We define functions  $f$  and  $g$  on  $\partial D$  by putting  $f = \chi_A = g$  on  $\partial D \cap \partial E$ ,  $f = \overline{H}_{\chi_A}^E$  on  $\partial D \cap E$ , and  $g = \underline{H}_{\chi_A}^E$  on  $\partial D \cap E$ . Since  $A$  is  $\omega_q^E$ -measurable, Theorem 6(a) shows that

$$\overline{H}_{\chi_A}^E(q) = \underline{H}_{\chi_A}^E(q) = \int_{\partial E} \chi_A d\omega_q^E = \omega_q^E(A).$$

Because  $0 \leq \chi_A \leq 1$  on  $\partial E$ , we have  $0 \leq \underline{H}_{\chi_A}^E \leq \overline{H}_{\chi_A}^E \leq 1$  on  $E$ , so that  $\underline{H}_{\chi_A}^E$  and  $\overline{H}_{\chi_A}^E$  are both temperatures on  $E$ , by Lemma 4 Corollary. It therefore follows from Lemma 10 that  $\overline{H}_f^D = \overline{H}_{\chi_A}^E$  and  $\underline{H}_g^D = \underline{H}_{\chi_A}^E$  on  $D$ . In particular,  $\overline{H}_f^D(q) = \omega_q^E(A) = \underline{H}_g^D(q)$ . Since  $g \leq f$  on  $\partial D$ , it follows that

$$\omega_q^E(A) = \underline{H}_g^D(q) \leq \underline{H}_f^D(q) \leq \overline{H}_f^D(q) = \omega_q^E(A).$$

In particular,  $\underline{H}_f^D(q) = \overline{H}_f^D(q)$  and is finite, so that Theorem 6(b) shows that  $f$  is  $\omega_q^D$ -measurable and

$$\overline{H}_f^D(q) = \int_{\partial D} f d\omega_q^D.$$

Since  $f \geq \chi_A$  on  $\partial D$ , it follows that

$$\omega_q^D(A) = \int_{\partial D} \chi_A d\omega_q^D \leq \int_{\partial D} f d\omega_q^D = \overline{H}_f^D(q) = \omega_q^E(A),$$

as required. If  $\partial_e D \subseteq \partial E$ , then  $f = \chi_A$  on  $\partial_e D$ , and the fact that  $\omega_q^D$  is supported in  $\partial_e D$  implies that

$$\omega_q^D(A) = \int_{\partial_e D} \chi_A d\omega_q^D = \int_{\partial_e D} f d\omega_q^D = \omega_q^E(A).$$

$\square$

*Remark.* The condition  $\partial_e D \subseteq \partial E$ , of Theorem 10, holds if  $D = \Lambda(p_0; E)$  for some point  $p_0 \in E$ , by [9, Lemma 8.4]. If  $D$  is a component of  $E$ , the condition also holds.

### 7. The Equality of Upper H-Solutions and Upper S-Solutions

The results of Section 6 illustrate the fact that we sometimes need to consider boundary functions that are not necessarily resolutive. Our final theorem shows that, for any boundary function  $f$ , the equality  $\overline{H}_f^E = U_f^E$  holds on  $E$ . Thus the PWB method does not distinguish between the two forms of the Dirichlet problem. The result and its proof were suggested by [2, Satz 4.1.7].

**Theorem 11.** *Let  $E$  be an open set, and let  $f$  be an extended real-valued function on  $\partial E$ . Then  $\overline{H}_f^E = U_f^E$  on  $E$ .*

**Proof.** Let  $p \in E$ . We denote by  $\Psi$  the class of all lower semicontinuous, lower bounded majorants of  $f$  on  $\partial E$ . Any element  $\psi$  of  $\Psi$  is  $\omega_p^E$ -measurable, so that Theorem 6(a) and Theorem 7 yield

$$\overline{H}_\psi^E(p) = \int_{\partial E} \psi d\omega_p^E = \int_{\partial E} \psi d\bar{\mu}_p^E = \int_{\partial_e E} \psi d\mu_p^E. \tag{7.1}$$

We take any function  $u \in \overline{\mathfrak{H}}_f^E$ , and define  $\phi$  by putting  $\phi(q) = \liminf_{r \rightarrow q} u(r)$  for all  $q \in \partial E$ . Then  $\phi \in \Psi$ , and also  $u \in \overline{\mathfrak{H}}_\phi^E$ . It follows that

$$\overline{H}_f^E(p) \leq \inf_{\psi \in \Psi} \overline{H}_\psi^E(p) \leq \overline{H}_\phi^E(p) \leq u(p).$$

Taking the infimum over all choices of  $u$ , and using (7.1), we obtain

$$\overline{H}_f^E(p) = \inf_{\psi \in \Psi} \overline{H}_\psi^E(p) = \inf_{\psi \in \Psi} \int_{\partial_e E} \psi d\mu_p^E. \tag{7.2}$$

We now denote by  $\Delta$  the class of all lower semicontinuous, lower bounded majorants of  $f$  on  $\partial_e E$ . Any element  $\delta$  of  $\Delta$  is  $\mu_p^E$ -measurable, so that [9, Theorem 8.32(a)] or [10, Theorem 4.6(a)] yields

$$U_\delta^E(p) = \int_{\partial_e E} \delta d\mu_p^E. \tag{7.3}$$

We take any function  $v \in \mathfrak{U}_f^E$ , and define  $\varphi$  by putting

$$\varphi(q) = \begin{cases} \liminf_{r \rightarrow q} v(r) & \text{if } q \in \partial_n E, \\ \liminf_{r \rightarrow q+} v(r) & \text{if } q \in \partial_{ss} E. \end{cases}$$

Then  $\varphi \in \Delta$ , and also  $v \in \mathfrak{U}_\varphi^E$ . It follows that

$$U_f^E(p) \leq \inf_{\delta \in \Delta} U_\delta^E(p) \leq U_\varphi^E(p) \leq v(p).$$

Taking the infimum over all choices of  $v$ , and using (7.3), we obtain

$$U_f^E(p) = \inf_{\delta \in \Delta} U_\delta^E(p) = \inf_{\delta \in \Delta} \int_{\partial_e E} \delta d\mu_p^E. \tag{7.4}$$

The class  $\Delta$  is precisely the class of restrictions to  $\partial_e E$  of the functions in  $\Psi$ , because if  $\delta \in \Delta$  we can extend its definition to get a function in  $\Psi$  by putting  $\delta = +\infty$  on  $\partial_s E$ , since  $\partial_e E$  is closed. It now follows from (7.2) and (7.4) that

$$\overline{H}_f^E(p) = \inf_{\psi \in \Psi} \int_{\partial_e E} \psi d\mu_p^E = \inf_{\delta \in \Delta} \int_{\partial_e E} \delta d\mu_p^E = U_f^E(p).$$

Since  $p$  is arbitrary, the result is proved.  $\square$

**Corollary.** *Let  $E$  be an open set, let  $D$  be an open subset of  $E$ , and let  $f$  be an extended real-valued function defined on  $\partial E$  for which  $U_f^E$  is a temperature. If  $g$  is defined on  $\partial D$  by*

$$g(q) = \begin{cases} f(q) & \text{if } q \in \partial D \cap \partial E, \\ U_f^E(q) & \text{if } q \in \partial D \cap E, \end{cases}$$

then  $U_g^D = U_f^E$  on  $D$ .

**Proof.** In view of Theorem 11, the corollary is just Lemma 10.  $\square$

*Remark.* A direct proof of the corollary is elusive.

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