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ON A BINOMIAL COEFFICIENT AND A PRODUCT OF PRIME NUMBERS

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Let p_n be the n-th prime number. We prove the following double-inequality. For all integers $k \geq 5$ we have

$$\exp[k(c_0 - \log\log k)] \leq \frac{\binom{k^2}{k}}{p_1 \cdot p_2 \cdots p_k} \leq \exp[k(c_1 - \log\log k)]$$

with the best possible constants

$$c_0 = \frac{1}{5}\log 23 + \log\log 5 = 1.10298\ldots$$

and

$$c_1 = \frac{1}{192}\log\binom{36864}{192} + \log\log 192 - \frac{1}{192}\log\left(p_1 \cdot p_2 \cdots p_{192}\right) = 2.04287\ldots.$$

This refines a result published by Gupta and Khare in 1977.

1. INTRODUCTION

The work on this note has been inspired by a remarkable short paper published by Gupta and Khare [**4**] in 1977. The authors presented a connection between the binomial coefficient $\binom{k^2}{k}$ and the product of the first k prime numbers.

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Proposition. *If* $k = 3, 4, \ldots, 1793$, *then*

$$\binom{k^2}{k} > p_1 \cdot p_2 \cdots p_k,$$

whereas, if $k \geq 1794$, *then*

$$\binom{k^2}{k} < p_1 \cdot p_2 \cdots p_k.$$

The Proposition implies an interesting number theoretical theorem: if $1794 \leq k \leq n \leq k^2$, then $\binom{n}{k}$ has less than k distinct prime divisors. This improves an earlier result of Erdös, Gupta and Khare [**3**].

From the Proposition we obtain

$$(1.1) \qquad Q_k = \frac{\binom{k^2}{k}}{p_1 \cdot p_2 \cdots p_k} < 1 \quad \text{for} \quad k \geq 1794.$$

It is natural to look for a refinement of (1.1). More precisely, we ask for the largest number c_0 and the smallest number c_1 such that the double-inequality

$$(1.2) \qquad \exp[k(c_0 - \log\log k)] \leq Q_k \leq \exp[k(c_1 - \log\log k)]$$

holds for all $k \geq 5$. It is our aim to solve this problem. In the next section, we demonstrate that (1.2) is valid with $c_0 = 1.10298\ldots$ and $c_1 = 2.04287\ldots$. This provides not only a positive lower bound for Q_k, but improves (1.1) for all $k \geq 2237$.

2. MAIN RESULT

In order to offer sharp bounds for Q_k we need the following estimates.

Lemma 1. *For* $k \geq 2$ *we have*

$$(2.1) \qquad k^2 \log k - \left(k^2 - k + \frac{1}{2}\right)\log(k-1) - \frac{1}{2}\log(2\pi) - \frac{1}{12k} - \frac{1}{12(k-1)k^2}$$
$$< \log\binom{k^2}{k} < k^2 \log k - \left(k^2 - k + \frac{1}{2}\right)\log(k-1) - \frac{1}{2}\log(2\pi).$$

This lemma is due to Sasvári [**9**]. Moreover, we need upper and lower bounds for the Chebyshev function

$$\theta(x) = \sum_{p \leq x} \log p,$$

where p runs over all prime numbers $\leq x$.

Lemma 2. *We have for* $k \geq 3$

$$\frac{1}{k}\theta(p_k) = \frac{1}{k}\sum_{j=1}^{k}\log(p_j) \geq \log k + \log\log k - 1 + \frac{\log\log k - 2.1454}{\log k} \tag{2.2}$$

and for $k \geq 126$

$$\frac{1}{k}\theta(p_k) \leq \log k + \log\log k - 1 + \frac{\log\log k - 1.9185}{\log k}\,. \tag{2.3}$$

A proof of Lemma 2 can be found in [**7**]; see also [**2**]. We are now in a position to present our main result.

Theorem. *For all integers* $k \geq 5$ *we have*

$$\exp[k(c_0 - \log\log k)] \leq \frac{\binom{k^2}{k}}{p_1 \cdot p_2 \cdots p_k} \leq \exp[k(c_1 - \log\log k)] \tag{2.4}$$

with the best possible constants

$$c_0 = \frac{1}{5}\log 23 + \log\log 5 = 1.10298\ldots \tag{2.5}$$

and

$$c_1 = \frac{1}{192}\log\binom{36864}{192} + \log\log 192 - \frac{1}{192}\log\left(p_1 \cdot p_2 \cdots p_{192}\right) = 2.04287\ldots. \tag{2.6}$$

Proof. A short calculation reveals that (2.4) is equivalent to

$$c_0 \leq f(k) \leq c_1$$

with

$$f(k) = \frac{1}{k}\log\binom{k^2}{k} + \log\log k - \frac{1}{k}\theta(p_k). \tag{2.7}$$

First, we prove that

$$1.9114 < f(k) \quad \text{for} \quad k \geq 126. \tag{2.8}$$

Let $k \geq 126$. The left-hand side of (2.1) and the elementary inequalities

$$\frac{1}{k} < \log k - \log(k-1) < \frac{1}{k-1}$$

imply

$$\log\binom{k^2}{k} > k + \left(k - \frac{1}{2}\right)\log(k-1) - \alpha_0 > k + \left(k - \frac{1}{2}\right)\log k - \alpha_1 \tag{2.9}$$

with

$$\alpha_0 = \frac{1}{2}\log(2\pi) + \frac{1}{12\cdot 126} + \frac{1}{12\cdot 125\cdot 126\cdot 126}$$

and

$$\alpha_1 = \alpha_0 + 1 + \frac{1}{2\cdot 125} = 1.9235\ldots.$$

Using (2.3), (2.7), and (2.9) leads to

$$f(k) > 2 - \left(\frac{\log k}{2k} + \frac{1.9236}{k} + \frac{\log\log k - 1.9185}{\log k}\right).$$

Since

$$\frac{\log k}{2k} + \frac{1.9236}{k} < 0.0345 \quad \text{and} \quad \frac{\log\log k - 1.9185}{\log k} < 0.0541,$$

we get

$$f(k) > 2 - 0.0345 - 0.0541 = 1.9114\ldots.$$

This settles (2.8). By direct computation we find

$$\min_{5\le k\le 125} f(k) = f(5) = 1.10298\ldots. \tag{2.10}$$

From (2.8) and (2.10) we conclude that the first inequality in (2.4) holds for $k \ge 5$ with the best possible constant given in (2.5).

Applying the second inequality in (2.1), (2.2), and (2.7) yields for $k \ge 652$:

$$\begin{aligned} f(k) &< 1 + (k-1)\log\left(1 + \frac{1}{k-1}\right) - \frac{\log\bigl(2\pi(k-1)\bigr)}{2k} - \frac{\log\log k - 2.1454}{\log k} \\ &< 2 - \frac{\log\log k - 2.1454}{\log k} < 2.0428. \end{aligned} \tag{2.11}$$

Moreover, we have

$$\max_{5\le k\le 651} f(k) = f(192) = 2.04287\ldots. \tag{2.12}$$

From (2.11) and (2.12) we obtain that the right-hand side of (2.4) is valid with the best possible constant c_1 given in (2.6). □

3. REMARKS

(i) The behaviour of $f(k)$ for large k is quite surprising. We have the limit relation

$$\lim_{k\to\infty} f(k) = 2,$$

although computer calculations show that $f(k)$ is decreasing in the range $1000 \le k \le 1\,500\,000$ with $f(10\,000) = 1.9908\ldots$.

(ii) More inequalities involving the product $p_1 \cdot p_2 \cdots p_k$ can be found in [**1**], [**5**, p. 246], [**6**].
(iii) Let $\psi = \Gamma'/\Gamma$ be the logarithmic derivative of Euler's gamma function. We define for $x \geq 2$:

$$\phi(x) = \log \frac{\Gamma(x^2+1)}{\Gamma(x+1)\Gamma(x^2-x+1)}.$$

Differentiation gives

$$\phi'(x) = 2x[\psi(x^2+1) - \psi(x^2-x+1)] + [\psi(x^2-x+1) - \psi(x+1)].$$

Since ψ is strictly increasing on $(0,\infty)$, we conclude that $\phi'(x)$ is positive for $x \geq 2$. Hence,

$$\phi(k) = \log \binom{k^2}{k} < \log \binom{(k+1)^2}{k+1} = \phi(k+1) \quad \text{for} \quad k \geq 2.$$

We have $\phi(1) = 0 < \log 6 = \phi(2)$. Thus we obtain: If $\mu \geq 0$, then the sequence

$$\Delta_k(\mu) = \binom{k^2}{k} \exp(\mu\, p_k)$$

is strictly increasing for $k \geq 1$. This result leads to the question: Does there exist a negative real number μ_0 and an integer k_0 such that $\Delta_k(\mu_0)$ is increasing for $k \geq k_0$? We show that the answer in "no".

We assume that

$$\Delta_k(\mu_0) \leq \Delta_{k+1}(\mu_0) \quad \text{for} \quad k \geq k_0.$$

This is equivalent to, say

$$-\mu_0 \leq \frac{\log \binom{(k+1)^2}{k+1} - \log \binom{k^2}{k}}{p_{k+1} - p_k} = \sigma_k. \tag{3.1}$$

Using the asymptotic formula

$$\log \binom{k^2}{k} = \left(k - \frac{1}{2}\right) \log k + k - \frac{1}{2}\big(1 + \log(2\pi)\big) + O(1/k),$$

see [**4**], we get

$$\log \binom{(k+1)^2}{k+1} - \log \binom{k^2}{k} = \log k + 2 + O(1/k). \tag{3.2}$$

Applying (3.2) and the limit relations

$$\lim_{k\to\infty} \frac{\log p_k}{\log k} = 1 \quad \text{and} \quad \liminf_{k\to\infty} \frac{\log p_k}{p_{k+1} - p_k} = 0,$$

see [**8**], [**10**], yields

$$\lim_{k\to\infty} \inf \sigma_k = \lim_{k\to\infty} \inf \left(\frac{\log p_k}{p_{k+1} - p_k} \cdot \frac{\log k + 2 + O(1/k)}{\log p_k} \right) = 0. \tag{3.3}$$

From (3.1) and (3.3) we conclude that $\mu_0 \geq 0$.

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REFERENCES

1. H. Alzer, C. Berg: *Some classes of completely monotonic functions,* II. Ramanujan J., **11** (2006), 225–248.
2. P. Dusart: *Autour de la fonction qui compte le nombre de nombres premiers.* Thèse, Université de Limoges, 1998.
3. P. Erdös, H. Gupta, S. P. Khare: *On the number of distinct prime divisors of* $\binom{n}{k}$. Util. Math., **10** (1976), 51–60.
4. H. Gupta, S. P. Khare: *On* $\binom{k^2}{k}$ *and the product of the first k primes.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., **577–598** (1977), 25–29.
5. D. S. Mitrinović, J. Sándor, B. Crstici: *Handbook of Number Theory.* Kluwer, Dordrecht, 1996.
6. L. Panaitopol: *An inequality involving prime numbers.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., **11** (2000), 33–35.
7. G. Robin: *Estimation de la fonction de Tchebychef* θ *sur le* k*-ième nombre premier et grandes valeurs de la fonction* $\omega(n)$*, nombre de diviseurs premiers de* n. Acta Arith., **42** (1983), 367–389.
8. J. B. Rosser, L. Schoenfeld: *Approximate formulas for some functions of prime numbers.* Illinois J. Math., **6** (1962), 64–94.
9. Z. Sasvári: *Inequalities for binomial coefficients.* J. Math. Anal. Appl., **236** (1999), 223–226.
10. E. Westzynthius: *Über die Verteilung der Zahlen die zu den n ersten Primzahlen teilerfremd sind.* Comm. Phys. Math. Soc. Fenn., Helsingfors, **5** (1931), 1–37.

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