NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 38 (2008), 87–91

A FAMILY OF LACUNARY PARTITION FUNCTIONS

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(Received April 2008)

Abstract. We offer a new family of lacunary partition functions by using interesting properties of indefinite quadratic forms. In particular, we obtain a family of colored partition functions by paraphrasing some old and new q-series identities.

1. Introduction and Statement of Results

A lacunary function is a function on $\mathbb{N}$ that is zero for almost all natural numbers. Several authors (e.g. [3, 8]) have observed that certain partition functions exhibit this property. In fact, there are numerous examples (see [1]) of partition functions with distinct parts that are lacunary, and follow from the Jacobi triple product identity. The most famous example of this phenomenon in the theory of partitions is Euler's Pentagonal number theorem. That is, if $p_e(n)$ (resp. $p_o(n)$) denote the number of partitions into distinct parts of n with even (resp. odd) number of parts, then

$$p_e(n) - p_o(n) = \begin{cases} (-1)^m, & n = m(3m \pm 1)/2, \\ 0 & otherwise. \end{cases} \tag{1}$$

It follows directly from (1) that $p_e(n) = p_o(n)$ for almost all natural n.

In 1986, Andrews [2] proved the q-series identity

$$\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n-1}) = \sum_{\substack{n\geq 0 \\ |j|\leq n}} (-1)^j q^{n(3n+1)/2-j^2}(1-q^{2n+1}), \tag{2}$$

by use of Bailey pairs. It turns out we can paraphrase the left side of (2) using colored partitions in the same flavor as [5]. Furthermore, by using known results about indefinite quadratic forms, we relate some of our partition functions to the arithmetic of $\mathbb{Z}[\sqrt{2}]$, another to $\mathbb{Z}[\sqrt{6}]$.

Theorem 1.1. *Let S denote the set of partitions into 2 distinct colors with one color appearing at most once, and the other color appearing at most once and in parts that are even. Let $p_e(S,n)$ (resp. $p_o(S,n)$) denote the number of partitions of n taken from S with an even (resp. odd) number of distinct parts. For $n \equiv 1 \pmod 8$, let $\nu(n)$ be the excess of the number of inequivalent solutions of $x^2 - 2y^2 = n$ in which $x + y \equiv 1 \pmod 4$ over the number in which $x + y \equiv 3$*

1991 *Mathematics Subject Classification* 11B37, 11B75, 11P99.
Key words and phrases: q-series, partitions, indefinite quadratic forms.

(mod 4). *Then* $p_e(S,n) - p_o(S,n) = \nu(8n+1)$.

Theoren 1.2. *Let* B *be the set of partitions into 3 distinct colors with one color appearing at most once, and in parts that are odd, and the other 2 colors appear at most once and in parts that are even. Let* $p_e(B,n)$ *(resp.* $p_o(B,n)$*) denote the number of partitions of n taken from* B *with an even (resp. odd) number of distinct even parts. For* $n \equiv 1 \pmod 8$, *let* $\mu(n)$ *be the excess of the number of inequivalent solutions of* $x^2 - 2y^2 = n$ *in which* $x + 4y \equiv 1$ *or* $3 \pmod 8$ *over the number in which* $x + 4y \equiv 5$ *or* $7 \pmod 8$. *Then* $p_e(B,n) - p_o(B,n) = \mu(8n+1)$.

The next theorem is the arithmetic interpretation of a result equivalent to an identity given by Rowell [9].

Theorem 1.3. *Let* W *denote the set of partitions into 3 distinct colors with one color appearing at most once, and in parts that are odd, and the other 2 colors appear at most once and in parts that are multiples of 4. Let* $p_e(W,n)$ *(resp.* $p_o(W,n)$*) denote the number of partitions of n taken from* W *with an even (resp. odd) number of distinct parts. For* $n \equiv 7 \pmod{24}$, *let* $\kappa(n)$ *be the excess of the number of inequivalent solutions of* $x^2-2y^2 = n$ *in which* $x+2y \equiv 1,\ 3,$ *or* $5 \pmod{12}$ *over the number in which* $x+2y \equiv -1,\ -3,$ *or* $-5 \pmod{12}$. *Then* $p_e(B,n)-p_o(B,n) = \kappa(7-24n)$.

Our next result gives a combinatorial interpretation of (2) when q is replaced by $-q$.

Theorem 1.4. *Let* Z *denote the set of partitions into 5 distinct colors with 3 colors appearing at most once and in parts that are odd, and the other 2 colors appear at most once and in parts that are even. Let* $p_e(Z,n)$ *(resp.* $p_o(Z,n)$*) denote the number of partitions of n taken from* Z *with an even (resp. odd) number of distinct even parts. For* $n \equiv 1 \pmod{24}$, *let* $\alpha_+(n)$ *be the number of inequivalent solutions of* $x^2 - 6y^2 = n$ *when* $x + 12y \equiv 1, 5, 7,$ *or* $11 \pmod{24}$ *and* $\alpha_-(n)$ *when* $x + 12y \equiv 13, 17, 19,$ *or* $23 \pmod{24}$. *Then* $p_e(Z,n) - p_o(Z,n) = \alpha_+(24n+1) - \alpha_-(24n+1)$.

2. *Proofs of Theorems*

In order to prove our partition theorems we first need to obtain the desired generating functions, which follow directly from inserting some known Bailey pairs into the Bailey lemma [8, 10]. It should also be noted that we will employ standard q-series notation [7].

We define a pair of sequences (α_n, β_n) to be a Bailey pair with respect to a if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_n}{(aq;q)_{n+r}(q;q)_{n-r}}. \tag{3}$$

The q-series identities established herein follow directly from:

Proposition 2.1. *If* (α_n, β_n) *form a Bailey pair with respect to a then*

$$\sum_{n=0}^{\infty}(z)_n(y)_n(aq/zy)^n\beta_n = \frac{(aq/z)_\infty(aq/y)_\infty}{(aq)_\infty(aq/zy)_\infty}\sum_{n=0}^{\infty}\frac{(z)_n(y)_n(aq/zy)^n\alpha_n}{(aq/z)_n(aq/y)_n}. \tag{4}$$

Proof of Theorem 1.1: First recall [2] that (α_n, β_n) form a Bailey pair with respect to q with

$$\alpha_n = \frac{q^{n(3n+1)/2}(1-q^{2n+1})}{1-q}\sum_{j=-n}^{n}(-1)^j q^{-j^2},$$

$$\beta_n = \frac{1}{(-q)_n}.$$

Inserting this pair into (4) with $z = -q$ and $y \to \infty$ gives

$$(q)_\infty(q^2;q^2)_\infty = \sum_{\substack{n\geq 0\\ |j|\leq n}}^{\infty}(-1)^j q^{n(2n+1)-j^2}(1-q^{2n+1}). \tag{5}$$

Taking S to be the set of partitions into 2 distinct colors with both colors appearing at most once and one color having only even parts, we have

$$(q;q)_\infty(q^2;q^2)_\infty = \sum_{n=0}^{\infty}(p_e(S,n)-p_o(S,n))q^n.$$

Replacing q by q^8 in (5) and multiply both sides by q to obtain

$$\sum_{\substack{n\geq 0\\ |j|\leq n}}^{\infty}(-1)^j(q^{(4n+1)^2-2(2j)^2}-q^{(4n+3)^2-2(2j)^2}). \tag{6}$$

Now, note (see [6] for a related generating function) that this generates the number of inequivalent solutions of $x^2-2y^2=k$ with norm $8k+1$, where x is odd, y even, and an additional weight. Thus, we only need to consider when (11) is weighted by $+1$ and -1. Now we have $+1$ when $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 4$ or $x \equiv 3 \pmod 4$ and $y \equiv 2 \pmod 4$. We have -1 when $x \equiv 1 \pmod 4$ and $y \equiv 2 \pmod 4$ or $x \equiv 3 \pmod 4$ and $y \equiv 0 \pmod 4$. This gives us $+1$ when $x+y \equiv 1 \pmod 4$, and -1 when $x+y \equiv 3 \pmod 4$.

Proof of Theorem 1.2: Replace q by $-q$ in (5), then replace q by q^8 and multiply both sides by q to get

$$(q^2;q^2)^2_\infty(-q;q^2)_\infty = \sum_{\substack{n\geq 0\\ |j|\leq n}}^{\infty}(-1)^n(q^{(4n+1)^2-2(2j)^2}+q^{(4n+3)^2-2(2j)^2}).$$

Now

$$(q^2;q^2)^2_\infty(-q;q^2)_\infty = \sum_{n=0}^{\infty}(p_e(B,n)-p_o(B,n))q^n.$$

The connection with $\mathbb{Z}[\sqrt{2}]$ follows in the same manner as in the previous proof. However, this time, we have $+1$ when $x \equiv 1 \pmod 8$ or $x \equiv 3 \pmod 8$, and -1

when $x \equiv 5 \pmod 8$ or $x \equiv 7 \pmod 8$. Inspection on $x + 4y$ gives the desired result.

Proof of Theorem 1.3: Inserting the pair (α_n, β_n) [2] with respect to q with

$$\alpha_n = q^{n(2n+1)} \frac{(1-q^{2n+1})}{(1-q)} \sum_{j=-n}^{n} (-1)^j q^{-j(3j+1)/2},$$

and

$$\beta_n = 1.$$

into (4), and letting $z, y \to \infty$, we obtain the identity

$$\sum_{n=0}^{\infty} q^{n(n+1)} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{n(3n+2)}(1-q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-j(3j+1)/2}. \quad (7)$$

The sum on the left side of (7) can be summed by means of [1, pg. 23, e.q. (2.2.13)], which leads to the identity

$$(q;q^2)_\infty (q^4;q^4)_\infty^2 = \sum_{n=0}^{\infty} q^{n(3n+2)}(1-q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-j(3j+1)/2}, \quad (8)$$

since,

$$(q;q)_\infty \frac{(q^4;q^4)_\infty}{(q^2;q^4)_\infty} = (q;q^2)_\infty (q^4;q^4)_\infty^2. \quad (9)$$

The right side of (9) is the generating function for $p_e(W,n) - p_o(W,n)$, and the right hand is to be treated in the same manner as in the previous proofs. Replacing q by q^{24} and multiplying both sides by q^7 gives the generating function for the number of inequivalent solutions of $x^2 - 2y^2 = k$ with norm $7 - 24k$ in which $x + 2y \equiv 1$, 3, or 5 $\pmod{12}$ over the number in which $x + 2y \equiv -1$, -3, or $-5 \pmod{12}$.

Proof of Theorem 1.4: First, we recall from the work of [3] that the right hand side of (2) generates the number of inequivalent solutions of $x^2 - 6y^2 = k$, with norm $24k+1$ weighted by $+1$ when $2x + 3y \equiv 2$ or $4 \pmod{12}$ and -1 when $2x + 3y \equiv 8$ or $10 \pmod{12}$. Now replacing q by $-q$ in (2) gives us

$$(-q;q^2)_\infty^3 (q^2;q^2)_\infty^2 = \sum_{\substack{n \geq 0 \\ |j| \leq n}} (-1)^{n(3n+1)/2} q^{n(3n+1)/2 - j^2}(1+q^{2n+1}).$$

Replacing q by q^{24} in the sum on the right and multiplying both sides by q gives

$$\sum_{\substack{n \geq 0 \\ |j| \leq n}} (-1)^{n(3n+1)/2} (q^{(6n+1)^2 - 6(2j)^2} + q^{(6n+5)^2 - 6(2j)^2}).$$

Now we only need to consider when $(-1)^{n(3n+1)/2}$ is $+1$ and -1. When $n \equiv 0$ or $1 \pmod 4$ we have $+1$, and -1 when $n \equiv 2$ or $3 \pmod 4$. The remainder of the proof requires us to inspect $x + 12y$, and we leave the details to the reader.

3. Concluding remarks

Here we were able to observe a family of partition functions where lacunarity is easily established by use of indefinite quadratic forms. It is to be expected that there are further examples of partition functions in this family that have parts congruent to some other modulus, which will be dealt with in another paper.

Using the pair in the proof of Theorem 1.3 we may also show that

$$(q;q)_\infty^2 = \sum_{n\geq 0}^{\infty} q^{n(2n+1)}(1-q^{2n+1}) \sum_{j=-n}^{n} (-1)^j q^{-j(3j+1)/2}, \tag{10}$$

which will be investigated in future study. It should also be noted that Rowell [9] has proven the q-series identities herein by means of his new general conjugate Bailey pair. Also, (10) is due L. J Rogers (see [9] for this reference).

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