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DISCRETE HILBERT TYPE INEQUALITY WITH NON–HOMOGENEOUS KERNEL

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Dedicated to the Memory of Professor D. S. Mitrinović (1908–1995)

A new version of discrete Hilbert type inequality is given where the kernel function is non–homogeneous. The main mathematical tools are the representation of the Dirichlet series by means of the Laplace integral, and the Hölder inequality with non–conjugated parameters. Numerous special cases are treated and conditional best constants are discussed.

1. INTRODUCTION AND PRELIMINARIES

Let ℓ_p be the space of all complex sequences $\boldsymbol{x} = (x_n)_{n=1}^\infty$ with $\|\boldsymbol{x}\|_p := \left(\sum\limits_{n=1}^\infty |x_n|^p\right)^{1/p} < +\infty$. The famous Hilbert's double series theorem, frequently called a discrete Hilbert inequality too, reads as follows. Let $\boldsymbol{a} = (a_n)_{n=1}^\infty \in \ell_p$, $\boldsymbol{b} = (b_n)_{n=1}^\infty \in \ell_q$ be nonnegative sequences and $1/p + 1/q = 1,\ p > 1$. Then

$$\sum_{m,n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|\boldsymbol{a}\|_p \|\boldsymbol{b}\|_q \,, \tag{1}$$

where constant $\dfrac{\pi}{\sin(\pi/p)}$ is the best possible [**5**, p. 253].

Fundamental contributions have been given to this classical inequality by Hardy [**4**], Mulholland [**14**], [**15**], Bonsall [**1**] and Levin [**12**]. Discrete Hilbert inequalities with non–homogeneous kernels were studied in [**2**], [**3**], [**7**], [**8**]–[**11**], [**16**], [**17**], [**19**]–[**24**].

As already pointed out in [**17**], the standard way in deriving Hilbert's inequality is to apply the Hölder inequality to a suitably transformed Hilbert type

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double sum expression, that is, to the bilinear form

$$\mathfrak{H}_K^{\boldsymbol{a},\boldsymbol{b}} := \sum_{m,n=1}^{\infty} K(m,n)\, a_m b_n \tag{2}$$

where $\boldsymbol{a}, \boldsymbol{b}$ are nonnegative; $K(\cdot,\cdot)$ we call kernel function of the double series (2). So, to obtain discrete HILBERT type inequalities (or in other words - double series theorems) one derives sharp upper bounds for $\mathfrak{H}_K^{\boldsymbol{a},\boldsymbol{b}}$ in terms of weighted ℓ_p–norms of $\boldsymbol{a}, \boldsymbol{b}$.

In this article we use an approach different from [**17**] to get more general and in the same time simpler discrete HILBERT type inequalities. Namely, our main goal here will be to establish simplest possible sharp upper bounds over $\mathfrak{H}_K^{\boldsymbol{a},\boldsymbol{b}}$ in terms of $\|\boldsymbol{a}\|_p, \|\boldsymbol{b}\|_q$, when $K(m,n) = (\lambda_m + \rho_n)^{-\mu}$ is non–homogeneous, that is, we are looking for a sharp estimate of the form

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(\lambda_m + \rho_n)^{\mu}} \le \mathsf{C}^{\star}\, \|\boldsymbol{a}\|_p \|\boldsymbol{b}\|_q\,. \tag{3}$$

Additionally, we will obtain inequalities like (3) by HÖLDER inequality with non–conjugate parameters, i.e. when $p, q > 1$ and $p^{-1} + q^{-1} \ge 1$, introducing the new incremental parameter

$$\Delta := \frac{1}{p} + \frac{1}{q} - 1 \ge 0\,.$$

The non–conjugated parameters were already considered in the literature (see [**1**] or LEVIN [**12**], for instance). However, our strong reduction requirements have to be balanced by sharp, but fairly complicated constant $\mathsf{C}^{\star}$ in (3). Therefore, specifying the functions λ and ρ from one, and p, q from other hand we simplify the HILBERT type inequality step–by–step into a set of Corollaries. All our derived upper bounds are new and sharp when $\Delta = 0$.

Note that here, and in what follows $\mathcal{I}(x) = x$ denotes the identity and the LAPLACE integral of the DIRICHLET series reads [**18**, §5]

$$\mathcal{D}_\lambda(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = x \int_0^{\infty} e^{-xt} \Bigg(\sum_{n=1}^{[\lambda^{-1}(t)]} a_n \Bigg)\, \mathrm{d}t \tag{4}$$

for positive monotone increasing $(\lambda_n)_{n=1}^{\infty}$ satisfying (5).

2. MAIN RESULT

We are ready to state our principal inequality result.

Theorem. *Suppose* $p, q > 1$, $\mu > 0$, $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}} \in \ell_p$, $\boldsymbol{b} = (b_n)_{n\in\mathbb{N}} \in \ell_q$ *are non-negative sequences and* λ, ρ *are positive monotone increasing functions satisfying the condition*

$$\lim_{x\to\infty} \lambda(x) = \lim_{x\to\infty} \rho(x) = \infty\,. \tag{5}$$

Then

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(\lambda_m+\rho_n)^\mu} \le \mathsf{C}_{p,q}^{\mu,\Delta}(\lambda,\rho)\,\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q\,, \tag{6}$$

where

$$\mathsf{C}_{p,q}^{\mu,\Delta}(\lambda,\rho) = \frac{q^{1/q}p^{1/p}}{\Gamma(\mu)}\int_0^\infty x^{\mu+\Delta}\left(\int_{\lambda_1}^\infty e^{-qxt}[\lambda^{-1}(t)]\,\mathrm{d}t\right)^{1/q} \times\left(\int_{\rho_1}^\infty e^{-pxu}[\rho^{-1}(u)]\,\mathrm{d}u\right)^{1/p}\mathrm{d}x\,. \tag{7}$$

The equality in (6) *appears for* $\lambda=\rho=\mathcal{I}$, $p=q=2$ *when*

$$\frac{a_m}{b_n} = C\,\delta_{mn} \qquad \big(m,n\in\mathbb{N}\big), \tag{8}$$

where C *is an absolute constant and* δ_{mn} *denotes the Kronecker's delta.*

Proof. First, we transform the double series by means of the Gamma function formula $\Gamma(\mu)A^{-\mu} = \int_0^\infty x^{\mu-1}e^{-Ax}\,\mathrm{d}x$. After splitting the kernel function into two Dirichlet series, we evaluate these Dirichlet series by the Hölder inequality with non–conjugated parameters p, q, $\min\{p,q\}>1$, $p^{-1}+q^{-1}\ge 1$ [**13**, p. 57]. These transformations result in

$$\begin{aligned}\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(\lambda_m+\rho_n)^\mu} &= \frac{1}{\Gamma(\mu)}\int_0^\infty x^{\mu-1}\left(\sum_{m=1}^\infty a_m e^{-\lambda_m x}\right)\left(\sum_{n=1}^\infty b_n e^{-\rho_n x}\right)\mathrm{d}x \\ &\le \frac{\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q}{\Gamma(\mu)}\int_0^\infty x^{\mu-1}\left(\sum_{m=1}^\infty e^{-\lambda_m qx}\right)^{1/q}\left(\sum_{n=1}^\infty e^{-\rho_n px}\right)^{1/p}\mathrm{d}x\,.\end{aligned} \tag{9}$$

Now, the inner–most Dirichlet series

$$\mathcal{D}_\lambda(x) = \sum_{m=1}^\infty e^{-\lambda_m qx}, \qquad \mathcal{D}_\rho(x) = \sum_{n=1}^\infty e^{-\rho_n px},$$

via (4) clearly become

$$\mathcal{D}_\lambda(x) = xq\int_0^\infty e^{-xqt}\left(\sum_{j=1}^{[\lambda^{-1}(t)]} 1\right)\mathrm{d}t = xq\int_0^\infty e^{-xqt}[\lambda^{-1}(t)]\,\mathrm{d}t,$$

that is

$$\mathcal{D}_\rho(x) = xp\int_0^\infty e^{-xpu}[\rho^{-1}(u)]\,\mathrm{d}u\,.$$

Collecting all these expressions the upper bound in (9) becomes

$$\begin{aligned}\sum_{m,n=1}^{\infty}\frac{a_m b_n}{(\lambda_m+\rho_n)^{\mu}} &\le \frac{\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q}{\Gamma(\mu)}\int_0^{\infty}x^{\mu-1}\left(qx\int_{\lambda_1}^{\infty}e^{-qxt}[\lambda^{-1}(t)]\,\mathrm{d}t\right)^{1/q}\\ &\qquad\times\left(px\int_{\rho_1}^{\infty}e^{-pxu}[\rho^{-1}(u)]\,\mathrm{d}u\right)^{1/p}\mathrm{d}x\\ &= \frac{q^{1/q}p^{1/p}\,\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q}{\Gamma(\mu)}\int_0^{\infty}x^{\mu+\Delta}\left(\int_{\lambda_1}^{\infty}e^{-qxt}[\lambda^{-1}(t)]\mathrm{d}t\right)^{1/q}\\ &\qquad\times\left(\int_{\rho_1}^{\infty}e^{-pxu}[\rho^{-1}(u)]\mathrm{d}u\right)^{1/p}\mathrm{d}x\,.\end{aligned}$$

Finally, we discuss when equality holds in (6). Let us denote L, R the left, respectively right side of (6). Then making use of $a_m\, b_n = C\,\delta_{mn}$ when $\lambda=\rho=\mathcal{I}$, $p=q=2$, we get

$$L=\sum_{m,n=1}^{\infty}\frac{a_m/b_n\cdot b_n^2}{(m+n)^{\mu}}=\sum_{m,n=1}^{\infty}\frac{C\,\delta_{mn}b_n^2}{(m+n)^{\mu}}=\sum_{m,n=1}^{\infty}\frac{b_n^2}{(2m)^{\mu}}=\frac{\zeta(\mu)}{2^{\mu}}\,\|\boldsymbol{b}\|_2^2=R\,.$$

This completes the proof of the theorem. □

3. SPECIAL CASES

In this chapter we specify the parameters p, q and the functions λ, ρ getting a set of corollaries of Theorem.

3.1. If we take $\lambda(x)=Ax^q$, $\rho(x)=Bx^p$, their inverses are $\lambda^{-1}(x)=(x/A)^{1/q}$, $\rho^{-1}(x)=(x/B)^{1/p}$ and the related constant becomes

$$\begin{aligned}\mathsf{C}_{p,q}^{\mu,\Delta}(Ax^q,Bx^p) &= \frac{p^{1/p}q^{1/q}}{\Gamma(\mu)}\int_0^{\infty}x^{\mu+\Delta}\left(\int_A^{\infty}e^{-qtx}\left[\left(\frac{t}{A}\right)^{1/q}\right]\mathrm{d}t\right)^{1/q}\\ &\qquad\times\left(\int_B^{\infty}e^{-pux}\left[\left(\frac{u}{B}\right)^{1/p}\right]\mathrm{d}u\right)^{1/p}\mathrm{d}x\\ &= \frac{(Aq)^{1/q}(Bp)^{1/p}}{\Gamma(\mu)}\int_0^{\infty}x^{\mu+\Delta}\left(\int_A^{\infty}e^{-qAtx}[t^{1/q}]\,\mathrm{d}t\right)^{1/q}\\ &\qquad\times\left(\int_B^{\infty}e^{-pBux}[u^{1/p}]\mathrm{d}u\right)^{1/p}\mathrm{d}x.\end{aligned}$$

The kernel K is obviously non–homogeneous for all $p\neq q$. So we have the following

Corollary 1. *Suppose $p, q > 1$, $\mu > 0$, and $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}} \in \ell_p$, $\boldsymbol{b} = (b_n)_{n\in\mathbb{N}} \in \ell_q$ nonnegative sequences. Then*

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(Am^q + Bn^p)^\mu} \le \mathsf{C}_{p,q}^{\mu,\Delta}(Ax^q, Bx^p)\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q \tag{10}$$

where the constant

$$\mathsf{C}_{p,q}^{\mu,\Delta}(Ax^q, Bx^p) = \frac{(Aq)^{1/q}(Bp)^{1/p}}{\Gamma(\mu)} \int_0^\infty x^{\mu+\Delta} \left(\int_A^\infty e^{-qAtx}[t^{1/q}]\, \mathrm{d}t \right)^{1/q} \times \left(\int_B^\infty e^{-pBux}[u^{1/p}]\, \mathrm{d}u \right)^{1/p} \mathrm{d}x\,.$$

3.2. Furthermore, if we take $\lambda = \rho = \mathcal{I}$ the kernel is homogeneous. The inequality becomes

$$\begin{aligned}\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(m+n)^\mu} &\le \frac{\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q}{\Gamma(\mu)} \int_0^\infty x^{\mu-1} \left(\sum_{m=1}^{\infty} e^{-mqx}\right)^{1/q} \left(\sum_{n=1}^{\infty} e^{-npx}\right)^{1/p} \mathrm{d}x \\ &= \frac{\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q}{\Gamma(\mu)} \int_0^\infty \frac{x^{\mu-1}}{(e^{qx}-1)^{1/q}(e^{px-1})^{1/p}}\, \mathrm{d}x\,.\end{aligned}$$

Corollary 2. *Suppose $p, q > 1$, $\mu > \Delta+1$ and let $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}} \in \ell_p$, $\boldsymbol{b} = (b_n)_{n\in\mathbb{N}} \in \ell_q$ be nonnegative sequences. Then*

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(m+n)^\mu} \le \mathsf{C}_{p,q}^{\mu,\Delta}(\mathcal{I},\mathcal{I})\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q \tag{11}$$

where

$$\mathsf{C}_{p,q}^{\mu,\Delta}(\mathcal{I},\mathcal{I}) = \frac{1}{\Gamma(\mu)} \int_0^\infty \frac{x^{\mu-1}}{(e^{qx}-1)^{1/q}(e^{px-1})^{1/p}}\, \mathrm{d}x\,.$$

3.3. When $\lambda(x) = \rho(x) = x^2$, the kernel $K(m^2, n^2)$ is homogeneous.

Corollary 3. *Suppose $p, q > 1$, $\mu > \Delta+1$ and let $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}} \in \ell_p$, $\boldsymbol{b} = (b_n)_{n\in\mathbb{N}} \in \ell_q$ be nonnegative sequences. Then*

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(m^2+n^2)^\mu} \le \mathsf{C}_{p,q}^{\mu,\Delta}(x^2, x^2)\|\boldsymbol{a}\|_p\|\boldsymbol{b}\|_q\,. \tag{12}$$

In this case

$$\mathsf{C}_{p,q}^{\mu,\Delta}(x^2, x^2) = \frac{1}{2^{\Delta+1}\Gamma(\mu)} \int_0^\infty x^{\mu-1}\big(\vartheta_3(0, e^{-px}) - 1\big)^{1/p}\big(\vartheta_3(0, e^{-qx}) - 1\big)^{1/q} \mathrm{d}x\,.$$

Here $\vartheta_3(\cdot\,,\cdot)$ *stands for the Jacobi Theta function*

$$\vartheta_3(u,\mathfrak{q}) = 1 + 2\sum_{n=1}^{\infty} \mathfrak{q}^{n^2}\cos(2n\pi u) \qquad \big(|\mathfrak{q}|<1\big).$$

Proof. By direct calculation we deduce

$$\begin{aligned}\mathsf{C}_{p,q}^{\mu,\Delta}(x^2,x^2) &= \frac{1}{\Gamma(\mu)}\int_0^\infty x^{\mu-1}\left(\sum_{n=1}^{\infty} e^{-n^2px}\right)^{1/p}\left(\sum_{m=1}^{\infty} e^{-m^2qx}\right)^{1/q}\mathrm{d}x\\ &= \frac{1}{\Gamma(\mu)}\int_0^\infty x^{\mu-1}\Big(\frac{\vartheta_3(0,e^{-px})-1}{2}\Big)^{1/p}\Big(\frac{\vartheta_3(0,e^{-qx})-1}{2}\Big)^{1/q}\mathrm{d}x\\ &= \frac{1}{2^{1/p+1/q}\Gamma(\mu)}\int_0^\infty x^{\mu-1}\big(\vartheta_3(0,e^{-px})-1\big)^{1/p}\big((\vartheta_3(0,e^{-qx})-1\big)^{1/q}\mathrm{d}x\,,\end{aligned}$$

which proves the desired statement. □

We point out that the case $p=q=2$ (such that means *a fortiori* $\Delta=0$), results in reduced constant

$$C_{2,2}^{\mu,0}(x^2,x^2) = \frac{1}{2\Gamma(\mu)}\int_0^\infty x^{\mu-1}\big(\vartheta_3(0,e^{-2x})-1\big)\mathrm{d}x = \frac{\zeta(2\mu)}{2^\mu} \tag{13}$$

such that can be easily verified by the formula [**6**]

$$\int_0^\infty t^{\alpha-1}\big(\vartheta_3(0,e^{-At})-1\big)\,\mathrm{d}t = \frac{2\Gamma(\alpha)}{A^\alpha}\,\zeta(2\alpha) \qquad \big(\Re\{A\}>0,\ \Re\{\alpha\}>1/2\big)\,.$$

3.4. If we take $\lambda\equiv\rho$, $p=q=2$, the homogeneity of the kernel depends on the nature of $\lambda(x)$. The constant reduces to

$$\mathsf{C}_{2,2}^{\mu,0}(\lambda,\lambda) = \frac{2}{\Gamma(\mu)}\int_0^\infty x^{\mu}\left(\int_{\lambda_1}^\infty e^{-2tx}[\lambda^{-1}(t)]\mathrm{d}t\right)\mathrm{d}x = \frac{\mu}{2^\mu}\int_{\lambda_1}^\infty \frac{[\lambda^{-1}(t)]}{t^{\mu+1}}\,\mathrm{d}t\,. \tag{14}$$

Corollary 4. *Suppose that* $\mu>0$, $\boldsymbol{a}=(a_n)_{n\in\mathbb{N}}, \boldsymbol{b}=(b_n)_{n\in\mathbb{N}}\in\ell_2$ *are nonnegative sequences, and* λ *a positive and monotone function that satisfies* (5). *Then*

$$\sum_{m,n=1}^{\infty}\frac{a_mb_n}{(\lambda_m+\lambda_n)^\mu} \le \mathsf{C}_{2,2}^{\mu,0}(\lambda,\lambda)\|\boldsymbol{a}\|_2\|\boldsymbol{b}\|_2\,. \tag{15}$$

Here $\mathsf{C}_{2,2}^{\mu,0}(\lambda,\lambda)$ *is given by* (14).

3.5. Finally, if we specify $\lambda=\rho=\mathcal{I}$, $p=q=2$, by Corollary 4. we get

Corollary 5. *Suppose that* $\mu>1$, *and* $\boldsymbol{a}=(a_n)_{n\in\mathbb{N}}$, $\boldsymbol{b}=(b_n)_{n\in\mathbb{N}}\in\ell_2$ *be nonnegative sequences. Then*

$$\sum_{m,n=1}^{\infty}\frac{a_mb_n}{(m+n)^\mu} \le 2^{-\mu}\,\zeta(\mu)\,\|\boldsymbol{a}\|_2\|\boldsymbol{b}\|_2\,. \tag{16}$$

Proof. This conclusion follows immediately by [**18**, p. 97, Corollary 6]. □

4. HOW SHARP IS $C_{p,q}^{\mu,\Delta}(\lambda,\rho)$?

It is of a great interest to find the best constant in non–conjugated parameter HILBERT type inequalities, generated by the HÖLDER inequality with the non–conjugated parameters. Unfortunately, our approach does not solve this well–known open problem.

I. PERIĆ gave us the following remark: "Being $\mu > 1$, $p = q = 2$, by means of (1) the following straightforward estimates hold

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(m+n)^\mu} < \sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \le \pi \|\boldsymbol{a}\|_2 \|\boldsymbol{b}\|_2 \,. \tag{17}$$

On the other hand

$$C_{2,2}^{1.1,0}(\mathcal{I},\mathcal{I}) = \frac{\zeta(1.1)}{2^{1.1}} \approx 4.937797 > \pi,$$

that is, (17) is sharper than (16)". Obviously, $C_{2,2}^{\mu,0}(\mathcal{I},\mathcal{I})$ decreases in μ. Let μ_0 be the unique root of

$$\zeta(\mu) = \pi\, 2^\mu \,.$$

Then, for all $\mu > \mu_0$ the upper bound (16) is sharper then (17), while for $\mu \in [1,\mu_0)$, the reverse result holds true. Let us mention that $\mu_0 \approx 1.156$.

However, the used estimation method is not efficient for general non–conjugated $p, q > 1$ (discussed in **3.2.** too). Indeed, we deduce

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(m+n)^\mu} < \sum_{m,n=1}^{\infty} \frac{a_m b_n}{m+n} \le \frac{\pi}{\sin(\pi/p)} \|\boldsymbol{a}\|_p \|\boldsymbol{b}\|_q \,. \tag{18}$$

Now, by the BERNOULLI inequality we get

$$\begin{aligned} C_{p,q}^{\mu,\Delta}(\mathcal{I},\mathcal{I}) &= \frac{1}{\Gamma(\mu)} \int_0^\infty x^{\mu-1} e^{-2x} \left(1-e^{-px}\right)^{-1/p} \left(1-e^{-qx}\right)^{-1/q} \mathrm{d}x \\ &\le \frac{1}{\Gamma(\mu)} \int_0^\infty x^{\mu-1} e^{-2x} \Big(1-\frac{1}{p}\, e^{-px}\Big)\Big(1-\frac{1}{q}\, e^{-qx}\Big)\, \mathrm{d}x \\ &= \frac{1}{2^\mu} - \frac{1}{p(2+p)^\mu} - \frac{1}{q(2+q)^\mu} + \frac{1}{pq(2+p+q)^\mu} \\ &< \frac{1}{2^\mu}\Big(1+\frac{1}{2^\mu}\Big) < \frac{\pi}{\sin(\pi/p)} \,. \end{aligned}$$

Accordingly, (11) is superior to (1) for all $\mu > 1$.

5. FINAL REMARKS

1. According to the authors' best knowledge, except (1), only (6) includes non–weighted norms $\|\boldsymbol{a}\|_p, \|\boldsymbol{b}\|_q$ in the literature. Moreover, applying different parameter Hölder inequalities in evaluating the Dirichlet series in (9), that is, first p, q, $p^{-1} + q^{-1} \geq 1$, then r, s, $r^{-1} + s^{-1} \geq 1$ we can easily (but artificially) generalize Theorem.

2. One of our main tasks was to express the upper bound upon $\mathfrak{H}_K^{\boldsymbol{a},\boldsymbol{b}}$ with non–homogeneous K in terms of $\|\boldsymbol{a}\|, \|\boldsymbol{b}\|$. Transforming both Dirichlet series in (9) in the manner:

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = \sum_{n=1}^{\infty} \left(\frac{a_n}{\phi(n)}\right) \cdot \left(\phi(n) e^{-\lambda_n x}\right),$$

ϕ certain convenient function, we can proceed to apply the Hölder inequality to the right–hand sum.

3. Taking in (6) $a_n \mapsto \phi(a_n)$, $b_m \mapsto \psi(b_m)$, ϕ, ψ suitable functions, one generalizes Theorem in another way.

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