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A CLASS OF SHANNON-MCMILLAN THEOREMS FOR NONHOMOGENEOUS MARKOV CHAINS FIELD INDEXED BY A HOMOGENEOUS TREE

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Abstract. In this paper, a class of Shannon-McMillan theorems for the nonhomogeneous Markov chains field on a homogeneous tree are discussed by constructing a nonnegative martingale. As corollaries, some Shannon-Mcmillan theorems for the homogeneous Markov chains field on a homogeneous tree and the nonhomogeneous Markov chain are obtained. A result which has been obtained is extended.

1. Introduction

Let T be a homogeneous tree on which each vertex has $N+1$ neighboring vertices. We first fix any vertex as the "root" and label it by 0. Let σ, τ be vertices of a tree. Write $\tau \leq \sigma$ if τ is on the unique path connecting 0 to σ, $|\sigma|$ for the number of edges on this path. For any two vertices σ, τ, denote $\sigma \wedge \tau$ the vertex farthest from 0 satisfying

$$\sigma \wedge \tau \leq \sigma, \ and\ \sigma \wedge \tau \leq \tau.$$

If $\sigma \neq 0$, then we let $\bar{\sigma}$ stand for the vertex satisfying $\bar{\sigma} \leq \sigma$ and $|\bar{\sigma}| = |\sigma| - 1$ (we refer to σ as a son of $\bar{\sigma}$). It is easy to see that the root has $N+1$ sons and all other vertices have N sons.

If $|\sigma| = n$, it is said to be on the nth level on a tree T. We denote by $T^{(n)}$ the subtree of T containing the vertices from level 0 (the root) to level n, and L_n set of all vertices on the level n. Let B be a subgraph of T. Denote $X^B = \{X_\sigma, \sigma \in B\}$, and denote by $|B|$ the number of vertices of B. Let $S(\sigma)$ be the set of all sons of vertices σ. It is easy to see that $|S(0)| = N+1$ and $|S(\sigma)| = N$, where $\sigma \neq 0$.

Definition 1(see[6]). Let T be a homogeneous tree, $S = \{s_0, s_1, s_2, \cdots\}$ be a countable state space, $\{X_\sigma, \sigma \in T\}$ be a collection of $S-$valued random variables defined on the measurable space $\{\Omega, \mathcal{F}\}$. Let

$$p = \{p(x), x \in S\} \tag{1}$$

be a distribution on S, and

$$P_n = (P_n(y|x)),\ \forall x, y \in S,\ n \geq 1 \tag{2}$$

be a strictly positive stochastic matrix on S^2. If for any vertices σ, τ, $\sigma \in L_n$,

$$P(X_\sigma = y | X_{\bar{\sigma}} = x,\ and\ X_\tau\ for\ \sigma \wedge \tau \leq \bar{\sigma}) \tag{3}$$

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$$= P(X_\sigma = y|X_{\bar{\sigma}} = x) = P_n(y|x)\ \forall x, y \in S,$$

and

$$P(X_0 = x) = p(x),\ \forall x \in S. \tag{4}$$

$\{X_\sigma, \sigma \in T\}$ will be called $S-$valued Markov chains indexed by a homogeneous tree with the initial distribution (1) and transition matrix (2).

Two special finite tree-indexed Markov chains are introduced in Kemeny et al.(1976[10]), Spitzer (1975[11]), and there the finite transition matrix is assumed to be positive and reversible to its stationary distribution, and this tree-indexed Markov chains ensure that the cylinder probabilities are independent of the direction we travel along a path. In this paper, we have no such assumption.

Let $\Omega = S^T$, $\omega = \omega(\cdot) \in \Omega$, where $\omega(\cdot)$ is a function defined on T and taking values in S, and $\mathcal{F}$ be the smallest Borel field containing all cylinder sets in Ω, μ be the probability measure on $(\Omega, \mathcal{F})$. Let $X = \{X_\sigma, \sigma \in T\}$ be the coordinate stochastic process defined on the measurable space $(\Omega, \mathcal{F})$; that is, for any $\omega = \{\omega(t), t \in T\}$, define

$$X_t(\omega) = \omega(t),\ t \in T^{(n)}$$

$$X^{T^{(n)}} \triangleq \{X_t, t \in T^{(n)}\},\ \mu(X^{T^{(n)}} = x^{T^{(n)}}) = \mu(x^{T^{(n)}}). \tag{5}$$

Now we give a definition of Markov chain fields on the tree T by using the cylinder distribution directly, which is a natural extension of the classical definition of Markov chains (see[9]).

Definition 2. Let $P_n = P_n(j|i)$ and $p = (p(s_1), p(s_2), \cdots)$ be defined as before, μ_P be a nonhomogeneous Markov measure on $(\Omega, \mathcal{F})$. If

$$\mu_P(x_0) = p(x_0) \tag{6}$$

$$\mu_P(x^{T^{(n)}}) = p(x_0) \prod_{k=0}^{n-1} \prod_{\sigma \in L_k} \prod_{\tau \in S(\sigma)} P_{k+1}(x_\tau|x_\sigma)\ n \geq 1, \tag{7}$$

then μ_P will be called a Markov chains field on the homogeneous tree T determined by the stochastic matrix P_n and the distribution p.

Let μ be an arbitrary probability measure defined as (5), *log* is the natural logarithm. Let

$$f_n(\omega) = -\frac{1}{|T^{(n)}|} \log \mu(X^{T^{(n)}}). \tag{8}$$

$f_n(\omega)$ is called the entropy density on subgraph $T^{(n)}$ with respect to μ. If $\mu = \mu_P$, then by (7),(8) we have

$$f_n(\omega) = -\frac{1}{|T^{(n)}|}[\log p(X_0) + \sum_{k=0}^{n-1} \sum_{\sigma \in L_k} \sum_{\tau \in S(\sigma)} \log P_{k+1}(x_\tau|x_\sigma)]. \tag{9}$$

The convergence of $f_n(\omega)$ in a sense (L_1 convergence, convergence in probability, or almost sure convergence) is called the Shannon-McMillan theorem or the asymptotic equipartition property(AEP) in information theory. The Shannon-McMillan theorem on the Markov chain has been studied extensively (see [**7**], cite2). In the recent years, with the development of information theory scholars get to study the Shannon-McMillan theorems for random field on the tree graph(see [**3**]). The tree models have recently drawn increasing interest from specialists in physics, probability and information theory. Berger and Ye (see [**1**]) have studied the existence of entropy rate for G-invariant random fields. Recently, Ye and Berger (see [**2**]) have also studied the ergodic property and Shannon-McMillan theorem for PPG-invariant random fields on trees. But their results only relate to convergence in probability. Yang and Liu (see [**12**], [**9**]) have recently studied a.s. convergence of Shannon-McMillan theorem for Markov chains indexed by a homogeneous tree and the generalized Cayley tree. But Yang and Liu's results only relate to the case of homogeneous Markov chains fields on trees in a finite state space, they haven't consider the case of nonhomogeneous Markov chains field on trees in a countable state space.

In this paper, we study a class of Shannon-McMillan theorem for nonhomogeneous Markov chains field which takes values in a countable alphabet set on the homogeneous tree. As corollaries, several Shannon-McMillan theorems for a homogeneous Markov chains field on a homogeneous tree and the general nonhomogeneous Markov chain are obtained. A result which has been obtained is extended.

2. Main Results and its Proof

Theorem 1. Let $X = \{X_\sigma, \sigma \in T\}$ be a nonhomogeneous Markov chains field on a homogeneous tree, $f_n(\omega)$ be defined as (9). Denote $\alpha > 0$. Let $H_k(X_\tau|X_\sigma)$ be the random conditional entropy of X_τ relative to X_σ on the measure μ_P, that is

$$H_k(X_\tau|X_\sigma) = -\sum_{x_\tau \in S} P_{k+1}(x_\tau|X_\sigma)\log P_{k+1}(x_\tau|X_\sigma)\ \sigma \in L_k,\ \tau \in S(\sigma),\ k \geq 0. \tag{10}$$

Set

$$b_\alpha = \limsup_{n\to\infty} \frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)} E[(\log P_{k+1}(X_\tau|X_\sigma))^2 P_{k+1}(X_\tau|X_\sigma)^{-\alpha}|X_\sigma] < \infty. \tag{11}$$

Then

$$\lim_{n\to\infty}[f_n(\omega) - \frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)} H_k(X_\tau|X_\sigma)] = 0.\ \mu_P - a.s. \tag{12}$$

Proof. On the probability space $(\Omega, \mathcal{F}, \mu_P)$, Let $\lambda > 0$ be a constant, let

$$Q_k(\lambda) = E[P_{k+1}(X_\tau|X_\sigma)^{-\lambda}|X_\sigma = x_\sigma] = \sum_{x_\tau\in S} P_{k+1}(x_\tau|x_\sigma)^{1-\lambda}, \tag{13}$$

$$q_k(\lambda; x_\tau, x_\sigma) = \frac{P_{k+1}(x_\tau|x_\sigma)^{1-\lambda}}{Q_k(\lambda)},\ x_\tau, x_\sigma \in S. \tag{14}$$

$$g(\lambda; x^{T^{(n)}}) = p(x_0)\prod_{k=0}^{n-1}\prod_{\sigma\in L_k}\prod_{\tau\in S(\sigma)} q_k(\lambda; x_\tau, x_\sigma). \tag{15}$$

Hence $g(\lambda; x^{T^{(n)}}),\ n = 1, 2, \cdots$ are a set of distribution functions. Set

$$t_n(\lambda, \omega) = \frac{g(\lambda; X^{T^{(n)}})}{\mu_P(X^{T^{(n)}})}. \tag{16}$$

Apparently $\{t_n(\lambda,\omega), F_n, n \geq 1\}$ is a nonnegative martingale which converges almost surely (see [**4**]). Thus, by Doob's martingale convergence theorem we have

$$\lim_{n\to\infty} t_n(\lambda,\omega) = t_\infty(\lambda,\omega) < \infty,\ \ \mu_P - a.s. \tag{17}$$

By (17) we have

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \log t_n(\lambda,\omega) \leq 0.\ \mu_P - a.s. \tag{18}$$

By (13)-(16), we have

$$\begin{aligned} & \frac{1}{|T^{(n)}|} \log t_n(\lambda,\omega) \\ = \ & \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} [-\lambda \log P_{k+1}(X_\tau|X_\sigma) - \log E(P_{k+1}(X_\tau|X_\sigma)^{-\lambda}|X_\sigma)]. \ (19) \end{aligned}$$

By (18) and (19) we have

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} [-\lambda \log P_{k+1}(X_\tau|X_\sigma) - \log E(P_{k+1}(X_\tau|X_\sigma)^{-\lambda}|X_\sigma)] \leq 0.$$

$$\mu_P - a.s. \tag{20}$$

It follows from (20) that

$$\begin{aligned} & \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} [-\lambda \log P_{k+1}(X_\tau|X_\sigma) - E(-\lambda \log P_{k+1}(X_\tau|X_\sigma)|X_\sigma)] \\ \leq \ & \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \big[\log E(P_{k+1}(X_\tau|X_\sigma)^{-\lambda}|X_\sigma) \\ & - E(-\lambda \log P_{k+1}(X_\tau|X_\sigma)|X_\sigma)].\ \mu_P - a.s. \end{aligned} \tag{21}$$

By the inequality $e^x - 1 - x \leq (1/2)x^2 e^{|x|}$, we have

$$x^{-\lambda} - 1 - (-\lambda)\log x \leq (1/2)\lambda^2 (\log x)^2 x^{-|\lambda|},\ \ 0 \leq x \leq 1. \tag{22}$$

Hence by (11), (21), (22) and the inequality $\log x \leq x - 1,\ (x \geq 0)$, in the case of $|\lambda| < \alpha$ we have

$$\begin{aligned} & \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} [-\lambda \log P_{k+1}(X_\tau|X_\sigma) - E(-\lambda \log P_{k+1}(X_\tau|X_\sigma)|X_\sigma)] \\ \leq \ & \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \big[E(P_{k+1}(X_\tau|X_\sigma)^{-\lambda}|X_\sigma) - 1 \\ & -E(-\lambda \log P_{k+1}(X_\tau|X_\sigma)|X_\sigma)] \\ \leq \ & \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} E[\frac{1}{2}\lambda^2 \log^2 P_{k+1}(X_\tau|X_\sigma) P_{k+1}(X_\tau|X_\sigma)^{-|\lambda|}|X_\sigma] \end{aligned}$$

$$\leq \frac{1}{2}\lambda^2 \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} E[\log^2 P_{k+1}(X_\tau|X_\sigma) P_{k+1}(X_\tau|X_\sigma)^{-\alpha}|X_\sigma]$$

$$= \frac{1}{2}\lambda^2 b_\alpha \ < \infty \ \mu_P - a.s. \tag{23}$$

In the case of $0 < \lambda < \alpha$, by (23) we have

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} [-\log P_{k+1}(X_\tau|X_\sigma) - E(-\log P_{k+1}(X_\tau|X_\sigma)|X_\sigma)] \leq \frac{1}{2}\lambda b_\alpha,$$

$$\mu_P - a.s. \tag{24}$$

Choose $0 < \lambda_i < \alpha,\ (i = 1, 2, \cdots)$ such that $\lambda_i \to 0\ (i \to \infty)$, then for all i we have by (24) that

$$\limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} [-\log P_{k+1}(X_\tau|X_\sigma) - E(-\log P_{k+1}(X_\tau|X_\sigma)|X_\sigma)] \leq 0.$$

$$\mu_P - a.s. \tag{25}$$

When $-\alpha < \lambda < 0$, by virtue of (25) it can be shown in a similar way that

$$\liminf_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} [-\log P_{k+1}(X_\tau|X_\sigma) - E(-\log P_{k+1}(X_\tau|X_\sigma)|X_\sigma)] \geq 0.$$

$$\mu_P - a.s. \tag{26}$$

It follows from (25) and (26) that

$$\lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} [-\log P_{k+1}(X_\tau|X_\sigma) - E(-\log P_{k+1}(X_\tau|X_\sigma)|X_\sigma)] = 0.$$

$$\mu_P - a.s. \tag{27}$$

Noticing that

$$\begin{aligned} H_k(X_\tau|X_\sigma) &= -\sum_{x_\tau\in S} P_{k+1}(x_\tau|X_\sigma) \log P_{k+1}(x_\tau|X_\sigma) \\ &= E(-\log P_{k+1}(X_\tau|X_\sigma)|X_\sigma),\ k \geq 0. \end{aligned}$$

It follows from (9) and (27) that

$$\begin{aligned} &\lim_{n\to\infty} [f_n(\omega) - \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} H_k(X_\tau|X_\sigma)] \\ &= \lim_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} [-\log P_{k+1}(X_\tau|X_\sigma) \\ &\quad -E(-\log P_{k+1}(X_\tau|X_\sigma)|X_\sigma)] = 0. \end{aligned} \tag{28}$$

We complete the proof of the theorem.

$\{X_\sigma, \sigma \in T\}$ will be called $S-$valued homogeneous Markov chains field indexed by a homogeneous tree if for all n,

$$P_n = P = (P(y|x)),\ \forall x, y \in S. \tag{29}$$

Corollary 1. Let $\{X_\sigma, \sigma \in T\}$ be a homogeneous Markov chains field indexed by a homogeneous tree, $f_n(\omega)$ and $H_k(X_\tau|X_\sigma)$ be defined by (9) and (10). Denote $0 < \alpha < 1$, if

$$\sum_{h\in S}\sum_{l\in S} \log^2 P(l|h)P(l|h)^{1-\alpha} < \infty. \tag{30}$$

Then

$$\lim_{n\to\infty}[f_n(\omega) - \frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)} H_k(X_\tau|X_\sigma)] = 0. \ \mu_P - a.s. \tag{31}$$

Proof. By (11), (29) and (30) we have

$$\begin{aligned}
b_\alpha &= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)} E[(\log(P(X_\tau|X_\sigma))^2 P(X_\tau|X_\sigma)^{-\alpha}|X_\sigma] \\
&= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\sum_{x_\tau\in S} (\log(P(x_\tau|X_\sigma))^2 P(x_\tau|X_\sigma)^{1-\alpha} \\
&= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\sum_{h\in S}\sum_{l\in S} \delta_h(X_\sigma)\log^2 P(l|h)P(l|h)^{1-\alpha} \\
&\le \limsup_{n\to\infty} \frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)}\sum_{h\in S}\sum_{l\in S} \log^2 P(l|h)P(l|h)^{1-\alpha} \\
&\le \sum_{h\in S}\sum_{l\in S}\limsup_{n\to\infty} \frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)} \log^2 P(l|h)P(l|h)^{1-\alpha} \\
&\le \sum_{h\in S}\sum_{l\in S}\limsup_{n\to\infty} \frac{|T^{(n)}|-1}{|T^{(n)}|} \log^2 P(l|h)P(l|h)^{1-\alpha} \\
&= \sum_{h\in S}\sum_{l\in S} \log^2 P(l|h)P(l|h)^{1-\alpha} < \infty. \qquad (32)
\end{aligned}$$

Therefore, (31) follows from Theorem 1.

3. Some Shannon-McMillan Theorems on a Finite States Space

Corollary 2. Let $X = \{X_\sigma, \sigma \in T\}$ be a nonhomogeneous Markov chains field on a homogeneous tree which takes values in the finite alphabet set $S = \{1, 2, \cdots, N\}$, $f_n(\omega)$ be defined as (9). Denote $0 < \alpha < 1$. Let $H_k(X_\tau|X_\sigma)$ be the random conditional entropy of X_τ relative to X_σ on the measure μ_P, that is

$$H_k(X_\tau|X_\sigma) = -\sum_{x_k=1}^{N} P_{k+1}(x_\tau|X_\sigma)\log P_{k+1}(x_\tau|X_\sigma), \ \sigma\in L_k, \ \tau\in S(\sigma), \ k\ge 0.$$

Then

$$\lim_{n\to\infty}[f_n(\omega) - \frac{1}{|T^{(n)}|}\sum_{k=0}^{n-1}\sum_{\sigma\in L_k}\sum_{\tau\in S(\sigma)} H_k(X_\tau|X_\sigma)] = 0. \ \mu_P - a.s. \tag{33}$$

Proof. Let $0 < \alpha < 1$, consider the function

$$\phi(x) = (\log x)^2 x^{1-\alpha},\ 0 < x \le 1,\ 0 < \alpha < 1.\ (\phi(0) = 0)$$

Let $\phi^{'}(x) = 0$, we get $x = e^{2/(\alpha-1)}$. Therefore

$$\max\{\phi(x), 0 \le x \le 1\} = \phi(e^{2/(\alpha-1)}) = (\frac{2}{\alpha-1})^2 e^{-2}. \tag{34}$$

By (11) and (34) we have

$$\begin{aligned}
b_\alpha &= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} E[(\log(P_{k+1}(X_\tau|X_\sigma))^2 P_{k+1}(X_\tau|X_\sigma)^{-\alpha}|X_\sigma] \\
&= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \sum_{x_\tau\in S} (\log(P_{k+1}(x_\tau|X_\sigma))^2 P_{k+1}(x_\tau|X_\sigma)^{1-\alpha} \\
&\le \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} \sum_{x_\tau=1}^{N} (\frac{2}{\alpha-1})^2 e^{-2} \\
&= \limsup_{n\to\infty} \frac{|T^{(n)}|-1}{|T^{(n)}|} \sum_{x_\tau=1}^{N} (\frac{2}{\alpha-1})^2 e^{-2} = N(\frac{2}{\alpha-1})^2 e^{-2} < \infty,\ \mu_P - a.s.
\end{aligned} \tag{35}$$

Hence (11) holds naturally. (33) follows from (12).

Corollary 3[7]. Let $\{X_n, n \ge 0\}$ be a nonhomogeneous Markov chain with the initial distribution and the transition probabilities as follows:

$$p(i) > 0,\ i \in S.$$

$$P_k(j|i) > 0,\ i, j \in S,\ k = 1, 2, \cdots .$$

Set

$$f_n(\omega) = -\frac{1}{n+1}[\log p(X_0) + \sum_{k=0}^{n-1} \log P_k(X_{k+1}|X_k)],$$

$$H_k(X_{k+1}|X_k) = -\sum_{x_{k+1}=1}^{N} P_{k+1}(x_{k+1}|X_k) \log P_{k+1}(x_{k+1}|X_k).$$

Then

$$\lim_{n\to\infty}[f_n(\omega) - \frac{1}{n+1} \sum_{k=0}^{n-1} H_k(X_{k+1}|X_k)] = 0.\ a.s. \tag{36}$$

Proof. Letting $|S(0)| = |S(\sigma)| = 1$, at this time the nonhomogeneous Markov chains field on the homogeneous tree is just the general nonhomogeneous Markov chain. (36) follows from Theorem 1.

4. Derivation Results

Theorem 2. Let $X = \{X_\sigma, \sigma \in T\}$ be a nonhomogeneous Markov chains field on a homogeneous tree which takes values in the countable alphabet set $S =$

$\{s_1, s_2, \cdots\}$, $f_n(\omega)$ be defined as (9). Denote $\alpha \geq 0$, $0 < C < 1$. Set

$$C_\alpha = \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} E[P_{k+1}(X_\tau|X_\sigma)^{-(2+\alpha)} I_{\{P_{k+1}(X_\tau|X_\sigma)\leq C\}}|X_\sigma] < \infty,$$

$$\mu_P - a.s. \tag{37}$$

Then

$$\lim_{n\to\infty}[f_n(\omega) - \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} H_k(X_\tau|X_\sigma)] = 0.\ \mu_P - a.s. \tag{38}$$

Proof. By (11) and (37), noticing $0 \geq \log x \geq 1 - 1/x$, $(1 > x > 0)$, denote $P_{k+1}(X_\tau|X_\sigma) = P_{k+1}$ in brief, we have

$$\begin{aligned}
b_\alpha &= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} E[\log^2 P_{k+1}(X_\tau|X_\sigma) P_{k+1}(X_\tau|X_\sigma)^{-\alpha}|X_\sigma] \\
&= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} E\left[(\log P_{k+1})^2 P_{k+1}^{-\alpha}(I_{\{P_{k+1}(X_\tau|X_\sigma)\leq C\}}\right. \\
&\quad \left. + I_{\{P_{k+1}(X_\tau|X_\sigma)>C\}}|X_\sigma\right] \\
&\leq \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} E[(\log P_{k+1})^2 P_{k+1}^{-\alpha} I_{\{P_{k+1}(X_\tau|X_\sigma)\leq C\}}|X_\sigma] \\
&\quad + \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} C^{-\alpha}(\log C)^2 \\
&\leq \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} E[(1 - \frac{1}{P_{k+1}})^2 P_{k+1}^{-\alpha} I_{\{P_{k+1}(X_\tau|X_\sigma)\leq C\}}|X_\sigma] \\
&\quad + \limsup_{n\to\infty} \frac{|T^{(n)}| - 1}{|T^{(n)}|} C^{-\alpha}(\log C)^2 \\
&= \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} E[(P_{k+1} - 1)^2 P_{k+1}^{-(2+\alpha)} I_{\{P_{k+1}(X_\tau|X_\sigma)\leq C\}}|X_\sigma] \\
&\quad + C^{-\alpha}(\log C)^2 \\
&\leq \limsup_{n\to\infty} \frac{1}{|T^{(n)}|} \sum_{k=0}^{n-1} \sum_{\sigma\in L_k} \sum_{\tau\in S(\sigma)} E[P_{k+1}^{-(2+\alpha)} I_{\{P_{k+1}(X_\tau|X_\sigma)\leq C\}}|X_\sigma] \\
&\quad + C^{-\alpha}(\log C)^2 < \infty.\ \mu_P - a.s.
\end{aligned} \tag{39}$$

Therefore (38) follows from Theorem 1.

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