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OSCILLATION CRITERIA FOR CERTAIN THIRD ORDER NONLINEAR DIFFERENCE EQUATIONS

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Dedicated to the Memory of Professor D. S. Mitrinović (1908–1995)

Some new criteria for the oscillation of all solutions of third order nonlinear difference equations of the form

$$\Delta\left(a(n)(\Delta^2 x(n))^\alpha\right) + q(n)f(x[g(n)]) = 0$$

and

$$\Delta\left(a(n)(\Delta^2 x(n))^\alpha\right) = q(n)f(x[g(n)]) + p(n)h(x[\sigma(n)])$$

with $\sum^\infty a^{-1/\alpha}(n) < \infty$ are established.

1. INTRODUCTION

We will study the oscillatory behavior of solutions of the nonlinear third order difference equations

$$\Delta\left(a(n)(\Delta^2 x(n))^\alpha\right) + q(n)f(x[g(n)]) = 0 \tag{1.1}$$

and

$$\Delta\left(a(n)(\Delta^2 x(n))^\alpha\right) = q(n)f(x[g(n)]) + p(n)h(x[\sigma(n)]), \tag{1.2}$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0+1, \cdots\}, n_0$ is a nonnegative integer, Δ is the forward difference operator $\Delta x(n) = x(n+1) - x(n)$, and $\{a(n)\}, \{p(n)\}, \{q(n)\}, \{g(n)\}$, and $\{\sigma(n)\}$ are sequences of real numbers.

The following conditions are always assumed to hold:

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(i) α is the ratio of two positive odd integers;

(ii) $a(n) > 0$ for $n \in \mathbb{N}(n_0)$ and

$$\sum_{n=n_0}^{\infty} a^{-1/\alpha}(n) < \infty; \tag{1.3}$$

(iii) $p(n), q(n) \geq 0$ for $n \in \mathbb{N}(n_0)$;

(iv) $g, \sigma : \mathbb{N}(n_0) \to \mathbb{Z}$ satisfy $g(n) < n, \sigma(n) > n, \Delta g(n) \geq 0$, and $\Delta\sigma(n) \geq 0$ for $n \in \mathbb{N}(n_0)$, and $\lim_{n\to\infty} g(n) = \infty$;

(v) $f, h \in C(\mathbb{R}, \mathbb{R}), xf(x) \geq 0, xh(x) \geq 0, f'(x) \geq 0$, and $h'(x) \geq 0$ for $x \neq 0$,

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \text{for} \quad xy > 0, \tag{1.4}$$

and

$$-h(-xy) \geq h(xy) \geq h(x)h(y) \quad \text{for} \quad xy > 0. \tag{1.5}$$

By a *solution* of equation $(1, i), i = 1, 2$, we mean a real sequence $\{x(n)\}$ defined on $\mathbb{N}(n_0)$, which satisfies equation $(1, i), i = 1, 2$. A nontrivial solution of equation $(1, i), i = 1, 2$, is said to be *nonoscillatory* if it is either eventually positive or eventually negative, and it is *oscillatory* otherwise. Equation $(1, i), i = 1, 2$, is said to be oscillatory if all its solutions are oscillatory.

The problem of determining the oscillation and nonoscillation of solutions of difference equations with deviating arguments has been a very active area of research in the last three decades. Much of the literature on the subject has been concerned with equations of types (1.1) and (1.2) when $\alpha = 1, a(t) = 1$, and/or equations of different orders. For typical results concerning this case, we refer the reader to [**1**–**7**, **10**] and the references cited therein. There is however much current interest in the study of the oscillatory behavior of equations (1.1) and (1.2) when $\alpha \neq 1$, and

$$\sum_{n=n_0\geq 0}^{\infty} a^{-1/\alpha}(n) = \infty;$$

see, for example, [**1**, **4**–**7**]. The purpose of this paper is to establish some new criteria for the oscillation of equations (1.1) and (1.2) when condition (1.3) holds. The results obtained here extend and improve many well–known oscillation criteria that have appeared in the literature for special cases of equations (1.1) and (1.2).

2. OSCILLATION CRITERIA FOR EQUATION (1.1)

In this section, we present some sufficient conditions for the oscillation of all solutions of equation (1.1). We begin with the following result.

Theorem 2.1. *Let conditions* (i)–(v), (1.3), *and* (1.4) *hold, and assume that there exist two sequences* $\{\xi(n)\}$ *and* $\{\eta(n)\}, \xi, \eta : \mathbb{N}(n_0) \to \mathbb{Z}$, *such that* $\Delta\xi(n) \geq$

$0, \Delta\eta(n) \geq 0$, *and* $g(n) < \xi(n) < \eta(n) < n-1$ *for* $n \in \mathbb{N}(n_0)$. *If both of the first order difference equations*

$$\Delta y(n) + cq(n) f\left(\sum_{k=n_1}^{g(n)-1}\left(\frac{k}{a^{1/\alpha}(n)}\right)\right) f\left(y^{1/\alpha}[g(n)]\right) = 0, \quad n_1 \in \mathbb{N}(n_0), \tag{2.1}$$

for any constant $0 < c < 1$, *and*

$$\Delta z(n) + q(n) f(\xi(n) - g(n)) f\left(\sum_{k=\xi(n)}^{\eta(n)-1} a^{-1/\alpha}(k)\right) \times f\Big(z^{1/\alpha}[\eta(n)]\Big) = 0, \quad n_1 \in \mathbb{N}(n_0), \tag{2.2}$$

are oscillatory, and

$$\sum_{\ell=n_1}^{\infty}\left(\frac{1}{a(\ell)}\sum_{k=n_1}^{\ell-1} q(k) f\left(g(k)\right) f\left(\sum_{s=g(k)}^{\infty} a^{-1/\alpha}(s)\right)\right)^{1/\alpha} = \infty, \tag{2.3}$$

then equation (1.1) *is oscillatory.*

Proof. Assume, for the sake of a contradiction, that equation (1.1) has a nonoscillatory solution $\{x(n)\}$ and that $\{x(n)\}$ is eventually positive. Then, there exists a positive integer $n_1 \geq n_0$ such that $x(n) > 0$ and $x[g(n)] > 0$ for $n \geq n_1$. From equation (1.1), we see that $\Delta\left(a(n)(\Delta^2 x(n))^\alpha\right) \leq 0$ for $n_1 \leq n \in \mathbb{N}(n_0)$. There exists $n_2 \in \mathbb{N}(n_0), n_2 \geq n_1$, such that $\Delta x(n)$ and $\Delta^2 x(n)$ are of fixed sign for $n \geq n_2$. There are the following four possibilities to consider.

(I) $\Delta^2 x(n) > 0$ and $\Delta x(n) > 0$ for $n \geq n_2$;

(II) $\Delta^2 x(n) > 0$ and $\Delta x(n) < 0$ for $n \geq n_2$;

(III) $\Delta^2 x(n) < 0$ and $\Delta x(n) > 0$ for $n \geq n_2$; and

(IV) $\Delta^2 x(n) < 0$ and $\Delta x(n) < 0$ for $n \geq n_2$.

We note that Case (IV) cannot hold. In fact, if $\Delta^2 x(n) < 0$ and $\Delta x(n) < 0$ for $n \geq n_2$, then $\lim_{n\to\infty} x(n) = -\infty$, which contradicts the positivity of $x(n)$. We now consider each case.

Case (I). There exist an integer $n_3 \in \mathbb{N}(n_0), n_3 \geq n_2$, and a constant $b, 0 < b < 1$, such that

$$\Delta x(n) \geq bn\Delta^2 x(n) = b\,\frac{n}{a^{1/\alpha}(n)} y^{1/\alpha}(n) \quad \text{for} \quad n \geq n_3, \tag{2.4}$$

where $y(n) = a(n)(\Delta^2 x(n))^\alpha$. Summing (2.4) from n_3 to $n-1 \geq n_3$, we have

$$x(n) \geq b\sum_{k=n_3}^{n-1} ka^{-1/\alpha}(k) y^{1/\alpha}(k) \geq b\left(\sum_{k=n_3}^{n-1} ka^{-1/\alpha}(k)\right) y^{1/\alpha}(k) \ \text{ for } \ n \geq n_3.$$

Now, there exists $n_4 \in \mathbb{N}(n_0), n_4 \geq n_3$, such that

$$x[g(n)] \geq b\left(\sum_{k=n_3}^{g(n)-1} ka^{-1/\alpha}(k)\right) y^{1/\alpha}[g(n)] \quad \text{for} \quad n \geq n_4. \tag{2.5}$$

Using (2.5) and (1.4) in equation (1.1), we obtain

$$\begin{aligned} -\Delta y(n) = q(n)f(x[g(n)]) \geq f(b)q(n)f&\left(\sum_{k=n_3}^{g(n)-1} ka^{-1/\alpha}(k)\right) \\ &\times f\left(y^{1/\alpha}[g(n)]\right) \quad \text{for } n \geq n_4. \end{aligned} \tag{2.6}$$

Summing the above inequality from $n \geq n_4$ to $u \geq n$ and letting $u \to \infty$, we have

$$y(n) \geq f(b) \sum_{s=n}^{\infty} q(s) f\left(\sum_{k=n_3}^{g(s)-1} ka^{-1/\alpha}(k)\right) f\left(y^{1/\alpha}[g(s)]\right).$$

The sequence $\{y(n)\}$ is obviously strictly decreasing for $n \geq n_4$. Hence, by the discrete analog of Theorem 1 in [**9**] (also see [**6**]), we conclude that there exists a positive solution $\{y(n)\}$ of equation (2.1) with $\lim_{n\to\infty} y(n) = 0$. This contradiction completes the proof for this case.

Case (II). For $t \geq s \geq n_0$, we have

$$x(t) - x(s) = \sum_{k=s}^{t-1} \Delta x(k),$$

or

$$x(s) \geq (t-s)(-\Delta x(t)).$$

With s and t replaced by $g(n)$ and $\xi(n)$ respectively, we see that

$$x[g(n)] \geq (\xi(n) - g(n))(-\Delta x[\xi(n)]) \quad \text{for} \quad n \geq n_2 \geq n_1. \tag{2.7}$$

Substituting (2.7) into equation (1.1) yields

$$-\left(a(n)(\Delta^2 x(n))^\alpha\right) = q(n)f(x[g(n)]) \geqq (n)f(\xi(n) - g(n))f(-x'[\xi(n)])$$

for $n \geq n_2$. Setting $z(n) = -\Delta x(n)$, we obtain

$$\Delta\left(a(n)(\Delta z(n))^\alpha\right) \geqq (n)f(\xi(n) - g(n))f(z[\xi(n)]) \quad \text{for} \quad n \geq n_2. \tag{2.8}$$

Clearly, $z(n) > 0$ and $\Delta z(n) < 0$ for $n \geq n_2$. Next, for $t \geq s \geq n_2$, we have

$$\begin{aligned} z(s) &\geq \sum_{k=s}^{t-1} -\Delta z(k) = \sum_{k=s}^{t-1} a^{-1/\alpha}(k)\left(-a(k)(\Delta z(k))^\alpha\right)^{1/\alpha} \\ &\geq \left(\sum_{k=s}^{t-1} a^{-1/\alpha}(k)\right)\left(-a(t)(\Delta z(t))^\alpha\right)^{1/\alpha}. \end{aligned}$$

Replacing s and t with $\xi(n)$ and $\eta(n)$ respectively, we obtain

$$z[\xi(n)] \geq \left(\sum_{k=\xi(n)}^{g(n)-1} a^{-1/\alpha}(k)\right) w^{1/\alpha}[\eta(n)] \quad \text{for} \quad n \geq n_3 \geq n_2, \tag{2.9}$$

where $w(n) = -a(n)(\Delta z(n))^\alpha$. Using (1.4) and (2.9) in (2.8), we have

$$\Delta w(n) + q(n) f(\xi(n) - g(n)) f\left(\sum_{k=\xi(n)}^{\eta(n)-1} a^{-1/\alpha}(k)\right) f\left(w^{1/\alpha}[\eta(n)]\right) \leq 0 \quad \text{for} \quad n \geq n_3.$$

The rest of the proof is similar to that of Case (I) above, and hence is omitted.

Case (III). There exist $n_2 \in \mathbb{N}(n_0), n_2 \geq n_1$, and a constant $0 < d < 1$ such that

$$x(n) \geq dn \Delta x(n) \quad \text{for} \quad n \geq n_2.$$

Hence, there exists $n_3 \in \mathbb{N}(n_0), n_3 \geq n_2$, such that

$$x[g(n)] \geq d g(n) v[g(n)] \quad \text{for} \quad n \geq n_3, \tag{2.10}$$

where $v(n) = \Delta x(n) > 0$. Using (2.10) and (1.4) in equation (1.1), we have

$$\Delta\left(a(n)(\Delta v(n))^\alpha\right) + f(d) q(n) f(g(n)) f(v[g(n)]) \leq 0 \quad \text{for} \quad n \geq n_3. \tag{2.11}$$

We see that $v(n) > 0$ and $\Delta v(n) < 0$ for $n \geq n_3$. Now, for $s \geq t \geq n_3$, it can easily be seen that

$$a(s)(-\Delta v(s))^\alpha \geq a(t)(-\Delta v(t))^\alpha,$$

or equivalently,

$$-a^{1/\alpha}(s) \Delta v(s) \geq -a^{1/\alpha}(t) \Delta v(t) \quad \text{for} \quad s \geq t \geq n_3. \tag{2.12}$$

Dividing (2.12) by $a^{-1/\alpha}(s)$ and summing from $t = n$ to $u - 1 \geq n \geq n_3$, we have

$$v(n) \geq v(n) - v(u) \geq \left(-a^{1/\alpha}(n) \Delta v(n)\right) \sum_{s=n}^{u-1} a^{-1/\alpha}(s).$$

Letting $u \to \infty$ in the above inequality gives

$$v(n) \geq -a^{-1/\alpha}(n) \Delta v(n) \left(\sum_{s=n}^{\infty} a^{-1/\alpha}(s)\right). \tag{2.13}$$

Combining (2.13) with the inequality

$$-a^{1/\alpha}(n) \Delta v(n) \geq -a^{-1/\alpha}(a_3) \Delta v(n_3) \quad \text{for} \quad n \geq n_3,$$

which is implied by (2.12), we find that

$$v(n) \geq -a^{1/\alpha}(t_3)\Delta v(n_3)\left(\sum_{s=n}^{\infty} a^{-1/\alpha}(s)\right) \text{ for } n \geq n_3.$$

Thus, there exist a constant $\overline{d} > 0$ and $n_4 \in \mathbb{N}(n_0), n_4 \geq n_3$, such that

$$v[g(t)] \geq \overline{d} \sum_{s=g(n)}^{\infty} a^{-1/\alpha}(s) \text{ for } n \geq n_4. \tag{2.14}$$

Summing inequality (2.11) from n_3 to $n-1 \geq n_3$ yields

$$f(d)\sum_{k=n_3}^{n-1} q(k)f(g(k))f(v[g(k)]) \leq a(n_3)(\Delta v(n_3))^{\alpha} - a(n)(\Delta v(n))^{\alpha}. \tag{2.15}$$

Using (2.14) and (1.4) in (2.15) gives

$$\overline{c}\left(\frac{1}{a(n)}\sum_{k=n_3}^{n-1} q(k)f(g(k))f\left(\sum_{s=g(k)}^{\infty} a^{-1/\alpha}(s)\right)\right)^{1/\alpha} \leq -\Delta v(n), \tag{2.16}$$

where $\overline{c} = (f(d)f(\overline{d}))^{1/\alpha}$. Summing (2.16) from n_3 to $n-1 \geq n_3$, we have

$$\overline{c}\sum_{\ell=n_3}^{n-1}\left(\frac{1}{a(\ell)}\sum_{k=n_3}^{\ell-1} q(k)f(g(k))f\left(\sum_{s=g(k)}^{\infty} a^{-1/\alpha}(s)\right)\right)^{1/\alpha} \leq v(n_3) - v(n) \leq v(t_3) < \infty.$$

Letting $n \to \infty$ in the above inequality, we obtain a contradiction to condition (2.3). This completes the proof of the theorem. □

If we combine equations (2.1) and (2.2) into one by letting

$$Q(n) = \min\left\{cq(n)f\left(\sum_{k=n_1}^{g(n)-1} ka^{-1/\alpha}(k)\right),\; q(n)f(\xi(n)-g(n))f\left(\sum_{k=\xi(n)}^{\eta(n)-1} a^{-1/\alpha}(k)\right)\right\} \tag{2.17}$$

for any constant $0 < c < 1$ and all $n_1 \in \mathbb{N}(n_0)$, then we can easily see that equations (2.1) and (2.2) can be replaced by

$$\Delta w(n) + Q(n)f\left(w^{1/\alpha}[\eta(n)]\right) = 0. \tag{2.18}$$

Remark 2.1. We note that the results of this paper are presented in a form that allows us to extract results for equation (1.1) that are valid in case

$$\sum^{\infty} a^{-1/\alpha}(n) = \infty. \tag{2.19}$$

The reason for this is that condition (1.3) is only needed in the proof of Case (III) above. Thus, we have the following result.

Theorem 2.2. *Let conditions* (i)–(v), (1.4), *and* (2.19) *hold, and assume that there exist two real sequences* $\{\xi(n)\}$ *and* $\{\eta(n)\}$ *such that* $\Delta\xi(n) \geq 0, \Delta\eta(n) \geq 0$, *and* $g(n) < \xi(n) < \eta(n) < n-1$ *for* $n \in \mathbb{N}(n_0)$. *If equation* (2.18) *is oscillatory, then equation* (1.1) *is oscillatory.*

Proof. The proof of Theorem 2.2 is exactly the same as that of Cases (I) and (II) in Theorem 2.1 and so is omitted. □

The following corollary is immediate.

Corollary 2.1. *Let conditions* (i)–(v), (1.3), (1.4), *and* (2.3) *hold, and assume that there exist real nondecreasing sequences* $\{\xi(n)\}$ *and* $\{\eta(n)\}$ *such that* $g(n) < \xi(n) < \eta(n) < n-1$ *for* $n \in \mathbb{N}(n_0)$. *Then* (1.1) *is oscillatory if one of the following conditions holds*:

(I_1) $\dfrac{f(u^{1/\alpha})}{u} \geq 1$ *for* $u \neq 0$, *and*

$$\limsup_{n\to\infty} \sum_{k=\eta(n)}^{n-1} Q(k) > 1;$$

(I_2) $\displaystyle\int_{\pm 0} \frac{du}{f(u^{1/\alpha})} < \infty$, *and*

$$\sum^{\infty} Q(n) = \infty;$$

(I_3) $\dfrac{u}{f(u^{1/\alpha})} \to 0$ *as* $u \to \infty$, *and*

$$\limsup_{n\to\infty} \sum_{k=\eta(n)}^{n-1} Q(k) > 0.$$

3. OSCILLATION CRITERIA FOR EQUATION (1.2)

The purpose of this section is to establish criteria for the oscillation of equation (1.2) which contains mixed nonlinearities and functional arguments.

Theorem 3.1. *Let conditions* (i)–(v) *and* (1.3)–(1.5) *hold, and assume that there exist nondecreasing sequences of real numbers* $\{\xi(n)\}, \{\rho(n)\}$, *and* $\{\theta(n)\}$ *such that* $g(n) < \xi(n) < n-1$ *and* $\sigma(n) > \rho(n) > \theta(n) > n+1$ *for* $n \in \mathbb{N}(n_0)$. *If the first order advanced difference equation*

$$\Delta y(n) - p(n)h(\sigma(n)-\rho(n))h\left(\frac{\rho(n)-\theta(n)}{a^{1/\alpha}(\theta(n))}\right)h\left(y^{1/\alpha}[\theta(n)]\right) = 0, \tag{3.1}$$

and the first order delay difference equation

$$\Delta w(n) + cq(n)f(g(n))f\left(\frac{\xi(n)-g(n)}{a^{1/\alpha}[\xi(n)]}\right)f\left(w^{1/\alpha}[\xi(n)]\right) = 0, \tag{3.2}$$

for every constant $0 < c < 1$ *are oscillatory, and*

$$(3.3)\qquad \sum_{u=n_0\geq 0}^{\infty}\left(\frac{1}{a(u)}\sum_{s=n_0}^{u-1} q(s)f(\xi(s)-g(s))f\left(\sum_{k=\xi(s)}^{\infty} a^{-1/\alpha}(k)\right)\right)^{1/\alpha} = \infty,$$

then equation (1.2) *is oscillatory.*

Proof. Let $\{x(n)\}$ be a nonoscillatory solution of equation (1.2), say, $x(n) > 0, x[g(n)] > 0$, and $x[\sigma(n)] > 0$ for $n \geq n_0 \geq 0$. It is easy to see that $\Delta x(n)$ and $\Delta^2 x(n)$ are of fixed sign for $n \geq n_1 \geq n_0$. Again there are four possibilities to consider.

(I) $\Delta^2 x(n) > 0$ and $\Delta x(n) > 0$ for $n \geq n_1$;

(II) $\Delta^2 x(n) < 0$ and $\Delta x(n) > 0$ for $n \geq n_1$;

(III) $\Delta^2 x(n) > 0$ and $\Delta x(n) < 0$ for $n \geq n_1$; and

(IV) $\Delta^2 x(n) < 0$ and $\Delta x(n) < 0$ for $n \geq n_1$.

Case (IV) cannot hold since $\Delta^2 x(n) < 0$ and $\Delta x(n) < 0$ for $n \geq n_1$ would imply $\lim_{n\to\infty} x(n) = -\infty$, which contradicts the positivity of $x(n)$.

Case (I). For $t \geq s \geq n_1$, we have

$$x(t) - x(s) = \sum_{k=s}^{t-1} \Delta x(k),$$

and hence

$$x(t) \geq (t-s)\Delta x(s).$$

Replacing t and s by $\sigma(n)$ and $\rho(n)$ respectively, we obtain

$$(3.4)\qquad x[\sigma(n)] \geq (\sigma(n) - \rho(n))\Delta x[\rho(n)] \quad \text{for} \quad n \geq n_2 \geq n_1,$$

and substituting into (1.2) gives

$$(3.5)\qquad \begin{aligned}\Delta\left(a(n)(\Delta^2 x(n))^{\alpha}\right) &\geq p(n)h(x[\sigma(n)])\\ &\geq p(n)h(\sigma(n)-\rho(n))h(\Delta x[\rho(n)]) \quad \text{for} \quad n \geq n_2.\end{aligned}$$

Setting $y(n) = \Delta x(n)$ in (3.5), we have

$$(3.6)\qquad \Delta\left(a(n)(\Delta y(n))^{\alpha}\right) \geq p(n)h(\sigma(n)-\rho(n))h(y[\rho(n)]) \quad \text{for} \quad n \geq n_2.$$

Once again, for $t \geq s \geq n_2$, we have $y(t) \geq (t-s)\Delta y(s)$, so replacing t and s by $\rho(n)$ and $\theta(n)$ respectively yields

$$y[\rho(n)] \geq (\rho(n) - \theta(n))\Delta y[\theta(n)] = \left(\frac{\rho(n) - \theta(n)}{a^{1/\alpha}[\theta(n)]}\right) z^{1/\alpha}[\theta(n)] \quad \text{for} \quad n \geq n_3 \geq n_2, \tag{3.7}$$

where $z(n) = a(n)(\Delta y(n))^\alpha$. Using (3.7) and (1.5) in (3.6) gives

$$\Delta z(n) \geq p(n)h(\sigma(n) - \rho(n))h\left(\frac{\rho(n) - \theta(n)}{a^{1/\alpha}[\theta(n)]}\right) h\left(z^{1/\alpha}[\theta(n)]\right) \quad \text{for} \quad n \geq n_3. \tag{3.8}$$

By known results in [**6**, **8**], we arrive at the desired contradiction.

Case (II). There exist $n_1 \geq n_0$ and a constant $0 < b < 1$ such that

$$x(n) \geq bn\Delta x(n) \quad \text{for} \quad n \geq n_1,$$

so there exists $n_2 \geq n_1$ such that

$$x[g(n)] \geq bg(n)y[g(n)] \quad \text{for} \quad n \geq n_2, \tag{3.9}$$

where $y(n) = \Delta x(n)$. From (3.9) and (1.4), equation (1.2) becomes

$$\begin{aligned} \Delta\left(a(n)(\Delta y(n))^\alpha\right) &= \Delta\left(a(n)(\Delta^2 x(n))^\alpha\right) = q(n)f(x[g(n)]) \\ &\geq f(b)q(n)f(g(n))f(y[g(n)]) \quad \text{for} \quad n \geq n_2. \end{aligned} \tag{3.10}$$

Clearly, $y(n) > 0$ and $\Delta y(n) < 0$ for $n \geq n_2$. Now for $t \geq s \geq n_2$, we have

$$y(s) \geq (t - s)(-\Delta y(t)).$$

Replacing s and t by $g(n)$ and $\xi(n)$ respectively, we obtain

$$y[g(n)] \geq (\xi(n) - g(n))(-\Delta y[\xi(n)]) \quad \text{for} \quad n \geq n_3 \geq n_2,$$

or

$$y[g(n)] \geq \left(\frac{\xi(n) - g(n)}{a^{1/\alpha}[\xi(n)]}\right) z^{1/\alpha}[\xi(n)] \quad \text{for} \quad n \geq n_3, \tag{3.11}$$

where $z(n) = a(n)(\Delta y(n))^\alpha$. Inserting (3.11) into (3.10), we see that

$$\Delta z(n) + f(b)q(n)f(g(n))f\left(\frac{\xi(n) - g(n)}{a^{1/\alpha}[\xi(n)]}\right) f\left(z^{1/\alpha}[\xi(n)]\right) \leq 0 \quad \text{for} \quad n \geq n_3.$$

The rest of the proof is similar to that of Case (I) of Theorem 2.1.

Case (III). For $t \geq s \geq n_1$, we have $x(s) \geq (t - s)(-\Delta x(t))$, and replacing s and t with $g(n)$ and $\xi(n)$ respectively gives

$$x[g(n)] \geq (\xi(n) - g(n))w[\xi(n)] \quad \text{for} \quad n \geq n_2 \geq n_1, \tag{3.12}$$

where $w(n) = -\Delta x(n)$. Using (3.12) in equation (1.2), we have

$$\begin{aligned} -\Delta\left(a(n)(\Delta w(n))^{\alpha}\right) &= \Delta\left(a(n)(\Delta^2 x(n))^{\alpha}\right) = q(n)f(x[g(n)]) \\ &\geq q(n)f(\xi(n)-g(n))f(w[\xi(n)]), \end{aligned}$$

or

$$\Delta\left(a(n)(\Delta w(n))^{\alpha}\right) + q(n)f(\xi(n)-g(n))f(w[\xi(n)]) \leq 0 \quad \text{for} \quad n \geq n_2.$$

The remainder of the proof is similar to that of Case (III) of Theorem 2.1 and is omitted. This completes the proof of the theorem. □

When condition (2.19) holds, we have the following immediate result.

Theorem 3.2. *Let conditions* (i)–(v), (1.4), (1.5), *and* (2.19) *hold, and assume that there exist nondecreasing sequences* $\{\xi(n)\}, \{\rho(n)\}$, *and* $\{\theta(n)\}$ *such that* $g(n) < \xi(n) < n-1$ *and* $\sigma(n) > \rho(n) > \theta(n) > n+1$ *for all* $n_0 \leq n \in \mathbb{N}(n_0)$. *If equations* (3.1) *and* (3.2) *are oscillatory, then equation* (1.2) *is oscillatory.*

From Theorem 3.1, the following corollary is immediate.

Corollary 3.1. *Let conditions* (i)–(v), (1.3)–(1.5), *and* (3.3) *hold, and assume that there exist nondecreasing real sequences* $\{\xi(n)\}, \{\rho(n)\}$, *and* $\{\theta(n)\}$ *such that* $g(n) < \xi(n) < n-1$ *and* $\sigma(n) > \rho(n) > \theta(n) > n+1$ *for* $n_0 \leq n \in \mathbb{N}(n_0)$. *Equation* (1.2) *is oscillatory if one of the following conditions holds*:

$(\mathrm{II})_1$ $\quad \dfrac{h(u^{1/\alpha})}{u} \geq 1$ *and* $\dfrac{f(u^{1/\alpha})}{u} \geq 1$ *for* $u \neq 0$,

$$\limsup_{n\to\infty} \sum_{k=n}^{\theta(n)-1} p(k)h(\sigma(k)-\rho(k))h\left(\frac{\rho(k)-\theta(k)}{a^{1/\alpha}[\theta(k)]}\right) > 1,$$

and

$$\limsup_{n\to\infty} \sum_{k=\xi(n)}^{n-1} q(k)f(g(k))f\left(\frac{\xi(k)-g(k)}{a^{1/\alpha}[\xi(k)]}\right) > 1;$$

$(\mathrm{II})_2$ $\quad \displaystyle\int^{\pm\infty} \frac{\mathrm{d}u}{h(u^{1/\alpha})} < \infty$ *and* $\displaystyle\int_{\pm 0} \frac{\mathrm{d}u}{f(u^{1/\alpha})} < \infty$,

$$\sum_{k=n_0}^{\infty} p(k)h(\sigma(k)-\rho(k))h\left(\frac{\rho(k)-\theta(k)}{a^{1/\alpha}[\theta(k)]}\right) = \infty,$$

and

$$\sum_{k=n_0}^{\infty} q(k)f(g(k))f\left(\frac{\xi(k)-g(k)}{a^{1/\alpha}[\xi(k)]}\right) = \infty;$$

$(\mathrm{II})_3$ $\quad \dfrac{u}{h(u^{1/\alpha})} \to 0$ *as* $u \to \infty$ *and* $\dfrac{u}{f(u^{1/\alpha})} \to 0$ *as* $u \to \infty$,

$$\limsup_{n\to\infty} \sum_{k=n}^{\theta(n)-1} p(k)h(\sigma(k)-\rho(k))h\left(\frac{\rho(k)-\theta(k)}{a^{1/\alpha}[\theta(k)]}\right) > 0,$$

and

$$\limsup_{n\to\infty} \sum_{k=\xi(n)}^{n-1} q(k)f(g(k))f\left(\frac{\xi(k)-g(k)}{a^{1/\alpha}[\xi(k)]}\right) > 0.$$

4. GENERAL REMARKS

1. The results of this paper are presented in a form which are essentially new and of a high degree of generality. They unify, improve, and extend many well–known oscillation criteria that have appeared in the literature for some special cases of equations (1.1) and (1.2).

2. We note that conditions (1.4) and (1.5) are automatically satisfied if we let $f(x) = x^\beta$ and $h(x) = x^\gamma$, where β and γ are ratio of positive odd integers.

3. The results of this paper can be easily extended to neutral difference equations of the form

$$\Delta\left(a(n)\left(\Delta^2(x(n)+c(n)x[\tau(n)])\right)^\alpha\right) + q(n)f(x[g(n)]) = 0$$

and

$$\Delta\left(a(n)\left(\Delta^2(x(n)+c(n)x[\tau(n)])\right)^\alpha\right) = q(n)f(x[g(n)]) + p(n)h(x[\sigma(n)]),$$

where $\{c(n)\}$ and $\{\tau(n)\}$ are sequences of real numbers and $\lim_{n\to\infty}\tau(n) = \infty$. The formulation of results for the above neutral equations are easy and the details are left to the reader.

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