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RELAXATIONS OF HALL'S CONDITION: OPTIMAL BATCH CODES WITH MULTIPLE QUERIES

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Combinatorial batch codes model the storage of a database on a given number of servers such that any k or fewer items can be retrieved by reading at most t items from each server. A combinatorial batch code with parameters n, k, m, t can be represented by a system $\mathcal{F}$ of n (not necessarily distinct) sets over an m-element underlying set X, such that for any k or fewer members of $\mathcal{F}$ there exists a system of representatives in which each element of X occurs with multiplicity at most t. The main purpose is to determine the minimum $N(n, k, m, t)$ of total data storage $\sum_{F\in\mathcal{F}} |F|$ over all combinatorial batch codes $\mathcal{F}$ with given parameters.

Previous papers concentrated on the case $t = 1$. Here we obtain the first nontrivial results on combinatorial batch codes with $t > 1$. We determine $N(n, k, m, t)$ for all cases with $k \le 3t$, and also for all cases where $n \ge t\binom{m}{\lceil k/t\rceil - 2}$. Our results can be considered equivalently as minimum total size $\sum_{F\in\mathcal{F}} |F|$ over all set systems $\mathcal{F}$ of given order m and size n, which satisfy a relaxed version of Hall's Condition; that is, $|\bigcup \mathcal{F}'| \ge |\mathcal{F}'|/t$ holds for every subsystem $\mathcal{F}' \subseteq \mathcal{F}$ of size at most k.

1. INTRODUCTION

Combinatorial batch codes and dual systems. Batch codes were introduced by Ishai, Kushilevitz, Ostrovsky and Sahai [**10**]. They represent the distributed storage of an n-element database on a set of m servers when any k or fewer data items can be recovered by submitting a limited number t of queries to each server. This model can be used for amortizing the computational cost in

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private information retrieval. Combinatorial batch code, studied in detail first by Paterson, Stinson and Wei [**13**], is the version of a batch code in which each server stores a subset of the database and decoding simply means reading items from servers. The latter model admits a purely combinatorial definition as a set system satisfying a requirement on systems of representatives. Therefore, it is in close connection with Hall-type conditions.

A *set system* $\mathcal{F}$ over an underlying set X is the collection of some nonempty subsets of X. Objects $x \in X$ are called *elements* whilst objects $F \in \mathcal{F}$ are referred to as *members*. Moreover, the *order* and the *size* of a system $\mathcal{F}$ are the number $|X|$ of elements and the number $|\mathcal{F}|$ of members, respectively. The *total size* of a system $\mathcal{F}$ is defined as $\sum_{F\in\mathcal{F}} |F|$. Throughout this paper, 'set system' is meant as a 'multisystem'; that is, repetitions are allowed, distinct members of the system may correspond to the same subset of the underlying set.

A combinatorial batch code with parameters n, k, m, t can be represented with its 'dual' set system (shortly, $\mathrm{CBC}^*(n, k, m, t)$-system) $\mathcal{F}$, where the m elements of the underlying set correspond to the m servers and the members of $\mathcal{F}$ correspond to the n items of data. A member $F_i \in \mathcal{F}$ then means the set of servers where the ith data item is stored. Hence, the total amount of data collectively stored by the m servers—which is the object of minimization—equals the total size of system $\mathcal{F}$. The formal definition of a $\mathrm{CBC}^*(n, k, m, t)$-system can be given as follows.

Definition 1. *For positive integers k and t, a set system $\mathcal{F}$ is a* $\mathrm{CBC}^*(k, t)$-system *if, for every subsystem $\mathcal{F}' = \{F_1, \dots, F_\ell\} \subseteq \mathcal{F}$ of size $1 \le \ell \le k$, there exist elements $x_1, \dots, x_\ell$ such that $x_i \in F_i$ holds for every $1 \le i \le \ell$ and each element of X has multiplicity at most t in $\{x_1, \dots, x_\ell\}$. A set system $\mathcal{F}$ over the underlying set X is called a* $\mathrm{CBC}^*(n, k, m, t)$-system *if $|\mathcal{F}| = n$, $|X| = m$, and $\mathcal{F}$ is a $CBC^*(k, t)$-system. Moreover, $N(n, k, m, t) := \min_{\mathcal{F}} \sum_{F\in\mathcal{F}} |F|$ denotes the minimum total size of a system taken over all $CBC^*(n, k, m, t)$-systems $\mathcal{F}$, subject to that there exists at least one such system.*

Note that if both $mt < k$ and $mt < n$ hold, no $\mathrm{CBC}^*(n, k, m, t)$-system exists. Otherwise, the system containing the underlying set X as member with multiplicity n is a $\mathrm{CBC}^*(n, k, m, t)$ and hence $N(n, k, m, t)$ is well-defined. We will assume throughout that n, k, m and t denote positive integers such that $mt \ge \min\{n, k\}$. Systems which are $\mathrm{CBC}^*(n, k, m, t)$ and have minimum total size $N(n, k, m, t)$ will be called *optimal*.

Hall-type conditions. Hall's Theorem [**9**] and related results on algorithms serve as basic tools in several branches of combinatorics and discrete optimization. Also, nonstandard Hall-type conditions and their consequences were intensively studied (see, e.g., [**6, 7, 8, 11, 12**]). Each earlier paper on combinatorial batch codes with $t = 1$ applied Hall's Condition. Here we use a relaxed version whose origin goes back to the works [**7, 8, 12**].

Definition 2. *We say that a set system* $\mathcal{F}$ *satisfies the* (k,t)-Hall Condition *(shortly,* (k,t)*-HC) if* $|\bigcup \mathcal{F}'| \geq |\mathcal{F}'|/t$ *holds for every subsystem* $\mathcal{F}' \subseteq \mathcal{F}$ *which contains at most* k *members.*

Results. In [**1, 2, 3, 4, 10, 13**] several results on combinatorial batch codes were obtained, moreover their connections with transversal matroids [**2**], unbalanced expander graphs [**10**] and binary constant-weight codes [**1**] were also pointed out. These papers considered—nearly exclusively—the case of $t = 1$, although some simple relations between combinatorial batch codes with $t > 1$ and those with $t = 1$ were established in [**10**].

In this paper we obtain the first nontrivial results for the case of general t. In Section 2 we prove the Equivalence Theorem, which is a three-sided characterization: beside the equivalence of the (k,t)-Hall Condition and the property of being a $\text{CBC}^*(k,t)$-system, the requirement can also be expressed in a form which implies that if $\lceil k/t \rceil = \lceil k'/t \rceil$ then a $\text{CBC}^*(k,t)$-system is a $\text{CBC}^*(k',t)$-system and vice versa. Some further basic properties and a cardinality-balancing transformation will be presented, too. In Section 3 and Section 4 we determine the minimum total size $N(n,k,m,t)$ for all parameters satisfying $n \geq t\binom{m}{\lceil k/t \rceil - 2}$ and for all cases where $k \leq 3t$, respectively. By the Equivalence Theorem, several methods developed originally for the case $t = 1$ can be applied for the general setting $t \geq 1$. Our proof techniques used here are similar to those in [**3**] and occasionally to those in [**1**] and [**13**], too. Some results proved here have been announced without proofs in [**5**].

2. SOME BASIC PROPERTIES

In this section we deal with three types of properties. First, we give three equivalent conditions for a system to be a $\text{CBC}^*(k,t)$. Then, we present some basic inequalities about the size distributions of members in a $\text{CBC}^*(n,k,m,t)$, and finally we show that for every four-tuple of parameters there exists an optimal $\text{CBC}^*(n,k,m,t)$ which either does not contain members larger than $\lceil k/t \rceil - 1$ or does not contain members smaller than $\lceil k/t \rceil - 1$.

In the following theorem, the equivalence of (i) and (ii) is a consequence of more general results on systems of representatives [**8, 12, 7**], hence we prove only the equivalence of (ii) and (iii).

Theorem 3. (**Equivalence Theorem**) *For all positive integers* k *and* t, *and for every set system* $\mathcal{F}$, *the following statements are equivalent:*

(i) $\mathcal{F}$ *is a* $CBC^*(k,t)$*-system.*

(ii) $\mathcal{F}$ *satisfies the* (k,t)*-Hall Condition.*

(iii) *For every* $\ell < \lceil k/t \rceil$ *and for every* ℓ*-element subset* X' *of the underlying set, at most* ℓt *members of* $\mathcal{F}$ *are subsets of* X'.

Proof. (ii) $\Leftrightarrow$ (iii) We prove the equivalence of the negations of (ii) and (iii). If (ii) does not hold, there exists a subsystem $\mathcal{F}' \subseteq \mathcal{F}$ of size $i \leq k$, for which the union $X' = \bigcup \mathcal{F}$ has at most $\lceil i/t \rceil - 1$ elements. That is, X' contains at least $i > t\,(\lceil i/t \rceil - 1) \geq t|X'|$ members of $\mathcal{F}$, and also $|X'| \leq \lceil k/t \rceil - 1$ is valid. This means that (iii) does not hold either. From the other direction, if a subset $X' \subseteq X$ of cardinality $\ell \leq \lceil k/t \rceil - 1$ contains more than ℓt members from $\mathcal{F}$, then the union of any $\ell t+1 \leq k$ of these members can contain at most $|X'| = \ell < \ell+1 = \lceil (\ell t+1)/t \rceil$ elements, which contradicts (ii). $\square$

Part (iii) of Theorem 3 expresses the (k,t)-Hall Condition referring only to $\lceil k/t \rceil$ and t as parameters. Hence, if an integer $t > 1$ is fixed, not the exact value of k but only $\lceil k/t \rceil$ is that really matters the meaning of (k,t)-HC. Particularly, it would suffice to determine the optimal total size $N(n,k,m,t)$ only for cases where k is divisible by t.

Corollary 4. *Assume that $\lceil k/t \rceil = \lceil k'/t \rceil$. Then, a system $\mathcal{F}$ is a CBC*(k,t)-system if and only if it is a CBC*(k',t)-system; moreover, $\mathcal{F}$ satisfies the (k,t)-Hall Condition if and only if it satisfies the (k',t)-Hall Condition. Particularly, if $\lceil k/t \rceil = \lceil k'/t \rceil$ then $N(n,k,m,t) = N(n,k',m,t)$ is valid for all n and m.*

From now on, also requirement (iii) from the Equivalence Theorem will be referred to as (k,t)-HC. Applying Theorem 3, the next necessary condition for systems satisfying (k,t)-HC is easy to verify. The analogous result for the special case of $t = 1$ first appeared in a proof of [**13**], and later it was stated in [**1**] and [**3**] as well.

Theorem 5. *Let $\mathcal{F}$ be a CBC*(n,k,m,t) and let ℓ_i denote the number of i-element members of $\mathcal{F}$, for every $1 \leq i \leq \lceil k/t \rceil$. Then,*

$$\sum_{i=1}^{\lceil k/t \rceil - 1} \ell_i \binom{m-i}{\lceil k/t \rceil - 1 - i} \leq t \left(\left\lceil \frac{k}{t} \right\rceil - 1 \right) \binom{m}{\lceil k/t \rceil - 1}.$$

Proof. We are going to estimate the number z of pairs (F, A) with $F \in \mathcal{F}$, $F \subseteq A \subseteq X$ and $|A| = \lceil k/t \rceil - 1$. Every i-element member F from $\mathcal{F}$ is contained in exactly $\binom{m-i}{\lceil k/t \rceil - 1 - i}$ such subsets A. Consequently, $z = \sum_{i=1}^{\lceil k/t \rceil - 1} \ell_i \binom{m-i}{\lceil k/t \rceil - 1 - i}$. On the other hand, since $\mathcal{F}$ satisfies (k,t)-HC, every $(\lceil k/t \rceil - 1)$-element $A \subseteq X$ contains at most $t(\lceil k/t \rceil - 1)$ members from $\mathcal{F}$. Therefore, $z \leq t(\lceil k/t \rceil - 1)\binom{m}{\lceil k/t \rceil - 1}$ and the inequality stated in the theorem follows.

Corollary 6. *Every CBC*(n,k,m,t) contains at most $t\,(\lceil k/t \rceil - 1)\binom{m}{\lceil k/t \rceil - 1}$ members of size not exceeding $\lceil k/t \rceil - 1$.*

Due to the Equivalence Theorem, we can take some observations on extensions of a CBC*(k,t)-system $\mathcal{F}$ with a new member $F \subseteq X$. First, since the fulfilment of (k,t)-HC depends only on members of size at most $\lceil k/t \rceil - 1$, the following statement clearly holds.

Observation 7. *If $\mathcal{F}$ is a CBC*(k,t)-system and $|F| \geq \lceil k/t \rceil$, then $\mathcal{F} \cup \{F\}$ is a CBC*(k,t)-system, as well. Therefore, an optimal CBC*(n,k,m,t)-system does not contain members of size greater than $\lceil k/t \rceil$.*

Second, since a member F of size $\lceil k/t \rceil - 1$ is not contained in a $(\lceil k/t \rceil - 1)$-element subset of X other than itself, the following statement is valid.

Proposition 8. *Let $\mathcal{F}$ be a CBC*(k,t)-system and $|F| = \lceil k/t \rceil - 1$. Then, $\mathcal{F} \cup \{F\}$ is a CBC*(k,t)-system if and only if F contains fewer than $t(\lceil k/t \rceil - 1)$ members from $\mathcal{F}$. Moreover, if ℓ_i denotes the number of members of size i in $\mathcal{F}$ (for each $1 \leq i \leq \lceil k/t \rceil - 1$), then $\mathcal{F}$ can be extended with L appropriately chosen new members each of cardinality $\lceil k/t \rceil - 1$, such that the system remains a CBC*(k,t), if and only if*

$$L \leq t\left(\left\lceil \frac{k}{t} \right\rceil - 1\right)\binom{m}{\lceil k/t \rceil - 1} - \sum_{i=1}^{\lceil k/t \rceil - 1} \ell_i \binom{m-i}{\lceil k/t \rceil - 1 - i}.$$

Next, we present a transformation which is applicable for two members of a CBC*(n,k,m,t) if one of them contains the other. Then, some (any) elements from the larger member can be transferred to the smaller one and the system remains a CBC*(n,k,m,t) with the same total size. This transformation was introduced in [**3**] (Proposition 1) for the case $t = 1$. In fact the proof remains the same for the general case $t \geq 1$, hence it is omitted here.

Proposition 9. [**3**] *Let $\mathcal{F}$ be a CBC*(n,k,m,t) with two members $F_1 \subset F_2$ for which $|F_1| + 2 \leq |F_2|$ and let A be a nonempty set such that $A \subset F_2 \setminus F_1$. Then, replacing F_1 and F_2 with $F_1' = F_1 \cup A$ and $F_2' = F_2 \setminus A$, the obtained system $\mathcal{F}'$ is a CBC*(n,k,m,t) as well, and the two systems $\mathcal{F}$ and $\mathcal{F}'$ have the same total size.*

We say that a CBC* is of *type* $[a,b]$ if the size of each $F \in \mathcal{F}$ satisfies $a \leq |F| \leq b$. Due to Observation 7, every optimal CBC*(n,k,m,t)-system is of type $[1, \lceil k/t \rceil]$. By Proposition 9 we can prove a stronger result for $\lceil k/t \rceil \geq 3$.

Proposition 10. *If $\lceil k/t \rceil \geq 3$, then for every optimal CBC*(n,k,m,t)-system $\mathcal{F}$, there exists an $\mathcal{F}'$ which is an optimal CBC*(n,k,m,t) as well, and has type either $[1, \lceil k/t \rceil - 1]$ or $[\lceil k/t \rceil - 1, \lceil k/t \rceil]$.*

Proof. Suppose that an optimal CBC*(n,k,m,t)-system $\mathcal{F}$ contains a member F_1 of size $\ell \leq \lceil k/t \rceil - 2$ and also a member F_2 of size $\lceil k/t \rceil$. Observation 7 implies that F_2 can be replaced with any $\lceil k/t \rceil$-element subset F_2' of the underlying set. Let us choose this new member such that $F_2' \supset F_1$. Now, applying the transformation described in Proposition 9, an optimal CBC*(n,k,m,t)-system $\mathcal{F}'$ is obtained which contains fewer members of size $\lceil k/t \rceil$ than $\mathcal{F}$ did. Repeated application of this procedure yields an optimal CBC*(n,k,m,t) of type either $[1, \lceil k/t \rceil - 1]$ or $[\lceil k/t \rceil - 1, \lceil k/t \rceil]$. □

In the simple cases listed in the following observation it is enough to take n singletons to obtain a $\mathrm{CBC}^*(n,k,m,t)$.

Observation 11. *If at least one of $n \le tm$ and $k \le t$ is valid, then $N(n,k,m,t) = n$.*

The next proposition is the generalization of Theorem 4 of [**13**].

Proposition 12. *For every four positive integers n,k,m and t, if $m = \lceil k/t \rceil$ and $n \ge tm$, then $N(n,k,m,t) = mn - tm(m-1)$.*

Proof. Under the given conditions consider a $\mathrm{CBC}^*(n,k,m,t)$-system $\mathcal{F}$. By (k,t)-HC, for every element x of the underlying set X, the $(m-1)$-element set $X \setminus \{x\}$ covers entirely at most $t(m-1)$ members of $\mathcal{F}$. Thus, x has to be involved in at least $n - t(m-1)$ members of $\mathcal{F}$. Therefore, counting the total size of the system by summing up the degrees of elements, $N(n,k,m,t) \ge m(n-t(m-1))$ must hold. On the other hand, let $\mathcal{F}^*$ be the system over the underlying set $X = \{x_1,\dots,x_m\}$, in which X is a member with multiplicity $n - tm$ and each singleton $\{x_i\}$ occurs with multiplicity t. Clearly, $\mathcal{F}^*$ is a $\mathrm{CBC}^*(n,k,m,t)$-system and its total size is exactly $tm + (n-tm)m = mn - tm(m-1)$. This verifies the statement.

3. OPTIMUM VALUES FOR $n \ge t\binom{m}{\lceil k/t \rceil - 2}$

Theorem 13. *If $m \ge \left\lceil \frac{k}{t} \right\rceil$ and $n > t\left(\left\lceil \frac{k}{t} \right\rceil - 1\right)\binom{m}{\lceil k/t \rceil - 1}$, then*

$$N(n,k,m,t) = n\left\lceil \frac{k}{t} \right\rceil - t\left(\left\lceil \frac{k}{t} \right\rceil - 1\right)\binom{m}{\lceil k/t \rceil - 1}.$$

Proof. Consider parameters n,k,m and t satisfying the conditions given in the theorem. Due to Corollary 6, the number of members of $\mathcal{F}$ which are of size smaller than $\lceil k/t \rceil$ is at most $t\left(\lceil k/t \rceil - 1\right)\binom{m}{\lceil k/t \rceil - 1}$. Thus, under the present conditions, system $\mathcal{F}$ cannot be of type $[1, \lceil k/t \rceil - 1]$. Then, Proposition 10 implies that there exists an optimal $\mathrm{CBC}^*(n,k,m,t)$-system $\mathcal{F}$ of type $[\lceil k/t \rceil - 1, \lceil k/t \rceil]$. The total size of $\mathcal{F}$ is precisely $n\lceil k/t \rceil - n'$ where n' denotes the number of $(\lceil k/t \rceil - 1)$-element members. Applying Corollary 6 again, we obtain

$$N(n,k,m,t) = n\left\lceil \frac{k}{t} \right\rceil - n' \ge n\left\lceil \frac{k}{t} \right\rceil - t\left(\left\lceil \frac{k}{t} \right\rceil - 1\right)\binom{m}{\lceil k/t \rceil - 1}.$$

On the other hand, take each $(\lceil k/t \rceil - 1)$-element subset of an m-element underlying set X with multiplicity $t\left(\lceil k/t \rceil - 1\right)$ and further $n - t\left(\lceil k/t \rceil - 1\right)\binom{m}{\lceil k/t \rceil - 1}$ subsets of X, each of cardinality $\lceil k/t \rceil$. This construction is clearly a CBC$^*(n,k,m,t)$-system and proves that $N(n,k,m,t) \le n\lceil k/t\rceil - t\left(\lceil k/t\rceil - 1\right)\binom{m}{\lceil k/t\rceil - 1}$. This verifies the theorem. $\square$

To obtain a formula for the second highest range of n, we will apply the following technical lemma proved in [**3**].

Lemma 14. [**3**] *For any three integers i, p, m, if $1 \le i \le p \le m-1$, then*

$$\left\lfloor \left(\binom{m-i}{p-i} - 1\right)/(m-p) \right\rfloor \ge p - i.$$

Theorem 15. *If $m \ge \left\lceil \frac{k}{t} \right\rceil \ge 3$ and $t\binom{m}{\lceil k/t\rceil - 2} \le n \le t\left(\left\lceil \frac{k}{t}\right\rceil - 1\right)\binom{m}{\lceil k/t\rceil - 1}$, then*

$$N(n,k,m,t) = n\left(\left\lceil \frac{k}{t}\right\rceil - 1\right) - \left\lfloor \frac{t\left(\left\lceil \frac{k}{t}\right\rceil - 1\right)\binom{m}{\lceil k/t\rceil - 1} - n}{m - \left\lceil \frac{k}{t}\right\rceil + 1} \right\rfloor.$$

Proof. If $m = \lceil k/t\rceil$, the statement yields $N(n,k,m,t) = mn - tm(m-1)$ which corresponds to Proposition 12. Hence, we assume that $m > \lceil k/t \rceil$. Let us introduce the notation

$$K := \left\lceil \frac{k}{t} \right\rceil, \qquad y := \left\lfloor \frac{t(K-1)\binom{m}{K-1} - n}{m-K+1} \right\rfloor.$$

We construct a CBC$^*(n,k,m,t)$-system $\mathcal{F}^*$ on an m-element underlying set X as follows. First, choose y sets, each of cardinality $K-2$, such that every $(K-2)$-element subset of X has multiplicity at most t. This can be done, since by the given condition, $t\binom{m}{K-2} \le n$ holds and hence,

$$y \le \frac{t(K-1)\binom{m}{K-1} - n}{m-K+1} \le \frac{t(m-K+2)\binom{m}{K-2} - t\binom{m}{K-2}}{m-K+1} = t\binom{m}{K-2}.$$

Since every $(K-2)$-element subset of X contains at most t members, and every $(K-1)$-element subset contains at most $t(K-1)$ members, the obtained system is a CBC$^*(k,t)$. Moreover, in view of Proposition 8, the following inequality proves that the system can be extended with $n-y$ members, each of cardinality $K-1$, such that a CBC$^*(n,k,m,t)$-system $\mathcal{F}^*$ is obtained.

$$t\,(K-1)\binom{m}{K-1} - \left\lfloor \frac{t(K-1)\binom{m}{K-1}-n}{m-K+1} \right\rfloor (m-K+2)$$
$$\geq t\,(K-1)\binom{m}{K-1} - \left(t\,(K-1)\binom{m}{K-1}-n\right) - y = n-y.$$

The total size of $\mathcal{F}^*$ is $n(K-1)-y$, hence this is an upper bound on $N(n,k,m,t)$.

Turning to the lower bound, by Proposition 10 there exists an optimal CBC$^*(n,k,m,t)$ of type either $[1,K-1]$ or $[K-1,K]$. But if a CBC$^*(n,k,m,t)$ belongs to the latter type and contains a member of size K as well, then its total size is greater than $n(K-1)-y$ and consequently it cannot be optimal. Thus, there exists an optimal CBC$^*(n,k,m,t)$-system $\mathcal{F}$ of type $[1,K-1]$.

For every $1 \leq i \leq K-1$, denote by ℓ_i the number of members of size i in $\mathcal{F}$. The total size of $\mathcal{F}$ is

$$\mathcal{S}(\mathcal{F}) = \sum_{i=1}^{K-1} i\ell_i = (K-1)n - \sum_{i=1}^{K-2} (K-1-i)\,\ell_i. \tag{1}$$

On the other hand, Theorem 5 yields

$$\ell_{K-1} + \sum_{i=1}^{K-2} \ell_i \binom{m-i}{K-1-i} \leq t(K-1)\binom{m}{K-1}.$$

Substituting $\ell_{K-1} = n - (\ell_1 + \cdots + \ell_{K-2})$, this implies

$$\sum_{i=1}^{K-2} \ell_i \left\lfloor \frac{\binom{m-i}{K-1-i}-1}{m-K+1} \right\rfloor \leq \left\lfloor \frac{t(K-1)\binom{m}{K-1}-n}{m-K+1} \right\rfloor = y. \tag{2}$$

Now, we verify that $\mathcal{S}(\mathcal{F}) \geq (K-1)n - y$ holds. With $p = K-1$, Lemma 14 states that for every $1 \leq i \leq K-2$

$$K-1-i \leq \left\lfloor \frac{\binom{m-i}{K-1-i}-1}{m-K+1} \right\rfloor$$

is valid. Together with (1) and (2) this implies

$$\mathcal{S}(\mathcal{F}) = (K-1)n - \sum_{i=1}^{K-2} (K-1-i)\,\ell_i \geq (K-1)n - \sum_{i=1}^{K-2} \ell_i \left\lfloor \frac{\binom{m-i}{K-1-i}-1}{m-K+1} \right\rfloor$$
$$\geq (K-1)n - y.$$

Therefore, $N(n,k,m,t) = \mathcal{S}(\mathcal{F}) \geq (K-1)n - y$ follows, which completes the proof of the theorem. $\square$

The results analogous to Theorems 13 and 15 with $t = 1$ were obtained in [**13**] and [**3**], respectively.

4. OPTIMUM VALUES FOR $k \leq 3t$

In this section we determine exact formulae for the minimum total size $N(n,k,m,t)$ of combinatorial batch codes for all cases when $k \leq 3t$ holds. Due to Observation 11, if $\lceil k/t \rceil = 1$ then $N(n,k,m,t) = n$. Applying results from the previous section, formulae for the remaining cases $t < k \leq 2t$ and $2t < k \leq 3t$ can be obtained.

Theorem 16. *If* $\left\lceil \frac{k}{t} \right\rceil = 2$ *and* $m \geq 2$, *then*

$$\begin{array}{lll} N(n,k,m,t) = n & \textit{if} & n \leq tm; \\ N(n,k,m,t) = 2n - tm & \textit{if} & n > tm. \end{array}$$

Proof Observation 11 and Theorem 13 together cover all possibilities for $\lceil k/t \rceil = 2$ and yield the formulae in the statement. $\square$

Theorem 17. *If* $\left\lceil \frac{k}{t} \right\rceil = 3$ *and* $m \geq 3$, *then*

$$\begin{array}{lll} N(n,k,m,t) = n & \textit{if} & n \leq tm; \\ N(n,k,m,t) = 2n - mt + \left\lceil \frac{n - mt}{m-2} \right\rceil & \textit{if} & tm < n \leq 2t\binom{m}{2}; \\ N(n,k,m,t) = 3n - 2t\binom{m}{2} & \textit{if} & 2t\binom{m}{2} < n. \end{array}$$

Proof. Observation 11 yields the first formula whilst Theorem 13 yields the third one, by a simple substitution. Moreover, the condition $tm < n \leq tm(m-1)$ corresponds to that in Theorem 15. After substituting $\lceil k/t \rceil = 3$, the following computation yields the second formula:

$$\begin{aligned} N(n,k,m,t) &= 2n - \left\lfloor \frac{2t\binom{m}{2} - n}{m-2} \right\rfloor \\ &= 2n - mt - \left\lfloor \frac{tm - n}{m-2} \right\rfloor = 2n - mt + \left\lceil \frac{n - mt}{m-2} \right\rceil . \end{aligned}$$

which concludes the proof. $\square$

For the particular case of $t = 1$ the theorems above yield a direct consequence of Theorem 8 from [**13**] and Theorem 1 from [**3**].

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