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LEBESGUE-STIELTJES INTEGRAL AND YOUNG'S INEQUALITY

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For non-decreasing real functions f and g, we consider the functional $T(f, g; I, J) = \int_I f(x)\,\mathrm{d}g(x) + \int_J g(x)\,\mathrm{d}f(x)$, where I and J are intervals with $J \subseteq I$. In particular case with $I = [a, t]$, $J = [a, s]$, $a < s \le t$ and $g(x) = x$, this reduces to the expression in classical Young's inequality. We survey some properties of Lebesgue-Stieltjes integrals and present a simple proof for change of variables. Further, we formulate a version of Young's inequality with respect to arbitrary positive measure on real line and discuss applications in probability and number theory.

1. INTRODUCTION AND FIRST GENERALIZATION

Let f be a continuous and increasing function defined on an interval $I \subset \mathbb{R}$ and let a, s, t be arbitrary points in I. A version of Young's inequality [**19**] states that

$$Y(f; a, s, t) := \int_a^t f(x)\,\mathrm{d}x + \int_{f(a)}^{f(s)} f^{-1}(x)\,\mathrm{d}x \ge tf(s) - af(a). \tag{1}$$

There has been a considerable amount of literature related to this classical inequality. It seems that the first strict analytic proofs were given in [**8**] and in [**5**]. A relation between Young's functional and integration by parts in Riemann-Stieltjes integral was used in in [**2**] and [**3**] for proving various versions of Young's inequality, and also in [**1**] to derive several interesting applications. The reverse, in the sense

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that if (1) holds for some functional $Y(f;a,s,t)$ and for all continuous increasing f, then Y is of the form as in (1), has also been a topic of research. Although in this article we do not address the question of reverse, some interesting contributions like [**5**], [**12**] or [**13**] are worth mentioning. Finally, let us recall that the topic is closely related to convexity, complementary Young's functions and Orlicz spaces.

The first recorded result on upper bounds for Young's functional Y was found by Merkle [**14**] in 1974; in the setup of (1) it reads

$$Y(f;a,s,t) \le f(a)(t-a) + a\big(f(s)-f(a)\big) + \max\Big((t-a)\big(f(t)-f(a)\big), (s-a)\big(f(s)-f(a)\big)\Big), \tag{2}$$

where $s > a, t > a$. In 2008, Minguzzi [**16**] discovered a closer upper bound also with $s > a, t > a$:

$$Y(f;a,s,t) \le tf(t) + sf(s) - af(a) - sf(t) \tag{3}$$

and observed that, given f and an $a \in I$, (1) holds for all $s,t > a$ if and only if (3) does, that is, the statements (1) and (3) are equivalent. Incidentally, the same result was stated by Witkowski in [**18**, Second proof of Theorem 1] as well as (together with the observation about equivalence) by Cerone [**6**] in 2009.

Extending the idea of [**3**], one can observe a more general functional:

$$\begin{aligned} S(f,g;a,s,t) &:= \int_a^t f(x)\,\mathrm{d}g(x) + \int_a^s g(x)\,\mathrm{d}f(x) \\ &= \int_a^t f(x)\,\mathrm{d}g(x) + \int_{f(a)}^{f(s)} g(f^{-1}(y))\,\mathrm{d}y, \end{aligned} \tag{4}$$

where g is an arbitrary continuous and increasing function and integrals are taken in the Riemann-Stieltjes sense. For $g(x) = x$ this reduces to the classical case (1). The next theorem presents our first generalization of (1) and (3).

Theorem 1.1. *Let f and g be continuous increasing functions defined on an interval $I \subset \mathbb{R}$ and let a,s,t be arbitrary points in I. Then*

$$\begin{aligned} g(t)f(s) - g(a)f(a) &\le S(f,g;a,s,t) \\ &\le g(t)f(t) + g(s)f(s) - g(a)f(a) - g(s)f(t), \end{aligned} \tag{5}$$

with equality in both sides if and only if $s = t$.

Proof. Using integration by parts, we have that $S(f,g;a,s,s) = f(s)g(s) - f(a)g(a)$ and so, for $s \le t$,

$$\begin{aligned} S(f,g;a,s,t) &= S(f,g;a,s,s) + \int_s^t f(x)\,\mathrm{d}g(x) \\ &\ge f(s)g(s) - f(a)g(a) + f(s)\big(g(t)-g(s)\big) = g(t)f(s) - g(a)f(a), \end{aligned}$$

which is the first inequality in (5). The equality is possible if an only if

$$\int_s^t \big(f(x) - f(s)\big)\,\mathrm{d}g(x) = 0,$$

that is, if and only if $s = t$. For $s > t$, the proof is based on the relation

$$S(f, g; a, s, t) = S(f, g; a, t, t) + \int_t^s g(x)\,\mathrm{d}f(x)$$

and then one proceeds as above.

For the second inequality in (5), we start with the observation that

$$S(f, g; a, s, t) + S(g, f; a, s, t) = f(t)g(t) - f(a)g(a) + f(s)g(s) - f(a)g(a)$$

and so,

$$S(f, g; a, s, t) = f(t)g(t) + f(s)g(s) - 2f(a)g(a) - S(g, f; a, s, t).$$

Applying here the first inequality in (5) with $S(g, f; a, s, t)$ we get the second one which concludes the proof. □

Let us remark that the left and right inequality in (5) follow from each other, so the equivalence between (1) and (3) is preserved in the above generalization.

In section 3 we offer an ultimate generalization, for the case when f and g are non-increasing and not necessarily continuous. To achieve this goal, we need the material of Section 2. Examples of applications are postponed to Section 4.

2. LEBESGUE-STIELTJES INTEGRAL: CHANGE OF VARIABLES AND INTEGRATION BY PARTS

In the proof of Theorem 1.1, we used integration by parts formula, and the representation in the second line of (4) is due to change of variables $y = f(x)$. These are extensions of formulas in Riemann integral calculus to Riemann-Stieltjes case, and proofs can be found in the classical text [**17**]. In this section we deal with Lebesgue-Stieltjes integrals of the form $\int f\,\mathrm{d}g$, where f and g are non-decreasing functions that may have jumps. The change of variables in this case is based on the notion of generalized inverse (defined bellow, see (8)) and we were not able to locate a strict proof in the existing literature, except a brief note in [**17**, p.124]. Here we provide a simple proof (Lemma 2.2) for our case with f and g being non-decreasing; it can be extended to a more general setup, but this will be addressed elsewhere. The second part of this section is devoted to integration by parts in Lebesgue-Stieltjes integrals.

For a non-decreasing function $g : \mathbb{R} \mapsto \mathbb{R}$ one can define a positive measure μ_g on the family of intervals $[a, b]$ by

(6) $$\mu_g([a, b]) := g(b_+) - g(a_-), \qquad (a \le b)$$

and extend it to the Borel sigma field on real line. The Lebesgue-Stieltjes (L-S) integral of a given measurable function f with respect to g is defined as the Lebesgue integral of f with respect to μ_g

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}g(x) := \int_{\mathbb{R}} f(x)\,\mathrm{d}\mu_g(x)$$

If this integral is finite, we say that f is g-integrable. Let us assume that f is g-integrable and non-negative. Consider a sequence of partitions of intervals $[0,n]$ on the y-axis with subintervals $J_{n,k} = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$, $k = 1, 2, \ldots, n2^n$, and points $y_{n,k} = \frac{k-1}{2^n}$. Then from the theory of Lebesgue integration it follows that

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}g(x) = \lim_{n\to+\infty} \sum_{k=1}^{n2^n} y_{n,k}\mu_g(f^{-1}(J_{n,k})). \tag{7}$$

Generally, we use the decomposition $f = f_+ - f_-$, where $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = \max\{-f(x), 0\}$.

The equality (6) can be related to Lebesgue measure on real line, since $\mu_g([a,b]) = g(b_+) - g(a_-) = \mathrm{Leb}\left([g(a_-), g(b_+)]\right)$. More generally, for an interval I define the interval I_g as $[g(a_-), g(b_+)]$, $[g(a_-), g(b_-)]$, $[g(a_+), g(b_+)]$, $[g(a_+), g(b_-)]$, if I is $[a,b]$, $[a,b)$, $(a,b]$ or (a,b), respectively. For unbounded intervals I the same convention can be kept, taking $g(\pm\infty)$ to be the limiting value of monotone function g.

Sometimes it is convenient to think about empty set as a special interval; for $I = \emptyset$ we define $I_g = [a,a]$ where a is any real number, say zero; then $\mu_g(\emptyset) = \mathrm{Leb}\,([a,a]) = 0$.

In what follows we need a uniform notation for ends of the closed interval I_g, so let us define $g_*(I)$ and $g^*(I)$ to be left and right endpoint of the closed interval I_g, that is, $I_g = [g_*(I), g^*(I)]$. Note that for a non-decreasing g, $g_*(I) \le g(x) \le g^*(I)$ for any $x \in I$. Then on an arbitrary interval I we may define

$$\int_I f(x)\,\mathrm{d}g(x) = \int_{\mathbb{R}} f(x)\,\mathrm{d}g_I(x),$$

where $g_I(x)$ is defined to be $g(x)$ if $g_*(I) \le g(x) \le g^*(I)$, and $g_*(I)$ or $g^*(I)$ if $g(x) < g_*(I)$ or $g(x) > g^*(I)$, respectively. If a non-decreasing f is defined only on I, we may extend it to $\mathbb{R}$ in a way analogous to previous case, and so integrals of the form $\int_I f(x)\,\mathrm{d}g(x)$ where f and g are non-decreasing functions can be always expressed via integrals of the form $\int_R f(x)\,\mathrm{d}g(x)$ with some other non-decreasing functions f and g defined on $\mathbb{R}$. Therefore, we may always assume that the functions f and g are defined on $\mathbb{R}$.

Now, the general formula analogous to (6) reads

$$\mu_g(I) = \mathrm{Leb}\,(I_g), \quad \text{i.e.} \quad \int_I \mathrm{d}g(x) = \int_{I_g} \mathrm{d}y,$$

which is in fact change of variables $y = g(x)$. We want to extend this formula for integrals of the form $\int f\,\mathrm{d}g$ with a more general f.

Let g be a non-decreasing (not necessarily continuous) function on I. Following [**7**] and [**3**], we say that x a generalized inverse of g at y, $x = g^{-1}(y)$, if

$$\sup\{t \mid g(t) < y\} \le x \le \inf\{t \mid g(t) > y\}. \tag{8}$$

The generalized inverse $g^{-1}(y)$ is not unique at a point y if and only if $g(I) = \{y\}$, where I is an interval with non-empty interior; in that case, any rule that will assign an $x \in \overline{I}$ yields a version of $g^{-1}(y)$. From (8) the following properties follow easily:

$$g(x) < y \implies x \le g^{-1}(y), \quad g(x) > y \implies x \ge g^{-1}(y); \tag{9}$$

$$g(x_-) < y < g(x_+) \implies g^{-1}(y) = x,\ g^{-1}(y) = x \implies g(x_-) \le y \le g(x_+); \tag{10}$$

$$g^{-1}(y) < x \implies y \le g(x_-), \quad g^{-1}(y) > x \implies y \ge g(x_+); \tag{11}$$

$$g^{-1}(y) \le x \implies y \le g(x_+), \quad g^{-1}(y) \ge x \implies y \ge g(x_-). \tag{12}$$

The next lemma gives another way of expressing $\mu_g(I)$, for an interval I.

Lemma 2.1. *Let g be a non-decreasing functions on $\mathbb{R}$ and let I be an interval. For any version of the generalized inverse g^{-1} it holds that $\{y \mid g^{-1}(y) \in I\} \subseteq I_g$ and*

$$\mu_g(I) = \mathrm{Leb}\,(I_g) = \mathrm{Leb}\,\big(\{y \mid g^{-1}(y) \in I\}\big). \tag{13}$$

Proof. The relations (11) and (12) show that for any $y \in \mathbb{R}$, the set $\{y \mid g^{-1}(y) \in I\}$ is a subset of I_g. The first equality in (13) is already explained and discussed above, so we need to prove the second equality only. To this end, given I and a fixed version of g^{-1}, define $J := \{y \mid g^{-1}(y) \in I\}$.

For a singleton $I = \{a\}$, we have that $I_g = [g(a_-), g(a_+)]$ and by the second property in (10) we have that $J \subseteq I_g$. The first property in (10) implies that $I_g \subseteq J \cup \{g(a_-), g(a_+)\}$, hence $\mathrm{Leb}\,(I_g) = \mathrm{Leb}\,(J)$.

If I is an open interval $I = (a, b)$ where $-\infty \le a < b \le +\infty$, then $I_g = [g(a_+), g(b_-)]$, and the property (11) shows that $J \subseteq I_g$, whereas the reverse of (12) implies that $I_g \subseteq J \cup \{g(a_+), g(b_-)\}$, and again $\mathrm{Leb}\,(I_g) = \mathrm{Leb}\,(J)$.

For other kinds of intervals, the statement follows from additivity. □

Now we can give the announced proof of change of variables formula in case when f is also non-decreasing.

Lemma 2.2. *Let f and g be non-decreasing functions on an interval I and let g^{-1} be any version of generalized inverse. Then*

$$\int_I f(x)\,\mathrm{d}g(x) = \int_{I_g} f\big(g^{-1}(y)\big)\,\mathrm{d}y. \tag{14}$$

Proof. Since f is non-decreasing, then $f^{-1}(J)$ is an interval (or empty set) for any interval J. Then we use Lemma 2.1 to show that terms of summation on the right hand side of (7) are equal for both integrals in (14); therefore, the integrals are equal. $\square$

In order to derive the ultimate version of Young's inequality, we will also need the following result on integration by parts for L-S integrals, due to E. Hewitt [**11**] (see also [**4**, Theorem 6.2.2.]).

Lemma 2.3. *Let f and g be real valued functions of finite variation defined on $\mathbb{R}$ and let I be an interval. If $D(I)$ is (at most countable) set of points $d \in I$ where both f and g are discontinuous, then*

$$\int_I f(x)\,\mathrm{d}g(x) + \int_I g(x)\,\mathrm{d}f(x) = \mu_{fg}(I) + \sum_{d\in D(I)} A(d), \tag{15}$$

where

$$A(d) = f(d)\big(g(d_+) - g(d_-)\big) + g(d)\big(f(d_+) - f(d_-)\big) - f(d_+)g(d_+) + f(d_-)g(d_-). \tag{16}$$

Since L-S integral in explicit notations depends on order of points and type of intervals, the explicit form of (15) is different for different types of intervals. For example, if $I = [a, b]$, $a < b$, then

$$\int_{[a,b]} f(x)\,\mathrm{d}g(x) + \int_{[a,b]} g(x)\,\mathrm{d}f(x) = f(b_+)g(b_+) - f(a_-)g(a_-) + \sum_{d\in D([a.b])} A(d). \tag{17}$$

In a special case when one of functions f, g is right continuous at d and both are non-decreasing (or both non-increasing)

$$A(d) = \big(f(d) - f(d_-)\big)\big(g(d) - g(d_-)\big) \geq 0. \tag{18}$$

Let us finally remark that, given a finite measure μ, we may define a right continuous function $g_r(x) = \mu(-\infty, x]$ so that $\mu = \mu_{g_r}$ in the sense of L-S measure. Hence for an arbitrary positive finite measure μ and a non-decreasing function f, we may take $g = g_r$ to make summands $A(d)$ non-negative.

3. YOUNG'S FUNCTIONAL-LOWER AND UPPER BOUNDS

In this section we assume that all functions are defined on $\mathbb{R}$, as explained in the previous section. We also allow a possibility that intervals of integration can be infinite, as far as the value of the integral is finite.

For a given interval I and non-decreasing function f, we define

$$\underline{f}(I) = \sup\{f(x) \mid (\forall y \in I)x < y\}, \quad \overline{f}(I) = \inf\{f(x) \mid (\forall y \in I)x > y\},$$

and $\underline{f}(I) = f(-\infty)$, or $\overline{f}(I) = f(+\infty)$ if I is unbounded from the left or from the right. Further, let

$$f_{\min}(I) = \inf\{f(x) \mid x \in I\}, \quad f_{\max}(I) = \sup\{f(x) \mid x \in I\},$$

where I is an arbitrary nonempty interval. We note that for any non-decreasing function f we have

$$\underline{f}(I) \le f_*(I) \le f_{\min}(I) \le f_{\max}(I) \le f^*(I) \le \overline{f}(I).$$

In this section we present lower and upper bounds for the functional

$$\begin{aligned} T(f,g;I,J) &:= \int_I f(x)\,\mathrm{d}g(x) + \int_J g(x)\,\mathrm{d}f(x) \\ &= \int_I f(x)\,\mathrm{d}g(x) + \int_{J_f} g\big(f^{-1}(u)\big)\,\mathrm{d}u \end{aligned} \tag{19}$$

where I and J are arbitrary intervals with $J \subseteq I$. For $I = J$, the functional (19) can be expressed via integration by parts formula (15), but as we already noted, the explicit form of (15) depends on the kind of intervals I and J. A particular case of (19) that corresponds to classical Young's inequality is with $I = [a,t]$, $J = [a,s]$, $a < s \le t$ and with $g(x) = x$.

With unbounded intervals I and J, we assume that $T(f,g,I,J)$, $T(f,g,I,I)$ and $T(f,g,J,J)$ are well defined and finite, and also we have to assume that all expressions that appear in right hand sides of formulas in statements of next two theorems make sense.

Theorem 3.1 (Lower bounds). *Let f and g be non-decreasing functions on $\mathbb{R}$ and let I, J be intervals such that $J \subseteq I$. Let $I \setminus J = A \cup B$, where A and B are disjoint intervals (possibly empty) defined by*

$$A = \{x \in I \mid (\forall y \in J) x < y\}, \qquad B = \{x \in I \mid (\forall y \in J) x > y\}.$$

We distinguish three cases:

(i) *If $\mu_g(A) > 0$, $\mu_g(B) > 0$, then*

$$T(f,g;I,J) \ge T(f,g;J,J) + f_{\min}(I)\big(g_*(J) - g_*(I)\big) + \overline{f}(J)\big(g^*(I) - g^*(J)\big). \tag{20}$$

(ii) *If $\mu_g(A) > 0$ and $\mu_g(B) = 0$, then*

$$T(f,g;I,J) \ge T(f,g;J,J) + f_{\min}(I)\big(g_*(J) - g_*(I)\big). \tag{21}$$

(iii) *If $\mu_g(A) = 0$ and $\mu_g(B) > 0$, then*

$$T(f,g;I,J) \ge T(f,g;J,J) + \overline{f}(J)\big(g^*(I) - g^*(J)\big). \tag{22}$$

In the particular case with $I = [a,t]$, $J = [a,s]$, $-\infty \le a < s \le t$, we have that

$$\begin{aligned} &\int_{[a,t]} f(x)\,\mathrm{d}g(x) + \int_{[a,s]} g(x)\,\mathrm{d}f(x) \\ &\qquad \ge f(s_+)g(t_+) - f(a_-)g(a_-) + \sum_{d \in D([a,s])} A(d), \end{aligned} \tag{23}$$

where $A(d)$ is defined as in (16). *Equality holds if $s = t$.*

Proof. For case (i), we start from

$$T(f,g;I,J) = T(f,g;J,J) + \int_{A\cup B} f(u)\,\mathrm{d}g(u) \tag{24}$$

and then (noting that A and B are disjoint)

$$\begin{aligned}\int_{A\cup B} f(u)\,\mathrm{d}g(u) &\geq f_{\min}(I)\mu_g(A) + \overline{f}(J)\mu_g(B)\\ &= f_{\min}(I)(g_*(J) - g_*(I)) + \overline{f}(J)\big(g^*(I) - g^*(J)\big),\end{aligned} \tag{25}$$

where the second line follows from noticing that $g^*(A) = g_*(J), g_*(A) = g_*(I)$ and analogously for B. Then (20) follows from (24) and (25). Proofs for (ii) and (iii) are similar.

The special case (23) follows from (22) for $I = [a,t]$, $J = [a,s]$, by using the integration by parts formula of Lemma for $T(f,g;J,J)$ written in an explicit form.

Theorem 3.2 (Upper bounds). *We assume the same conditions and notations as in Theorem* 3.1.

(i) *If $\mu_f(A) > 0$, $\mu_f(B) > 0$, then*

$$T(f,g;I,J) \leq T(f,g;I,I) - g_{\min}(I)\big(f_*(J) - f_*(I)\big) - \overline{g}(J)\big(f^*(I) - f^*(J)\big), \tag{26}$$

(ii) *If $\mu_f(A) > 0$, $\mu_f(B) = 0$, then*

$$T(f,g;I,J) \leq T(f,g;I,I) - g_{\min}(I)\big(f_*(J) - f_*(I)\big) \tag{27}$$

(iii) *If $\mu_f(A) = 0$, $\mu_f(B) > 0$, then*

$$T(f,g;I,J) \leq T(f,g;I,I) - \overline{g}(J)\big(f^*(I) - f^*(J)\big) \tag{28}$$

In the particular case with $I = [a,t]$, $J = [a,s]$, $-\infty \leq a < s \leq t$, we have that

$$\begin{aligned}&\int_{[a,t]} f(x)\,\mathrm{d}g(x) + \int_{[a,s]} g(x)\,\mathrm{d}f(x)\\ &\leq f(t_+)g(t_+) + f(s_+)g(s_+) - f(a_-)g(a_-) - f(t_+)g(s_+) + \sum_{d\in D([a,t])} A(d).\end{aligned} \tag{29}$$

Equality holds if $s = t$.

Proof. Note first that

$$T(f,g;I,J) + T(g,f,I,J) = T(f,g;I,I) + T(f,g;J,J).$$

By Theorem 3.1, we have that $T(gI,J) \geq T(g,f,J,J) + R(g,f,I,J)$, where R depends on the assumed case with respect to positivity of $\mu_f(A)$ and $\mu_f(B)$. Th,f; en using the above equality we conclude that

$$\begin{aligned}T(f,g,I,J) &\leq T(f,g;I,I) + T(f,g;J,J) - T(g,f,J,J) - R(g,f,I,J)\\ &= T(f,g;I,I) - R(g,f,I,J).\end{aligned}$$

Writing down the explicit expressions for R, we get the statements for cases (i)–(iii). The upper bound for the special case follows upon writing $T(f,g,I,I)$ in the explicit form.

Remark 3.3. A simple observation that (25) can be re-phrased in terms of upper bounds, yields an immediate upper bound in Young's inequality. The bound obtained in such a way is in general different than the one of Theorem , but these two bonds agree if f and g are continuous functions.

Remark 3.4. It is difficult to state universal necessary conditions for equality in theorems 3.1 and 3.2. As an illustration, consider a function $g(x) = \mathrm{sgn}\,(x-c)$, for a fixed $c \in \mathbb{R}$, and let f be a continuous non-decreasing function. Then, for $a < c < s < t$, we have that

$$S(f,g;s,t) = \int_{[a,t]} f(x)\,\mathrm{d}g(x) + \int_{[a,s]} g(x)\,\mathrm{d}f(x) = f(s) + f(a),$$

which gives equality with both lower and the upper bound. If $a < s < c < t$, then $S(f,g;s,t) = 2f(c) + f(a) - f(s)$, the lower bound is $f(s)+f(a)$ and the upper bound is $2f(t)+f(a)-f(s)$ and equality with the lower bound occurs iff $f(c) = f(s)$, while the equality with the upper bound occurs iff $f(s) = f(t)$.

4. EXAMPLES

4.1. Young's inequalities with respect to a measure

Let μ be a positive finite and countably additive measure on Borel subsets of $\mathbb{R}$. Along the lines of Section 2, we define a right continuous function $G(x) = \mu\{(-\infty, x]\}$. For a non-decreasing function f we have that

$$T(f,G;I,J) = \int_I f(x)\,\mathrm{d}\mu(x) + \int_J G(x)\,\mathrm{d}f(x)$$

Then we may observe Young's inequalities as in Section 3. In order to simplify notations, let us assume that μ is the probability distribution of a random variable Y on some abstract probability space $(\Omega, \mathcal{F}, P)$; then G is the cumulative distribution function of Y, and taking $I = (-\infty, t]$, $J = (-\infty, s]$ and assuming that $f(-\infty) = 0$, we have that

$$\begin{aligned} (30)\qquad & f(s_+)G(t) + \sum_{d\in D((-\infty,s])} \big(f(d) - f(d_-)\big)P(Y=d) \\ & \le \mathrm{E}\,\big(f(Y)\cdot I_{\{Y\le t\}}\big) + \int_{[0,f(s_+)]} G\big(f^{-1}(u)\big)\,\mathrm{d}u \\ & \le f(t_+)\big(G(t)-G(s)\big) + f(s_+)G(s) + \sum_{d\in D((-\infty,t])} \big(f(d)-f(d_-)\big)P(Y=d). \end{aligned}$$

This holds for a general probability measure μ.

Now suppose that we have another probability measure that corresponds to a random variable X, with a probability distribution function F. Rewriting (30) in this setup, f being replaced with F, we get

$$\begin{aligned} F(s)G(t) &+ \sum_{d \in D((-\infty,s])} P(X=d)P(Y=d) \\ &\le \mathrm{E}\,\big(F(Y)\cdot I_{\{Y\le t\}}\big) + \mathrm{E}\,\big(G(X)\cdot I_{\{X\le s\}}\big) \\ &\le F(t)\big(G(t)-G(s)\big) + F(s)G(s) + \sum_{d\in D((-\infty,t])} P(X=d)P(Y=d). \end{aligned}$$

The case $X = Y$, $F = G$, $s = t = +\infty$ gives a simple formula

$$\mathrm{E}\,F(X) = \frac{1}{2} + \frac{1}{2}\sum_{d\in D} P^2(X=d), \tag{31}$$

where D is the set of atoms of probability distribution for X. Proceeding further, let F^{-1} be any version of inverse; then (31) implies that

$$F^{-1}\big(\mathrm{E}\,F(X)\big) \ge F^{-1}\left(\frac{1}{2}\right) \in \mathrm{Med}\,F,$$

where $\mathrm{Med}\,F$ denotes the median set (closed interval or a singleton), which is defined as the set of all possible values of $F^{-1}(1/2)$.

By (8), $F^{-1} \le \overline{F}^{\,-1}$, where

$$\overline{F}^{\,-1}(y) = \inf\{t \mid F(t) > y\}$$

and so, if m is any median of X, then

$$m \le \overline{F}^{\,-1}\big(\mathrm{E}\,F(X)\big).$$

This relation seems not to have been recorded in the literature (see for example [**9**] or [**15**] for some results and facts about one dimensional medians).

4.2. A summation formula

In this part, the letters i, j, k, m, n will be reserved for integers only. The following result was proved in [**10**]: If f is a strictly increasing positive function on $[0, +\infty)$, $0 < f(1) \le 1$, then for any positive integer n it holds that

$$\sum_{j=1}^{n}[f(j)] + \sum_{k=1}^{[f(n)]}[f^{-1}(k)] = n[f(n)] + K(1,n), \tag{32}$$

where K is the number of integer points $j \in [1, n]$ such that $f(j)$ is an integer. A generalized version of this result can be derived using the Lebesgue-Stieltjes integral as follows.

We start with (17) with $g(x) = [x]$ and replacing f with $[f]$ (where $[x]$ is the greatest integer $\le x$). To be consistent with usual notations, we may use $[f](x)$ for

$[f(x)]$; note that $[f](x_\pm) = \lim_{t\to x_\pm}[f(t)]$ and $[f(x_\pm)] = [\lim_{t\to x_\pm} f(t)]$ need not be equal. For simplicity take $a = m$ and $b = n$ where m and n are integers. Then we have,

$$\int_{[m,n]}[f](x)\,\mathrm{d}[x] + \int_{[m,n]}[x]\,\mathrm{d}[f](x) \\ = n\cdot[f](n_+) - (m-1)\cdot[f](m_-) + K(m,n),$$

where by (18),

$$K(m,n) = \sum_{j=m}^{n}\big([f](j) - [f](j_-)\big).$$

Now let us fix the version of inverse to be the smallest one:

$$f^{-1}(y) = \sup\{t \mid f(t) < y\}$$

and also assume that f is right continuous. With these assumptions, it follows that the mapping $x \mapsto [f](x)$ is right continuous. Further, if

$$J(x) := [f](x_+) - [f](x_-) = [f](x) - [f](x_-),$$

then the value of $J(x)$ equals the number of integers $k \in ([f](x_-), [f](x)]$, or, in another way, the number of integers k with the property that $f^{-1}(k) = x$. Then

$$\int_{[m,n]}[x]\,\mathrm{d}[f](x) = \sum_{x\in[m,n]\,:\,[f](x)-[f](x_-)>0}[x]\big([f](x) - [f](x_-)\big) \\ = \sum_{k\in([f](m_-),[f](n)]}[f^{-1}(k)]$$

and as an immediate consequence we have that

$$\sum_{j=m}^{n}[f(j)] + \sum_{k=[f](m_-)+1}^{[f](n)}[f^{-1}(k)] \\ = n\cdot[f](n) - (m-1)\cdot[f](m_-) + K(m,n), \tag{33}$$

where $K(m,n)$ is the number of integers $k \in ([f](m_-), [f](n)]$ with the property that $f^{-1}(k)$ is also an integer. This is a generalization of (32) for right continuous non-decreasing functions f. The formula could be useful in number theory, for example, to find a numerical value of $K(m,n)$ for some functions f.

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