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APPROXIMATION BY FABER-LAURENT RATIONAL FUNCTIONS ON DOUBLY CONNECTED DOMAINS

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Abstract. Let B be a doubly-connected domain bounded by two Dini-smooth curves. In this work, we prove some direct theorems of approximation theory in weighted rearrangement invariant Smirnov spaces $E_X(B,\omega)$ defined on B. For this, approximation properties of the Faber-Laurent rational series expansions are used.

1. Introduction and Main Results

Let B be a doubly connected domain in the complex plane C, bounded by two rectifiable Jordan curves Γ_1 and Γ_2 (the closed curve Γ_2 in the closed curve Γ_1). Without loss of generality we suppose that $0 \in Int\Gamma_2$. Let $B_1^0 := Int\Gamma_1$, $B_1^\infty := Ext\Gamma_1$, $B_2^0 := Int\Gamma_2, B_2^\infty := Ext\Gamma_2$. Further, we set

$$\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}, \quad \mathbb{D} := Int\mathbb{T}, \quad \mathbb{D}^- := Ext\mathbb{T}.$$

We denote by $\omega = \varphi(z)$ and $\omega = \varphi_1(z)$ the conformal mappings of B_1^∞ and B_2^0 onto D^- normalized as

$$\varphi(\infty) = \infty, \quad \lim_{z\to\infty} \tfrac{\varphi(z)}{z} = 1$$

and

$$\varphi_1(0) = \infty, \quad \lim_{z\to 0} z\varphi_1(z) = 1$$

respectively, and let ψ and ψ_1 be inverses of φ and φ_1.

We assume that G is a simply-connected domain with a rectifiable Jordan boundary Γ and $G^- := Ext\Gamma$.

Let $L_p(\Gamma)$ and $E_p(G)$ $(1 \le p < \infty)$ be the Lebesgue space of measurable complex valued functions on Γ and the Smirnov class of analytic functions in G, respectively. It is known that every function $f \in E_1(G)$ has non-tangential boundary values a. e. on Γ and if we use the same notation for the non-tangential boundary value of f, then $f \in L_1(\Gamma)$.

Let $\Gamma \subset C$ be a closed rectifiable Jordan curve with the Lebesgue length measure $|d\tau|$ and let $X(\Gamma)$ be rearrangement invariant space over Γ, generated by a

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rearrangement invariant function norm ρ, with associate space $X'(\Gamma)$. For each $f \in X(\Gamma)$ we define

$$\|f\|_{X(\Gamma)} := \rho(|f|), \quad f \in X(\Gamma).$$

A rearrangement invariant space $X(\Gamma)$ equipped with norm $\|.\|_{X(\Gamma)}$ is a Banach space [**5**, pp. 3-5].

It is well known that

$$\|f\|_{X(\Gamma)} = \sup \left\{ \int_{\Gamma} |fg| \, |d\tau| : g \in X'(\Gamma), \|g\|_{X'(\Gamma)} \leq 1 \right\} \tag{1}$$

and

$$\|g\|_{X'(\Gamma)} = \sup \left\{ \int_{\Gamma} |fg| \, |d\tau| : f \in X(\Gamma), \|f\|_{X(\Gamma)} \leq 1 \right\}.$$

hold.

If $f \in X$ and $g \in X'$, then fg is summable and the Hölder inequality

$$\int_{\Gamma} |fg| \, |d\tau| \leq \|f\|_{X(\Gamma)} \|g\|_{X'(\Gamma)}$$

holds [**5**, p. 9].

For definitions and fundamental properties of general rearrangement invariant spaces we refer to [**5**].

A measurable function $\omega : \Gamma \to [0, \infty]$ is called a weight function if the set $\omega^{-1}(\{0, \infty\})$ has Lebesgue measure zero. Let $X(\Gamma)$ be a rearrangement invariant space and ω be a weight function on Γ. The space of measurable functions $f : \Gamma \to \mathbb{C}$ for which $f\omega \in X(\Gamma)$ is denoted by $X(\Gamma, \omega)$. $X(\Gamma, \omega)$ is a Banach space with respect to the norm

$$\|f\|_{X(\Gamma,\omega)} := \|f\omega\|_{X(\Gamma)},$$

and is called a weighted rearrangement invariant space.

If $\omega \in X(\Gamma)$ and $1/\omega \in X'(\Gamma)$, then $X(\Gamma, \omega)$ is a Banach function space and from the Hölder's inequality we have

$$L_\infty(\Gamma) \subset X(\Gamma, \omega) \subset L_1(\Gamma).$$

By the Luxemburg representation theorem [**5**, Theorem 4.10, p.62], there is a unique rearrangement invariant function norm $\overline{\rho}$ over Lebesgue measure space $([0, |\Gamma|], m)$, where $|\Gamma|$ is the Lebesgue length of Γ, such that $\rho(f) = \overline{\rho}(f^*)$ for all non-negative and almost everywhere (a. e.) finite measurable functions f defined on Γ. Here f^* denotes the non-increasing rearrangement of f [**5**, p. 39]. The rearrangement invariant space over $([0, |\Gamma|], m)$ generated by $\overline{\rho}$ is called the Luxemburg representation of $X(\Gamma)$ and is denoted by $\overline{X}$.

Let f be a non-negative, almost everywhere finite and measurable function on $[0, |\Gamma|]$. For each $x > 0$ we consider the dilatation operator H_x defined by

$$(H_x f)(t) := \begin{cases} f(xt), & xt \in [0, |\Gamma|] \\ 0, & xt \notin [0, |\Gamma|] \end{cases}, \quad t \in [0, |\Gamma|].$$

It is known [**5**, p. 165] that $H_{1/x} \in B\left(\overline{X}\right)$ for each $x > 0$, where $B\left(\overline{X}\right)$ is the Banach algebra of bounded linear operators on $\overline{X}$. Let $h_X(x)$ denote the operator norm of $H_{1/x}$ i.e., $h_X(x) := \left\|H_{1/x}\right\|_{B(\overline{X})}$.

The numbers

$$\alpha_X := \lim_{x\to 0} \tfrac{\log h_X(x)}{\log x}, \quad \beta_X := \lim_{x\to\infty} \tfrac{\log h_X(x)}{\log x}$$

are called lower and upper Boyd indices of the rearrangement invariant space $X(\Gamma)$, respectively. The Boyd indices α_X and β_X are said to be nontrivial if $0 < \alpha_X \leq \beta_X < 1$.

Let be ω weight on Γ. Now we denote by $E_X(G,\omega)$ the class of functions $f \in E_1(G)$ for which the boundary function f belongs to $X(\Gamma,\omega)$. The class of functions $E_X(G,\omega)$ will be called weighted rearrangement invariant Smirnov space with respect to domain G ([**14**]). Obviously, the class $E_X(G,\omega)$ is wider than weighted Smirnov classes $E_p(G,\omega)$ and as well as the weighted Smirnov-Orlicz classes $E_M(G,\omega)$ given in [**13**]. Each function $f \in E_X(G,\omega)$ has a non-tangential boundary values a. e. on Γ.

Definition 1. A smooth Jordan curve Γ is called Dini-smooth, if the function $\theta(s)$, the angle between the tangent line and the positive real axis expressed as a function of arclength s, has modulus of continuity $\Omega(\theta,s)$ satisfying the Dini-condition

$$\int_0^\delta \frac{\Omega(\theta,s)}{s} ds < \infty, \quad \delta > 0.$$

For $z \in \Gamma$ and $\varepsilon > 0$, we denote by $\Gamma(z,\varepsilon)$ the portion of Γ in the open disk of radius ε centered at z, i. e.,

$$\Gamma(z,\varepsilon) := \{t \in \Gamma : |t-z| < \varepsilon\}.$$

For fixed $1 < p < \infty$ and $1/p + 1/q = 1$. A weight function ω belongs to the Muckenhoupt class $A_p(\Gamma)$ if

$$\sup_{z\in\Gamma} \sup_{\varepsilon>0} \left(\frac{1}{\varepsilon} \int_{\Gamma(z,\varepsilon)} \omega(x)^p |dx|\right)^{1/p} \left(\frac{1}{\varepsilon} \int_{\Gamma(z,\varepsilon)} \omega(x)^{-q} |dx|\right)^{1/q} < \infty.$$

Let $X(\mathbb{T})$ be a reflexive rearrangement invariant space with non-trivial Boyd indices α_X and β_X, and be ω weight function such that $\omega \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$. For given function $g \in X(\mathbb{T},\omega)$ we define the shift operator σ_h

$$\sigma_h g(x) := \tfrac{1}{2h} \int_{-h}^{h} g\left(we^{it}\right) dt, \quad 0 < h < \pi, w \in \mathbb{T},$$

and later r-th modulus of smoothness $\Omega^r(\cdot,\delta)_{X,\omega}$ $(r = 1, 2, ...)$

$$\Omega_r(g,\delta)_{X,\omega} := \sup_{\substack{0\leq h_i\leq\delta \\ 1\leq i\leq r}} \left\| \prod_{i=1}^{r} (I - \sigma_{h_i}) g \right\|_{X(\mathbb{T},\omega)}, \quad \delta > 0$$

where I is the identity operator. Since the operator σ_h is a bounded linear operator in $X(\mathbb{T},\omega)$ [**10**, Lemma 2.2], this modulus of smoothness is well defined.

In this definition we use as shift the mean value operator σ_h and the modulus of smoothness $\Omega_r(\cdot,\delta)_{X,\omega}$ in such way since the space $X(\mathbb{T},\omega)$ is non-invariant, in general, under the usual shift $g(\cdot)\to g(\cdot+h)$.

If Γ_1 and Γ_2 are Dini-smooth, then from the results in [**28**], it follows that

$$\begin{aligned} 0 &< c_1 \le |\varphi'(z)| \le c_2 < \infty, \quad 0 < c_3 \le |\varphi_1'(z)| \le c_4 < \infty, \\ 0 &< c_5 \le |\psi'(w)| \le c_6 < \infty, \quad 0 < c_7 \le |\psi_1'(w)| \le c_8 < \infty \end{aligned} \tag{2}$$

where the constants $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ and c_8 which are independent of $z \in \overline{B}$ and $|w| \ge 1$.

We will say that the doubly connected domain B is bounded by the Dini-smooth curve if the domains B_1^0 and B_2^0 are bounded by the closed Dini-smooth curves.

Let Γ_i $(i=1,2)$ be a Dini-smooth curve and be ω a weight on $\Gamma_1 \cup \Gamma_2^-$. We can consider ω as weight on Γ_1 and Γ_2 separately. We associate with ω, the following two weights defined on $\mathbb{T}$ by $\omega_0 := \omega \circ \psi$ and $\omega_1 := \omega \circ \psi_1$, and let $f_0 := f \circ \psi$ for $f \in X(\Gamma_1,\omega)$ and let $f_1 := f \circ \psi_1$ for $f \in X(\Gamma_2,\omega)$. Then by (2) we get $f_0 \in X(\mathbb{T},\omega_0)$ and $f_1 \in X(\mathbb{T},\omega_1)$. Using the non-tangential boundary values of f_0^+ and f_1^+ on $\mathbb{T}$, we define

$$\Omega_r^{\Gamma}(f,\delta)_{X,\omega} := \Omega_r\left(f_0^+,\delta\right)_{X,\omega_0}, \quad \widetilde{\Omega}_r^{\Gamma}(f,\delta)_{X,\omega} := \Omega_r\left(f_1^+,\delta\right)_{X,\omega_1}, \quad \delta > 0, \tag{3}$$

for $r = 1, 2, \ldots$.

For $f_0 \in X(\mathbb{T},\omega)$, since $\omega_0 \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, Lemma 1 and Theorem 4 in [**14**] we can deduce that $f_0^+ \in E_X(\mathbb{D},\omega_0)$ and $f_0^- \in E_X(\mathbb{D}^-,\omega_0)$ such that $f_0^-(\infty) = \infty$. Similarly, since $f_1 \in X(\mathbb{T},\omega)$ and $\omega_1 \in A_{1/\alpha_X}(\mathbb{T}) \cap A_{1/\beta_X}(\mathbb{T})$, we have $f_1^+ \in E_X(\mathbb{D},\omega_1)$ and $f_1^- \in E_X(\mathbb{D}^-,\omega_1)$ such that $f_1^-(\infty) = 0$. Moreover for two functions $f_0, f_1 \in X(\mathbb{T},\omega)$ we have

$$f_0(t) = f_0^+(t) - f_0^-(t), \quad f_1(t) = f_1^+(t) - f_1^-(t) \tag{4}$$

a.e. on $\mathbb{T}$.

Let us take

$$L_r := \{z : |\varphi(z)| = r\}, \quad L_R := \{z : |\varphi_1(z)| = R\}$$

for $r, R > 1$. φ has the Laurent expansion in some neighbourhood of the point $z = \infty$ has the form

$$\varphi(z) = \alpha z + \alpha_0 + \tfrac{\alpha_1}{z} + \tfrac{\alpha_2}{z^2} + \ldots + \tfrac{\alpha_n}{z^n} + \ldots$$

and by this we get

$$[\varphi(z)]^k = \alpha^k z^k + \sum_{i=0}^{k-1} \alpha_{k,i} z^i + \sum_{k<0} \alpha_{k,i} z^i.$$

The polynomial

$$F_k(z) = \alpha^k z^k + \sum_{i=0}^{k-1} \alpha_{k,i} z^i$$

is called the Faber polynomial of order k for the domain B_1^0.

The function φ_1 has an expansion in some neigbourhood of the origin:

$$\varphi_1 = \tfrac{1}{z} + \beta_0 + \beta_1 z + \ldots + \beta_n z^n + \ldots.$$

Raising this function to the power k, we obtain

$$\left[\varphi_1(z)\right]^k = \widetilde{F}_k\left(\tfrac{1}{z}\right) - \widetilde{E}_k(z), \quad z \in B_2^0,$$

where $\tilde{F}_k\left(\frac{1}{z}\right)$ denotes the polynomial of negative powers of z and the term $\tilde{E}_k(z)$ contains non-negative powers of z; hence this is analytic function in the domain B_2^0.

If a function $f(z)$ is analytic in a doubly connected domain bounded by the curves L_r and L_R, then the following Faber-Laurent series expansion holds [**21, 25**] :

$$f(z) = \sum_{k=0}^{\infty} \alpha_k F_k(z) + \sum_{k=1}^{\infty} \tilde{\alpha}_k \widetilde{F}_k\left(\tfrac{1}{z}\right) \tag{5}$$

where

$$\alpha_k = \frac{1}{2\pi i} \int\limits_{L_{r_1}} \frac{f(z)\varphi'(z)}{[\varphi(z)]^{k+1}} dz = \frac{1}{2\pi i} \int\limits_{|t|=r_1} \frac{f(\psi(t))}{t^{k+1}} dt, \quad 1 < r_1 < r$$

and

$$\tilde{\alpha}_k = \frac{1}{2\pi i} \int\limits_{L_{R_1}} \frac{f(z)\varphi_1'(z)}{\left[\varphi_1(z)\right]^{k+1}} dz = \frac{1}{2\pi i} \int\limits_{|t|=R_1} \frac{f\left(\psi_1(t)\right)}{t^{k+1}} dt, \quad 1 < R_1 < R. \tag{6}$$

The rational function

$$R_n(f,z) := \sum_{k=0}^{n} \alpha_k F_k(z) + \sum_{k=1}^{n} \tilde{\alpha}_k \widetilde{F}_k\left(\tfrac{1}{z}\right) \tag{7}$$

is called the Faber-Laurent rational function of degree n of f.

Since series of Faber polynomials are generalizations of Taylor series to the case of a simply connected domain, it is natural to consider the construction of similar generalization of Laurent series to the case of a doubly-connected domain.

Let $\Gamma := \Gamma_1 \cup \Gamma_2^-$, where Γ_1 and Γ_2 curves are rectifiable Jordan curves, and let be ω a weight on Γ. $X(\Gamma, \omega)$ is a weighted rearrangement invariant space on Γ and B is doubly connected domain bounded by the Dini-smooth curve Γ_1 and Γ_2. Let

$$E_X(B,\omega) := \{f \in E_1(B) : f \in X(\Gamma,\omega)\}.$$

where $E_1(B)$ is the Smirnov class of analytic functions in B ([**7**, pp. 182-183]).

Definition 2. The class $E_X(B,\omega)$ is called weighted rearrangement invariant Smirnov space with respect to doubly-connected domain B.

In the literature there are many results on direct and converse approximation theorems in different spaces defined on simply connected domain of complex plane. The some direct and converse theorems in weighted and non-weighted rearrangement invariant space, defined simply-connected domain were proved in [**2**], [**9**],[**10**] and [**14**]. These spaces are sufficiently wide; the Lebesgue, Orlicz, Lorentz spaces are examples of rearrangement invariant space. These problems in the different subspaces of the rearrangement invariant space were investigated by several authors. The direct and converse theorems of approximation in the weighted or non-weighted Lebesgue and Smirnov spaces have been studied in [**3**], [**4**], [**11**], [**12**] and [**15**]. Similar results in Smirnov-Orlicz and Orlicz spaces were researched in [**1**], [**16**], [**18**], [**19**], [**22**], and [**30**]. Some theorems in weighted Lorentz spaces were obtained in

[**23**] and [**29**]. All of these results have been obtained under different restrictive conditions on boundary of domains.

In this work, we prove a direct theorem approximation theory in the weighted rearrangement invariant Smirnov spaces, defined in the doubly-connected domain. Similar direct theorem weighted rearrangement invariant space,defined simply-connected domain, were proved by Israfilov and Akgun ([**14**]) in the case that Γ is a closed Dini-smooth curve. A direct theorem in Smirnov-Orlicz spaces with doubly-connected domain has been proved by Jafarov ([**17**]). Similar problems were studied in [**21**], [**26**], and [**27**].

Our main result is given in the following theorem:

Theorem 1. *Let B be a finite doubly-connected domain with the Dini-smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2^-$, $X(\Gamma)$ be a reflexive rearrangement invariant space with nontrivial Boyd indices α_X and β_X, $\omega \in A_{1/\alpha_X}(\Gamma) \cap A_{1/\beta_X}(\Gamma)$, $f \in E_X(B,\omega)$ and r be a natural number. Then there is a constant $c_9 > 0$ such that*

$$\|f - R_n(f,z)\|_{X(\Gamma,\omega)} \leq c_9 \left\{ \Omega_r^\Gamma (f, 1/n+1)_{X,\omega} + \widetilde{\Omega}_r^\Gamma (f, 1/n+1)_{X,\omega} \right\}$$

for every natural number n, where $R_n(f,z)$ is the $n-th$ Faber-Laurent rational function of f.

In weighted case the theorem 1 has not been known before, even not for the spaces $L_p(\Gamma)$, $1 < p < \infty$. We use $c, c_1, c_2, ...$ to denote constants (which may, in general, differ in different relations) depending only on numbers that are not important for the question of interest.

2. Auxiliary Results

We shall exploit for $F_k(z)$ and $\tilde{F}_k\left(\frac{1}{z}\right)$ the following integral representations hold [**24**]:

1. If $z \in IntL_r$, then

$$F_k(z) = \frac{1}{2\pi i} \int\limits_{L_r} \frac{[\varphi(\varsigma)]^k}{\varsigma - z} d\varsigma = \frac{1}{2\pi i} \int\limits_{|t|=r} \frac{\psi'(t)\, t^k}{\psi(t) - z} dt. \tag{8}$$

2. If $z \in ExtL_r$, then

$$F_k(z) = [\varphi(z)]^k + \frac{1}{2\pi i} \int\limits_{L_r} \frac{[\varphi(\varsigma)]^k}{\varsigma - z} d\varsigma. \tag{9}$$

3. If $z \in IntL_R$, then

$$\widetilde{F}_k\left(\frac{1}{z}\right) = \left[\varphi_1(z)\right]^k - \frac{1}{2\pi i} \int\limits_{L_R} \frac{\left[\varphi_1(\zeta)\right]^k}{\zeta - z} d\zeta. \tag{10}$$

4. If $z \in ExtL_R$, then

$$\widetilde{F}_k\left(\frac{1}{z}\right) = -\frac{1}{2\pi i} \int\limits_{L_R} \frac{\left[\varphi_1(\zeta)\right]^k}{\zeta - z} d\zeta = -\frac{1}{2\pi i} \int\limits_{|t|=R} \frac{\psi_1'(t)\, t^k}{\psi_1(t) - z} dt \tag{11}$$

Let G be a finite domain in the complex plane with a rectifiable Jordan curve Γ and $f \in L_1(\Gamma)$, Then the functions f^+ and f^- defined by

$$f^{+}(z) := \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\varsigma)}{\varsigma - z}d\varsigma, \quad z \in G$$

and

$$f^{-}(z) := \frac{1}{2\pi i}\int_{\Gamma}\frac{f(\varsigma)}{\varsigma - z}d\varsigma, \quad z \in G^{-}$$

are analytic in G and G^{-}, respectively, and $f^{-}(\infty) = 0$.

The Cauchy singular integral of $f \in L_1(\Gamma)$ at $z \in \Gamma$ is defined by

$$S_{\Gamma} f(z) := \lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int\limits_{\Gamma\backslash\Gamma(z,\varepsilon)}\frac{f(\varsigma)}{\varsigma - z}d\varsigma.$$

It is known that this limit exists for almost every $z \in \Gamma$ ([**6**, pp. 117-144.]).

The functions f^{+} and f^{-} have non-tangential limits a.e. on Γ, and the formulae

$$f^{+}(z) = S_{\Gamma} f(z) + \frac{1}{2}f(z), \quad f^{-}(z) = S_{\Gamma} f(z) - \frac{1}{2}f(z)$$

holds a. e. on Γ ([**8**, p. 431]), and hence

$$f = f - f^{-}$$

a. e. on Γ.

For $f \in L_1(\Gamma)$, we associate the function $S_{\Gamma} f$ taking the value $S_{\Gamma} f(z)$ exists a.e. on Γ. The linear operator S_{Γ} defined in such way is called the Cauchy singular operator.

Lemma 1 ([**20**])**.** *Let $X(\Gamma)$ be an reflexive rearrangement invariant space with non-trivial Boyd indices α_X and β_X. If a weight ω belongs to Muckenhoupt classes $A_{1/\alpha_X}(\Gamma)$ and $A_{1/\beta_X}(\Gamma)$, then the singular operator S_{Γ} is bounded on $X(\Gamma,\omega)$, i. e.,*

$$\|S_{\Gamma}(f)\|_{X(\Gamma,\omega)} \le c_{10}\|f\|_{X(\Gamma,\omega)}, \quad f \in X(\Gamma,\omega) \tag{12}$$

holds with a constant $c_{10} > 0$ independent of f.

Lemma 2 ([**10**])**.** If α_X and β_X, are nontrivial and $\omega \in A_{1/\alpha_X}(\Gamma) \cap A_{1/\beta_X}(\Gamma)$, then there exists a constant $c_{11} > 0$ such that for every natural number n,

$$\|g - T_n(g)\|_{X(\Gamma,\omega)} \le c_{11}\Omega_r(g, 1/n+1)_{X,\omega}, \quad g \in E_X(\mathbb{D},\omega)$$

where $r = 1, 2, 3, ...$ and $T_n(g)$ is n-th partial sum of the Taylor series of g at the origin.

For $z \in B$ use of Cauchy theorem, gives

$$f(z) = \frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\varsigma)}{\varsigma - z}d\varsigma - \frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\zeta)}{\zeta - z}d\zeta.$$

If $z \in Int\Gamma_2$ and $z \in Ext\Gamma_1$, then

$$\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\varsigma)}{\varsigma - z}d\varsigma - \frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\zeta)}{\zeta - z}d\zeta = 0. \tag{13}$$

Let define

$$I_1(z) := \frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\varsigma)}{\varsigma - z}d\varsigma, \quad I_2(z) := \frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\zeta)}{\zeta - z}d\zeta.$$

The function $I_1(z)$ determines the functions $I_1^+(z)$ and $I_1^-(z)$ while the function $I_2(z)$ determines the functions $I_2^+(z)$ and $I_2^-(z)$. The functions $I_1^+(z)$ and $I_1^-(z)$ are analytic in $Int\Gamma_1$ and $Ext\Gamma_1$, respectively. Also, the functions $I_2^+(z)$ and $I_2^-(z)$ are analytic in $Int\Gamma_2$ and $Ext\Gamma_2$, respectively.

3. Proof of Theorem 1

Let $\Gamma = \Gamma_1 \cup \Gamma_2^-$ and $f \in E_X(B,\omega)$. It is easily seen that

$$\|f - R_n(f,\cdot)\|_{X(\Gamma,\omega)} \leq \|f - R_n(f,\cdot)\|_{X(\Gamma_1,\omega)} + \|f - R_n(f,\cdot)\|_{X(\Gamma_2,\omega)}. \tag{14}$$

Due to the conditions of Theorem 1., we can take curves Γ_1, Γ_2 as the curves of integration in the formulas (8)−(11) and (6), respectively. Let $f \in E_X(B,\omega)$. Then $f_0 \in X(\mathbb{T},\omega_0)$, $f_1 \in X(\mathbb{T},\omega_1)$ and by (4)

$$f(\varsigma) = f_0^+(\varphi(\varsigma)) - f_0^-(\varphi(\varsigma)), \quad f(\zeta) = f_1^+(\varphi_1(\zeta)) - f_1^-(\varphi_1(\zeta)). \tag{15}$$

For $z' \in Ext\Gamma_1$, then from (9) and (15) we obtain

$$\begin{aligned} \sum_{k=0}^{n}\alpha_k F_k(z') &= \sum_{k=0}^{n}\alpha_k[\varphi(z')]^k + \frac{1}{2\pi i}\int_{\Gamma_1}\frac{\sum_{k=0}^{n}\alpha_k[\varphi(\varsigma)]^k}{\varsigma - z'}d\varsigma \\ &= \sum_{k=0}^{n}\alpha_k[\varphi(z')]^k + \frac{1}{2\pi i}\int_{\Gamma_1}\frac{\sum_{k=0}^{n}\alpha_k[\varphi(\varsigma)]^k - f_0^+[\varphi(\varsigma)]}{\varsigma - z'}d\varsigma \\ &\quad + \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(\varsigma)}{\varsigma - z'}d\varsigma - f_0^-(\varphi(z')). \end{aligned} \tag{16}$$

If $z' \in Ext\Gamma_2$, by (11) and (15) we get

$$\begin{aligned} \sum_{k=1}^{n}\tilde{\alpha}_k\widetilde{F}_k\left(\frac{1}{z'}\right) &= -\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\sum_{k=1}^{n}\tilde{\alpha}_k[\varphi_1(\zeta)]^k}{\zeta - z'}d\varsigma \\ &= \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f_1^+(\varphi_1(\zeta)) - \sum_{k=1}^{n}\tilde{\alpha}_k[\varphi_1(\zeta)]^k}{\zeta - z'}d\varsigma - \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f(\zeta)}{\zeta - z'}d\zeta. \end{aligned} \tag{17}$$

The use of (16) , (17) and (13) for $z' \in Ext\Gamma_1$ gives

$$\begin{aligned} &\sum_{k=0}^{n}\alpha_k F_k(z') + \sum_{k=1}^{n}\tilde{\alpha}_k\widetilde{F}_k\left(\tfrac{1}{z'}\right) \\ = &\sum_{k=0}^{n}\alpha_k[\varphi(z')]^k - f_0^-(\varphi(z')) - \frac{1}{2\pi i}\int_{\Gamma_1}\frac{f_0^+[\varphi(\varsigma)] - \sum_{k=0}^{n}\alpha_k[\varphi(\varsigma)]^k}{\varsigma - z'}d\varsigma \\ &+ \frac{1}{2\pi i}\int_{\Gamma_2}\frac{f_1^+(\varphi_1(\zeta)) - \sum_{k=1}^{n}\tilde{\alpha}_k[\varphi_1(\zeta)]^k}{\zeta - z'}d\varsigma. \end{aligned}$$

Taking the limit as $z' \to z \in \Gamma_1$ along all non-tangential paths outside Γ_1, we reach

$$\begin{aligned} & f(z) - \sum_{k=0}^{n} \alpha_k F_k(z) - \sum_{k=1}^{n} \tilde{\alpha}_k \widetilde{F}_k\left(\tfrac{1}{z}\right) \qquad (18) \\ = \; & f_0^+(\varphi(z)) - \sum_{k=0}^{n} \alpha_k [\varphi(z)]^k + \frac{1}{2}\left[f_0^+[\varphi(z)] - \sum_{k=0}^{n} \alpha_k [\varphi(z)]^k \right] \\ & + S_{\Gamma_1}\left[f_0^+ \circ \varphi - \sum_{k=0}^{n} \alpha_k \varphi^k \right] - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f_1^+\left(\varphi_1(\zeta)\right) - \sum_{k=1}^{n} \tilde{\alpha}_k \left[\varphi_1(\zeta)\right]^k}{\zeta - z'} d\varsigma \end{aligned}$$

a.e. on Γ_1.

Now using (18), Minkowski's inequality and the boundedness of S_{Γ_1} (12), we get

$$\begin{aligned} & \|f - R_n(f,z)\|_{X(\Gamma_1,\omega)} \qquad (19) \\ \leq \; & c_{12} \left\| f_0^+(t) - \sum_{k=0}^{n} \alpha_k t^k \right\|_{X\left(\mathbb{T},\omega_0\right)} + c_{13} \left\| f_1^+(t) - \sum_{k=1}^{n} \tilde{\alpha}_k t^k \right\|_{X\left(\mathbb{T},\omega_1\right)}. \end{aligned}$$

That is, the Faber−Laurent coefficients α_k and $\tilde{\alpha}_k$ of the function f are the Taylor coefficients of the functions f_0^+ and f_1^+, respectively. Then by (3) and (19), Lemma 2 we have

$$\|f - R_n(f,z)\|_{X(\Gamma_1,\omega)} \leq c_{14} \left\{ \Omega_r^{\Gamma}(f, 1/n+1)_{X,\omega} + \widetilde{\Omega}_r^{\Gamma}(f, 1/n+1)_{X,\omega} \right\}. \qquad (20)$$

Let $z'' \in Int\Gamma_2$. Then by (10) and (15) we get

$$\begin{aligned} \sum_{k=1}^{n} \tilde{\alpha}_k \tilde{F}_k\left(\tfrac{1}{z''}\right) &= \sum_{k=1}^{n} \tilde{\alpha}_k \left[\varphi_1(z'')\right]^k - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\sum_{k=1}^{n} \tilde{\alpha}_k \left[\varphi_1(\zeta)\right]^k}{\zeta - z''} d\zeta \qquad (21) \\ &= \sum_{k=1}^{n} \tilde{\alpha}_k \left[\varphi_1(z'')\right]^k - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{\sum_{k=1}^{n} \tilde{\alpha}_k \left[\varphi_1(\zeta)\right]^k - f_1^+\left(\varphi_1(\zeta)\right)}{\zeta - z''} d\zeta \\ & \quad - \frac{1}{2\pi i} \int_{\Gamma_2} \frac{f(\zeta)}{\zeta - z''} d\zeta - f_1^-\left(\varphi_1(z'')\right) \end{aligned}$$

For $z'' \in Int\Gamma_1$, from (8) and (15) we have

$$\begin{aligned} \sum_{k=0}^{n} \alpha_k F_k(z'') &= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\sum_{k=0}^{n} \alpha_k [\varphi(\varsigma)]^k}{\varsigma - z''} d\varsigma \qquad (22) \\ &= \frac{1}{2\pi i} \int_{\Gamma_1} \frac{\sum_{k=0}^{n} \alpha_k [\varphi(\varsigma)]^k - f_0^+(\varphi(\varsigma))}{\varsigma - z''} d\varsigma + \frac{1}{2\pi i} \int_{\Gamma_1} \frac{f(\varsigma)}{\varsigma - z''} d\varsigma. \end{aligned}$$

For $z'' \in Int\Gamma_1$, using (21) and (22) we obtain

$$\begin{aligned}\sum_{k=0}^{n}\alpha_k F_k\left(z''\right)+\sum_{k=1}^{n}\tilde{\alpha}_k\widetilde{F}_k\left(\tfrac{1}{z''}\right) &= \frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\sum_{k=0}^{n}\alpha_k\left[\varphi\left(\varsigma\right)\right]^k-f_0^{+}\left(\varphi\left(\varsigma\right)\right)}{\varsigma-z''}d\varsigma \\ &\quad -\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{\sum_{k=1}^{n}\tilde{\alpha}_k\left[\varphi_1\left(\zeta\right)\right]^k-f_1^{+}\left(\varphi_1\left(\zeta\right)\right)}{\zeta-z''}d\zeta \\ &\quad -f_1^{-}\left(\varphi_1\left(z''\right)\right)+\sum_{k=1}^{n}\tilde{\alpha}_k\left[\varphi_1\left(z''\right)\right]^k\end{aligned}$$

Taking the limit as $z'' \to z \in \Gamma_2$ along all non-tangential paths inside Γ_2, we obtain

$$\begin{aligned} & f\left(z\right)-\sum_{k=0}^{n}\alpha_k F_k\left(z\right)-\sum_{k=1}^{n}\tilde{\alpha}_k\widetilde{F}_k\left(\tfrac{1}{z}\right) \\ = \;& f_1^{+}\left(\varphi_1\left(z\right)\right)-\frac{1}{2}\left[\sum_{k=1}^{n}\tilde{\alpha}_k\left[\varphi_1\left(z\right)\right]^k-f_1^{+}\left(\varphi_1\left(z\right)\right)\right] \\ & -S_{\Gamma_2}\left[\sum_{k=1}^{n}\tilde{\alpha}_k\varphi_1^k-\left(f_1^{+}\circ\varphi_1\right)\right] \\ & -\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{\sum_{k=0}^{n}\alpha_k\left[\varphi\left(\varsigma\right)\right]^k-f_0^{+}\left(\varphi\left(\varsigma\right)\right)}{\varsigma-z''}d\varsigma \end{aligned} \tag{23}$$

a. e. on Γ_2.

Using (23), Minkowski's inequality and the boundedness of S_{Γ_2} (12), we obtain

$$\begin{aligned} & \left\|f-R_n\left(f,z\right)\right\|_{X\left(\Gamma_2,\omega\right)} \\ \leq \;& c_{15}\left\|f_1^{+}\left(t\right)-\sum_{k=1}^{n}\tilde{\alpha}_k t^k\right\|_{X\left(\mathbb{T},\omega_1\right)}+c_{16}\left\|f_0^{+}\left(t\right)-\sum_{k=0}^{n}\alpha_k t^k\right\|_{X\left(\mathbb{T},\omega_0\right)}. \end{aligned} \tag{24}$$

Then taking into account (3), (24) and Lemma 2, we conclude that

$$\left\|f-R_n\left(f,z\right)\right\|_{X\left(\Gamma_1,\omega\right)} \leq c_{17}\left\{\widetilde{\Omega}_r^{\Gamma}\left(f,1/n+1\right)_{X,\omega}+\Omega_r^{\Gamma}\left(f,1/n+1\right)_{X,\omega}\right\}. \tag{25}$$

Hence (20) and (25) complete proof of Theorem 1. ∎

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