NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 47 (2017), 141-150

COMPOSITION OPERATORS FROM WEIGHTED BERGMAN SPACES TO S^p SPACES

Waleed Al-Rawashdeh

(Received March 1, 2017)

Abstract. Let φ be an analytic self-map of the open unit disk $\mathbb{D}$. The operator given by $(C_\varphi f)(z) = f(\varphi(z))$, for $z \in \mathbb{D}$ and f analytic on $\mathbb{D}$ is called composition operator. For each $p \geq 1$, let S^p be the space of analytic functions on $\mathbb{D}$ whose derivatives belong to the Hardy space H^p. In this paper, we study composition operators between weighted Bergman space A^p_α and the space S^q for $1 \leq p, q < \infty$. We characterize the boundedness and compactness of these composition operators. We also give a lower bound for the essential norm of these operators.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. For $0 < p < \infty$ and $-1 < \alpha < \infty$, the weighted Bergman space $A^p_\alpha(\mathbb{D})$ consists of all analytic functions f on $\mathbb{D}$ that satisfy

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty,$$

where $dA_\alpha(z) = (1+\alpha)\left(1-|z|^2\right)^\alpha dA(z)$ is the weighted Lebesgue area measure on $\mathbb{D}$. For $0 < p < \infty$, the Hardy space $H^p(\mathbb{D})$ consists of all analytic functions f on $\mathbb{D}$ that satisfy

$$\|f\|^p_{H^p} = \sup_{0<r<1} \int_{\partial\mathbb{D}} |f(r\zeta)|^p d\sigma(\zeta) < \infty,$$

where σ is the normalized Lebesgue measure on the boundary of the unit disk. Let φ be a nonconstant analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined on the space $H(\mathbb{D})$ of all analytic functions on $\mathbb{D}$ by $(C_\varphi f)(z) = f(\varphi(z))$, for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. Let ψ be an analytic function on $\mathbb{D}$, the weighted composition operator $W_{\psi,\varphi}$ is defined on the space $H(\mathbb{D})$ by $(W_{\psi,\varphi}f)(z) = \psi(z)C_\varphi f(z) = \psi(z)f(\varphi(z))$, for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. The boundedness and compactness of the (weighted) composition operators have been studied by many authors, see for example the monographs [**3**], [**6**], [**9**], [**18**], [**21**], [**22**] and the references therein for the overview of the field as of the early 1990s.

For f belongs to $H^p(\mathbb{D})$, it is well known from Fatou's theorem that the radial limit $f^*(\zeta) = \lim_{r\to 1^-} f(r\zeta)$ exists for almost all ζ on $\partial\mathbb{D}$. Moreover,

$$\|f\|^p_{H^p} = \int_{\partial\mathbb{D}} |f^*(\zeta)|^p d\sigma(\zeta),$$

2010 *Mathematics Subject Classification* Primary 47B33, 47B38, 30H10, 30H20; Secondary 47B37, 30H05, 32C15.

Key words and phrases: Composition operators, Bergman spaces, Hardy spaces, S^p spaces, compact operators, essential norm, and vanishing Carleson measure .

for all finite values of p. Moreover, for $f \in H^p$ and φ self-analytic map of $\mathbb{D}$, we know that $(f \circ \varphi)^* = f^* \circ \varphi^*$ a.e. on $\partial\mathbb{D}$, see for example [**17**]. We often use the standard abuse of notation of writing f instead of f^*. For $1 \leq p < \infty$, we denote by S^p the space of all analytic functions f on the unit disk $\mathbb{D}$ whose derivative f' lies in H^p, endowed with the norm

$$\|f\|_{S^p} = |f(0)| + \|f'\|_{H^p}.$$

One can easily show that S^p is a Banach space with respect to this norm.

The boundedness and compactness of the (weighted) composition operators on the space S^p have been studied by many authors, see for example [**1**], [**4**], [**12**], [**14**], [**15**], [**16**] and the references therein. In this paper, we characterize boundedness and compactness of composition operators that act from weighted Bergman spaces A^p_α to S^q spaces for $1 \leq p, q < \infty$. Moreover, we find a lower bound for the essential norm of composition operator from A^p_α into S^q spaces, $1 < p \leq q$.

The results we obtain about composition operators will be given in terms of a certain measure, which we define next. For $1 \leq p < \infty$, we define for any $\varphi \in S^p$ a finite positive Borel measure $\mu_{\varphi,p}$ on $\overline{\mathbb{D}}$ as:

$$\mu_{\varphi,p}(E) = \int_{\varphi^{-1}(E)\cap\partial\mathbb{D}} |\varphi'|^p \, d\sigma,$$

where E is a Borel subset of the closed unit disk $\overline{\mathbb{D}}$, and where σ is the normalized Lebesgue measure on the boundary of the unit disk $\partial\mathbb{D}$. Thus by using ([**8**], Theorem III.10.4) we get the following change of variable formula,

$$\int_{\overline{\mathbb{D}}} g d\mu_{\varphi,p} = \int_{\partial\mathbb{D}} |\varphi'|^p \, g(\varphi) d\sigma,$$

where g is an arbitrary measurable positive function on $\overline{\mathbb{D}}$.

For $r \in (0,1)$ and $w \in \mathbb{D}$, the pseudo-hyperbolic metric ρ on $\mathbb{D}$ is defined by $\rho(z,w) = |\varphi_w(z)|$ where $\varphi_w(z) = \frac{w-z}{1-\overline{w}z}$, for $z \in \mathbb{D}$. Moreover the pseudo-hyperbolic disk, whose pseudo-hyperbolic center is w and pseudo-hyperbolic radius is r, is defined as $D(w,r) = \{z \in \mathbb{D} : \rho(z,w) < r\}$. For further details on pseudo-hyperbolic metric see [**7**], for example. Throughout this paper the notation $A \approx B$ means that there is a positive constant C independent of A and B such that $C^{-1}B \leq A \leq CB$.

Lemma 1.1. *Suppose $0 < r < 1$, $p > 0$, and $\alpha > -1$. Then there exists a positive constant $C = C(p,r,\alpha)$ such that*

$$|f(z)|^p \leq \frac{C}{(1-|z|^2)^{2+\alpha}} \int_{D(z,r)} |f(w)|^p \, dA_\alpha(w),$$

for all $f \in H(\mathbb{D})$ and all $z \in \mathbb{D}$.

The proof of Lemma 1.1 can be easily seen by using Lemma 2.12 and Lemma 2.14 in [**9**], so the proof is not given here. In what follows, for $a \in \mathbb{D}$, $D(a)$ denotes the disk whose pseudo-hyperbolic center is a and pseudo-hyperbolic radius is r. The following lemma is a standard estimate in Bergman space, which is a result of Luecking ([**11**], Lemma 2.1).

Lemma 1.2. *Let $a \in \mathbb{D}$ and let r be fixed, $0 < r < 1$. There exists a constant C such that if f is analytic in $\mathbb{D}$ and $p \geq 1$ then*

$$|f^{(n)}(a)|^p \leq \left(C2^n \frac{(n+2)n!}{(1-r)^{n+2}} \right)^p \frac{\int_{D(a)} |f|^p dA}{\left(r(1-|a|^2)\right)^{2+nq}}.$$

Luecking characterized positive measures μ with the property $\|f^{(n)}\|_{L^q(\mu)} \leq \|f\|_{A^p_\alpha}$. The following result is a special case of Luecking's result ([**11**], Theorem 2.2) for $n = 1$.

Theorem 1.3. *Let $1 \leq p \leq q$ and let $\alpha > -1$. Let $\mu \geq 0$ be a finite measure on $\mathbb{D}$. The following are equivalent.*
(1) *There exists a constant C such that $\|f'\|_{L^q(\mu)} \leq C\|f\|_{A^p_\alpha}$ for all $f \in A^p_\alpha$.*
(2) $\mu(D(a)) = O\left(\left(1 - |a|^2\right)^{q(\alpha+2+p)/p} \right)$ *as $|a| \to 1$.*

Luecking ([**10**], Theorem 1), for the case $1 \leq q < p$, used Khinchine's inequality to obtain a version of Theorem 1.3 for $f^{(n)}$, where f in the unweighted Bergamn space A^p_0 . Here we are interested in the case $n = 1$ and $f \in A^p_\alpha$. Therefore, we present next Theorem 1.4 which is a slight modification of Luecking's result, so the proof's details are omitted.

Theorem 1.4. *Let $1 \leq q < p$ and let $\alpha > -1$. Let $\mu \geq 0$ be a finite measure on $\mathbb{D}$. Let $L(z) = \left(1 - |z|^2\right)^{-(\alpha+2+q)} \mu(D(z))$. The following are equivalent.*
(1) *There exists a constant C such that $\|f'\|_{L^q(\mu)} \leq C\|f\|_{A^p_\alpha}$ for all $f \in A^p_\alpha$.*
(2) $L \in L^{p/(p-q)}(A^p_\alpha)$.

Finally before stating our results, we show the following lemma whose proof can be obtained by adapting the proof of ([**3**], Proposition 3.11).

Lemma 1.5. *For $1 \leq p, q < \infty$ and $\alpha > -1$. Let φ be an analytic self-map of $\mathbb{D}$ such that C_φ is bounded from $A^p_\alpha(\mathbb{D})$ into S^q. Then C_φ is compact if and only if whenever $\{f_n\}$ is bounded sequence in $A^p_\alpha(\mathbb{D})$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $\|C_\varphi(f_n)\|_{S^q} \to 0$.*

2. Boundedness and Compactness of Composition Operators

In the following results we show the boundedness and compactness of the composition operators $C_\varphi : A^p_\alpha \to S^q$ can be characterized in terms of the finite positive Borel measure $\mu_{\varphi,p}$ defined in the previous section.

Theorem 2.1. *Let $1 \leq p \leq q$, and let $\alpha > -1$. Let φ be an analytic self-map of $\mathbb{D}$ and $\varphi \in S^q$. Then C_φ is bounded from A^p_α into S^q if and only if $\mu_{\varphi,q}(D(a)) = O\left(\left(1 - |a|^2\right)^{(\alpha+2+p)q/p} \right)$ as $|a| \to 1$.*

Proof. Suppose $f \in A^p_\alpha$ and $\varphi \in S^q$. Then, by using change of variables formula we get

$$\begin{aligned}\|C_\varphi f\|_{S^q}^q &\geq \|(f\circ\varphi)'\|_{H^q}^q \\ &= \int_{\partial\mathbb{D}} |f'(\varphi(z))|^q\,|\varphi'(z)|^q\,d\sigma(z) \\ &= \int_{\overline{\mathbb{D}}} |f'(w)|^q\,d\mu_{\varphi,q}(w) \\ &= \|f'\|_{L^q(\mu_{\varphi,q})}^q.\end{aligned}$$

If $C_\varphi : A_\alpha^p \to S^q$ is bounded, then there exists a positive constant C such that

$$\|f'\|_{L^q(\mu_{\varphi,q})} \leq \|C_\varphi f\|_{S^q} \leq C\|f\|_{A_\alpha^p}.$$

Therefore we get the desired result by using Theorem 1.3.

Conversely, by the closed graph theorem, it suffices to show that $C_\varphi(f) \in S^q$ whenever $f \in A_\alpha^p$. From the hypothesis on $\mu_{\varphi,q}$, Theorem 1.3 implies

$$\|f'\|_{L^q(\mu_{\varphi,q})} \leq C_1\|f\|_{A_\alpha^p}, \tag{1}$$

for some constant C_1. On the other hand, we can find a constant C_2 such that

$$\begin{aligned}\|C_\varphi(f)\|_{S^q} &= |f(\varphi(0))| + \|(f\circ\varphi)'\|_{H^q} \\ &\leq \|(f\circ\varphi)'\|_{H^q} + C_2\|f\|_{A_\alpha^p}.\end{aligned}$$

Therefore, we get

$$\|C_\varphi(f)\|_{S^q} \leq \|f'\|_{L^q(\mu_{\varphi,p})} + C_2\|f\|_{A_\alpha^p}. \tag{2}$$

From (1) and (2), we get $\|C_\varphi(f)\|_{S^q} \leq (C_1 + C_2)\|f\|_{A_\alpha^p}$. Which completes the proof. □

Recall that an operator is said to be compact if it takes bounded sets to sets with compact closure. In the next theorem we are following operator-theoretic wisdom: If a "big-oh" condition determines when an operator is bounded, then the corresponding "little-oh" condition determines when it is compact.

Theorem 2.2. *Let $1 \leq p \leq q$ and $\alpha > -1$. Let φ be an analytic self-map of $\mathbb{D}$ and $\varphi \in S^q$. Then C_φ is compact from A_α^p into S^q if and only if*

$$\mu_{\varphi,q}(D(a)) = o\left(1-|a|^2\right)^{(\alpha+2+p)q/p} \qquad as \quad |a| \to 1. \tag{3}$$

Proof. Suppose that $C_\varphi : A_\alpha^p \to S^q$ is compact. For $a \in \mathbb{D}$, define

$$K_a(z) = \frac{\left(1-|a|^2\right)^{(\alpha+2)/p}}{(1-\overline{a}z)^{2(\alpha+2)/p}}.$$

It is clear that $\|K_a\|_{A_\alpha^p} \approx 1$, and K_a converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. The compactness of C_φ yields that for $\epsilon > 0$, there exists $r \in (0,1)$ such that $\|C_\varphi(K_a)\|_{S^q}^q < \epsilon$ for $|a| > r$. Then for $|a| > r$

$$\begin{aligned}\int_{D(a)} |K_a'(z)|^q\,d\mu_{\varphi,q}(z) &\leq \int_{\overline{\mathbb{D}}} |K_a'(z)|^q\,d\mu_{\varphi,q}(z) \\ &= \int_{\partial\mathbb{D}} |K_a'(\varphi(w))|^q\,|\varphi'(w)|^q\,d\sigma(w) \\ &\leq \|C_\varphi(K_a)\|_{S^q}^q < \epsilon.\end{aligned}$$

On the other hand, it is easy to see for $z \in D(a)$

$$|K_a'(z)| \approx \frac{1}{(1-|a|^2)^{(\alpha+2+p)/p}}.$$

Thus, above inequalities yield for all a with $|a| > r$

$$\mu_{\varphi,q}\left(D(a)\right) < \epsilon \left(1-|a|^2\right)^{(\alpha+2+p)q/p},$$

which gives the desired result as $|a| \to 1$.

Conversely, suppose that condition (3) holds. Let $\{f_n\}$ be a bounded sequence in A^p_α with $f_n \to 0$ uniformly compact subsets of $\mathbb{D}$. To show C_φ is compact, it suffices to show

$$\|C_\varphi(f_n)\|^q_{S^q} \le C\|f_n'\|^q_{L^q(\mu_{\varphi,q})} \to 0 \quad as \quad n \to \infty.$$

Now, by using Lemma 1.2 we have

$$|f_n'(z)|^q \le \frac{C_1}{(1-|z|^2)^{2+q}} \int_{D(z)} |f_n(w)|^q dA(w).$$

Integrate the last inequality with respect to $\mu_{\varphi,q}$ and then use Fubini's Theorem to get

$$\|f_n'\|^q_{L^q(\mu_{\varphi,q})} \le C_1 \int_{\mathbb{D}} \frac{\mu_{\varphi,q}(D(w))}{(1-|w|^2)^{2+q}} |f_n(w)|^q dA(w).$$

From Lemma 1.1, we have for $f \in A^p_\alpha$ and $w \in \mathbb{D}$

$$|f_n(w)| \le \frac{C_2\|f_n\|_{A^p_\alpha}}{(1-|w|^2)^{(2+\alpha)/p}}.$$

Therefore, we get

$$\|f'\|^q_{L^q(\mu_{\varphi,q})} \le C_1C_2\|f_n\|^{q-p}_{A^p_\alpha} \int_{\mathbb{D}} |f_n(w)|^p \frac{\mu_{\varphi,q}(D(w))}{(1-|w|^2)^{(\alpha q+2q+pq-\alpha p)/p}} dA(w).$$

Condition (3) implies that for a given $\epsilon > 0$, there exists $r \in (0,1)$ such that

$$\int_{|w|>r} |f_n(w)|^p \frac{\mu_{\varphi,q}(D(w))}{(1-|w|^2)^{(\alpha+2+p)q/p}} (1-|w|^2)^\alpha dA(w) \le \epsilon\|f_n\|^p_{A^p_\alpha}. \tag{4}$$

On the other hand, since $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, namely $|w| \le r$, we can find constants C_3 and C_4 such that for large n

$$\begin{aligned}
&\int_{\mathbb{D}} |f_n(w)|^p \frac{\mu_{\varphi,q}(D(w))}{(1-|w|^2)^{(\alpha q+2q+pq-\alpha p)/p}} dA(w)\\
&\le \epsilon C_3 \int_{|w|\le r} \mu_{\varphi,q}(D(w))dA(w)\\
&\le \epsilon C_3 \int_{\mathbb{D}} \mu_{\varphi,q}(\mathbb{D})dA(w)\\
&= \epsilon C_3C_4.
\end{aligned} \tag{5}$$

Hence, from (4) and (5) we have

$$\|f'\|^q_{L^q(\mu_{\varphi,q})} \le \epsilon C_1C_2M^{q-p}\left(M^p + C_3C_4\right),$$

which completes the proof. □

The next theorem characterizes boundedness and compactness of C_φ that maps A^p_α into S^q when $1 \le q < p$.

Theorem 2.3. *Let $1 \le q < p$ and $\alpha > -1$. Let φ be an analytic self-map of $\mathbb{D}$ and $\varphi \in S^q$. Let $L(z) = \left(1-|z|^2\right)^{-(\alpha+q+2)} \mu_{\varphi,q}(D(z))$. Then the following are equivalent.*
(1) *C_φ is bounded from A^p_α into S^q.*
(2) *C_φ is compact from A^p_α into S^q.*
(3) *$L \in L^{p/(p-q)}(\mathbb{D}, dA_\alpha)$.*

Proof. It is clear that (2) implies (1). Using change of variable formula, then use Theorem 1.4 we get the equivalence of (1) and (3). It remains to verify that (3) implies (2). Let $\{f_n\}$ be a bounded sequence in A^p_α with $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. As in Theorem 2.2 it suffices to show that

$$\lim_{n\to\infty} \|f'_n\|^q_{L^q(\mu_{\varphi,q})} = 0.$$

Following similar argument as that in Theorem 2.2 we have

$$\begin{aligned}
\|f'_n\|^q_{L^q_{\mu_{\varphi,q}}} &\le C \int_{\mathbb{D}} \frac{1}{(1-|z|^2)^{2+q}} \int_{D(z)} |f_n(w)|^q dA(w) d_{\mu_{\varphi,q}}(z) \\
&\le C \int_{\mathbb{D}} \frac{\mu_{\varphi,q}(D(z))}{(1-|z|^2)^{(\alpha+2+q)}} |f_n(z)|^q \left(1-|z|^2\right)^\alpha dA(z) \\
&= C \int_{\mathbb{D}} |f_n(z)|^q L(z) dA_\alpha(z).
\end{aligned}$$

Suppose $L(z) \in L^{p/(p-q)}(\mathbb{D}, dA_\alpha)$. Then for $\epsilon > 0$, there exists $r \in (0,1)$ such that

$$\int_{\mathbb{D}} L^{p/(p-q)}(z) dA_\alpha(z) < \epsilon^{p/(p-q)}.$$

Using Hölder's inequality we get,

$$\begin{aligned}
&\int_{|z|>r} |f_n(z)|^q L(z) dA_\alpha(z) \\
&\le \left(\int_{|z|>r} |f_n(z)|^p dA_\alpha(z)\right)^{q/p} \left(\int_{|z|>r} L^{p/(p-q)}(z) dA_\alpha(z)\right)^{(p-q)/p} \\
&\le \epsilon \|f_n\|^q_{A^p_\alpha}. \qquad (6)
\end{aligned}$$

Since $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, $|f_n(z)| < \epsilon$ for all z such that $|z| \le r$ and for large n. Thus,

$$\begin{aligned}
\int_{|z|\le r} |f_n(z)|^q L(z) dA_\alpha(z) &\le \epsilon \int_{|z|\le r} L(z) dA_\alpha(z) \\
&= \epsilon \int_{|z|\le r} \frac{\mu_{\varphi,q}(D(z))}{(1-|z|^2)^{(\alpha+2+q)}} dA_\alpha(z) \\
&\le \epsilon C \int_{|z|\le r} \mu_{\varphi,q}(D(z)) dA_\alpha(z) \\
&\le \epsilon C \int_{\mathbb{D}} \mu_{\varphi,q}(\overline{\mathbb{D}}) dA_\alpha(z)
\end{aligned}$$

$$< \epsilon C, \tag{7}$$

where the finiteness of $\mu_{\varphi,q}(\overline{\mathbb{D}})$ follows from $\varphi \in S^q$. Hence, from (6) and (7) we get the desired result. □

3. Essential Norm of a Composition Operator

The essential norm of an operator, denoted by $\|.\|_e$, is its distance in the operator norm from the ideal of compact operators. So for any composition operator $C_\varphi : A^p_\alpha \to S^q$, we define

$$\|C_\varphi\|_e = \inf_{K \in \mathbb{K}} \|C_\varphi - K\|,$$

where $\mathbb{K} = \mathbb{K}(A^p_\alpha, S^q)$ is the set of compact operators acting from A^p_α into S^q. It is well known that a bounded operator T is compact if and only if $\|T\|_e = 0$, so that estimates of essential norm lead for a compactness of a composition operator. The essential norm has been studied by many authors in spaces of analytic functions, see for example [**2**], [**5**], [**13**], [**19**] and the related references therein.

For any ζ in the unit circle and $0 < r < 1$, the Carleson squares are defined as: $S(\zeta, r) = \{z \in \mathbb{D} : |1 - \langle z, \zeta\rangle| < r\}$. Let μ be a positive Borel measure on $\mathbb{D}$, and let X be a Banach space of analytic functions on $\mathbb{D}$. Then for $q > 0$, μ is called vanishing $(X, q)-$Carleson measure if

$$\lim_{k\to\infty} \int_{\mathbb{D}} |f_k|^q d\mu = 0,$$

whenever $\{f_k\}$ is a bounded sequence in X that converges to 0 uniformly on compact subsets of $\mathbb{D}$. As a special case of ([**20**], p.71) we have the next Lemma 3.1.

Lemma 3.1. *Suppose $0 < p \le q < \infty$, $\alpha > -1$ and μ is a positive Borel measure on $\mathbb{D}$. Then the following are equivalent.*
(1) *μ is vanishing $(A^p_\alpha, q)-$Carleson measure.*
(2) *The limit*

$$\lim_{r\to 0^+} \frac{\mu(S(\zeta, r))}{r^{(2+\alpha)q/p}} = 0,$$

holds uniformly for $\zeta \in \mathbb{S}$.

The following theorem gives a lower bound for the essential norm of a composition operator.

Theorem 3.2. *Let $1 < p \le q$ and $\alpha > -1$. Let φ be an analytic self-map of $\mathbb{D}$ and $\varphi \in S^q$. If C_φ is bounded from A^p_α into S^q, then there exists a constant C such that*

$$\|C_\varphi\|^q_e \ge C \limsup_{|a|\to 1} \int_{\partial\mathbb{D}} \frac{|\varphi'(w)|^q (1 - |a|^2)^{(2+\alpha)q/p}}{|1 - \overline{a}\varphi(w)|^{(2\alpha+4+p)q/p}} d\sigma(w).$$

Proof. For any $a \in \mathbb{D}$, consider the function

$$f_a(z) = \frac{(1 - |a|^2)^{(\alpha+2)/p}}{(1 - \overline{a}z)^{2(\alpha+2)/p}}.$$

It is clear that $f_a \in A^p_\alpha$ and $\|f_a\|_{A^p_\alpha} \approx 1$. Moreover, $f_a \to 0$ uniformly on compact subsets of $\mathbb{D}$. For the moment fix a compact operator K acting from A^p_α into S^q.

Then $\|Kf_a\|_{S^q} \to 0$ as $|a| \to 1$. Thus,

$$\begin{aligned}\|C_\varphi - K\| &\geq \limsup_{|a|\to 1} \|(C_\varphi - K)f_a\|_{S^q} \\ &\geq \limsup_{|a|\to 1} (\|C_\varphi f_a\|_{S^q} - \|Kf_a\|_{S^q}) \\ &= \limsup_{|a|\to 1} \|C_\varphi f_a\|_{S^q}.\end{aligned}$$

Upon taking the infimum of both side of this inequality over all compact operators K, we obtain

$$\begin{aligned}\|C_\varphi - K\|_e^q &\geq \limsup_{|a|\to 1} \|C_\varphi f_a\|_{S^q} \\ &= \limsup_{|a|\to 1} \left(|f_a(\varphi(0))|^q + \int_{\partial\mathbb{D}} |f_a'(\varphi(z))|^q |\varphi'(z)|^q d\sigma(z) \right) \\ &= \limsup_{|a|\to 1} \int_{\overline{\mathbb{D}}} |f_a'(w)|^q d\mu_{\varphi,q}(w), \end{aligned} \tag{8}$$

where the last line follows from the change of variable formula and the fact that $f_a(\varphi(0)) \to 0$ as $|a| \to 1$. Using (8) and the derivative of f_a give a desirable result. □

We use the essential norm and vanishing Carleson measure to prove our main result in this section, which characterizes the compactness of a composition operator.

Theorem 3.3. *Let $1 < p \leq q$ and $\alpha > -1$. Let φ be an analytic self-map of $\mathbb{D}$ and $\varphi \in S^q$. Suppose C_φ is bounded from A^p_α into S^q. Then C_φ is compact if and only if*

$$\limsup_{|a|\to 1} \int_{\partial\mathbb{D}} \frac{|\varphi'(w)|^q (1-|a|^2)^{(2+\alpha)q/p}}{|1-\overline{a}\varphi(w)|^{(2\alpha+4+p)q/p}} d\sigma(w) = 0.$$

Proof. The necessary condition follows from Theorem 3.2. We consider the sufficient condition, first we prove that $\mu_{\varphi,q}$ is vanishing $(A^p_\alpha, q)-$Carleson measure. Suppose the hypothesis holds. If $\zeta \in \partial\mathbb{D}$ and $r \in (0,1)$, then by using change of variable formula we get for all $a \in \mathbb{D}$

$$\limsup_{|a|\to 1} \int_{S(\zeta,r)} \frac{(1-|a|^2)^{(2+\alpha)q/p}}{|1-\overline{a}w|^{(2\alpha+4+p)q/p}} d\mu_{\varphi,q}(w) = 0.$$

Hence, for all $\epsilon > 0$, there exists $r \in (0,1)$ such that for $|a| > r$

$$\left| \int_{S(\zeta,r)} \frac{(1-|a|^2)^{(2+\alpha)q/p}}{|1-\overline{a}w|^{(2\alpha+4+p)q/p}} d\mu_{\varphi,q}(w) \right| < \epsilon. \tag{9}$$

By taking $a = (1-r)\zeta$, it is easy to see

$$\frac{(1-|a|^2)^{(2+\alpha)q/p}}{|1-\overline{a}w|^{(2\alpha+4+p)q/p}} \geq \frac{2^{-(2\alpha+4+p)q/p}}{r^{(\alpha+2)q/p}}. \tag{10}$$

So, from (9) and (10) we get for all $\zeta \in \partial\mathbb{D}$

$$\begin{aligned}\epsilon &> \int_{S(\zeta,r)} \frac{2^{-(2\alpha+4+p)q/p}}{r^{(\alpha+2)q/p}} d\mu_{\varphi,q} \\ &\geq C \ \frac{\mu_{\varphi,q}\left(S(\zeta,r)\right)}{r^{(\alpha+2)q/p}},\end{aligned}$$

which gives $\lim_{r\to 0^+} \frac{\mu_{\varphi,q}\left(S(\zeta,r)\right)}{r^{(\alpha+2)q/p}} = 0$. Therefore, by using Lemma 3.1, $\mu_{\varphi,q}$ is a vanishing $(A^p_\alpha, q)-$Carleson measure. Hence using the definition of vanishing Carleson measure, for any bounded sequence $\{f_n\}$ in A^p_α with $f_n \to 0$ on compact subsets of $\mathbb{D}$, consider the function $g_n \in S^q$ such that $g'_n = f_n$ and $g_n(0) = 0$. Then

$$\begin{aligned}0 &= \lim_{n\to\infty} \int_{\mathbb{D}} |f_n(z)|^q d\mu_{\varphi,q}(z) \\ &= \lim_{n\to\infty} \int_{\partial\mathbb{D}} |g'_n(\varphi(z))|^q |\varphi'(z)|^q d\sigma(z) \\ &= \lim_{n\to\infty} \|C_\varphi(g_n)\|^q_{S^q}.\end{aligned}$$

Therefore, by using Lemma 1.5, we get C_φ is compact as desired. $\square$

Acknowledgement. The author is very grateful to the referee for his/her valuable comments and suggestions which improved the presentation of this paper.

References

[1] S.C. Arora, M. Mukherjee and A. Panigrahi, *Weighted composition orperators on the space* $S^p(\mathbb{D})$, Bull. Calcutta Math. Soc., **88** (1996), 151–154.
[2] J. Bonet, P. Domański and M. Lindström, *Essential norm and weak compactness of composition operators on weighted Banach spaces of analytic functions*, Canad. Math. Bull., **42** (1999), 139–148.
[3] C.C. Cowen and B.D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC press, Boca Raton, 1995.
[4] M.D. Contreras and A.G. Hernádez-Díaz, *Weighted composition operators on spaces of functions with derivatives in a Hardy space*, J. Operator Theory, **52** (2003), 173–184.
[5] Z. Cucković and R. Zhao, *Essential norm estimates of weighted composition operators between Bergman spaces on strongly pseudoconvex domains*, Math. Proc. Cambridge Philos. Soc., **142** (2007), 525–533.
[6] P. Duren and A. Schuster, *Bergman Spaces*, American Mathematical Society, 2004.
[7] P. Duren and R. Weir, *The pseudohyperbolic metric and Bergman spaces in the ball*, Trans. Amer. Math. Soc., **359** (2007), 63–76.
[8] N. Dunford and T. Schwartz, *Linear Operators, Part 1*, Interscience, New York, 1967.
[9] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman Spaces*, Springer-Verlag, New York, 2000.
[10] D.H. Luecking, *Embedding theorems for spaces of analytic functions via Khinchine's inequality*, Mich. Math. J., **40** (1993), 333–358.

[11] D.H. Luecking, *Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives*, Amer. J. Math., **107** (1985), 85–111.
[12] B.D. MacCluer, *Composition operators on S^p*, Houston J. Math., **13** (1987), 245–254.
[13] B.D. MacCluer and J.H. Shapiro, *Angular derivatives and compact composition operators on the Hardy and Bergman spaces,* Canad. J. Math., **38** (1986), 878–906.
[14] W. Al-Rawashdeh, *Weighted composition operators between weighted Bergman and S^p spaces*, Bull. Math. Anal. Appl., **5** (2013), 54–64.
[15] R. Roan, *Composition operators on the space of functions with H^p-derivative*, Houston J. Math., **4** (1978), 423–438.
[16] R. Roan, *Generators and Composition Operators*, Thesis, Univrsity of Michigan, Ann Arbor, 1976.
[17] J. Ryff, *Subordinate H^p functions*, Duke Math. J., **33** (1966), 347–354.
[18] J.H. Shapiro, *Composition Operators and Classical Function Theory*, Universitext: Tracts in Mathematics. Springer-Verlag, New York, 1993.
[19] J.H. Shapiro, *The essential norm of a composition operators*, Annals of Mathematics, **125** (1987), 375–404.
[20] R. Zhao and K. Zhu, *Theory of Bergman Spaces in the Unit Ball of $\mathbb{C}^n$*, Mémoire de la SMF 115, Société Mathématique de France, France, 2008.
[21] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer-Verlag, New York, 2005.
[22] K. Zhu, *Operator Theory in Function Spaces*, Monograghs and Textbooks in Pure and Applied Mathematics, 139. Marcel Dekker, Inc., New York, 1990.

Waleed Al-Rawashdeh,
Department of Mathematical Sciences
Montana Tech of The University of Montana
Montana 59701, USA.
walrawashdeh@mtech.edu