ON A CLASS OF SEMICOMMUTATIVE RINGS

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Abstract. Let R be a ring with identity and an ideal I. In this paper, we introduce a class of rings generalizing semicommutative rings which is called I-semicommutative. The ring R is called I-*semicommutative* whenever $ab = 0$ implies $aRb \subseteq I$ for any a, $b \in R$. We investigate general properties of I-semicommutative rings and show that several results of semicommutative rings and J-semicommutative rings can be extended to I-semicommutative rings for this general settings.

1. Introduction

Throughout this paper, a ring means an associative ring with identity. We write $U(R)$ for the set of all units in R, $T_n(R)$ stands for the ring of all $n \times n$ triangular matrices over a ring R. A ring R is called *semicommutative* if for any a, $b \in R$, $ab = 0$ implies $aRb = 0$, this ring is also called *ZI ring* in [9] and [13], while, in [24], R is said to be *central semicommutative* if $ab = 0$ implies aRb is central in R. And in [17] a ring R is called *weakly semicommutative*, if for any a, $b \in R$, $ab = 0$ implies arb is a nilpotent element for each $r \in R$. Another generalization is made in [6], in which a ring R is called *nil-semicommutative-II* if a, $b \in R$ satisfy $ab \in \mathrm{Nil}(R)$, then $arb \in \mathrm{Nil}(R)$ for any $r \in R$ where $\mathrm{Nil}(R)$ is the set of all nilpotent elements of R. A similar concept is nil-semicommutativity is investigated in [20], in which it is said that a ring R is *nil-semicommutative-I* if for all nilpotent elements a, b of R, $ab = 0$ implies $aRb = 0$.

Every semicommutative ring is central semicommutative, weakly semicommutative, nil-semicommutative-I and nil-semicommutative-II. These classes of rings have been getting much attention, see namely, [6, 11, 20, 24, 25, 26]. Their relations with other classes of rings, such as Abelian rings, reduced rings, Armendariz rings and others, have been studied in the past. Also, another generalization of semicommutative rings is given in [33], called J-semicommutative rings. Let $J(R)$ denote the Jacobson radical of R. A ring R is *J-semicommutative* if $ab = 0$ implies $aRb \subseteq J(R)$. It is well known that $J(R)$ has some nice properties to help us to get good results. Motivated by these considerations, a natural question arises here when we replace $J(R)$ by an ideal I of R. In this paper, a new kind of rings which behave like semicommutative rings is considered. We investigate the semicommutativity of the ring relative to an ideal I. These are called I-semicommutative rings. That is, a ring R with an ideal I is called I-*semicommutative* provided $ab = 0$ implies $aRb \subseteq I$. In the second section of this paper, occasionally, we focus on the

general case of the ideal I fixed in R. We prove that if R is von Neumann regular, then every I-semicommutative ring for which I is a quasiregular ideal is semicommutative. If R is an abelian I-semipotent ring and I is a semiprime ideal, then R is I-semicommutative. In the third section, the I-semicommutativity of certain extensions of rings is studied. It is proved that R is an I-semicommutative ring if and only if $T_n(R)$ is $T_n(I)$-semicommutative for all positive integer n. Also a necessary and sufficient conditions for the ideal extensions, Dorroh extensions and Nagata extensions to be I-semicommutative are studied. The last section is devoted to exchange J-semicommutative rings. It is proved that if R is a J-semicommutative exchange ring, then $R/J(R)$ is commutative if and only if for each positive integer n, the set $GL_n(R)$ of all invertible $n \times n$ matrices is closed under transposition if and only if for all $a, b, c \in U(R)$, $c + [a, b] \in U(R)$ if and only if for all $a, b \in U(R)$, $[a, b] \in J(R)$, where $U(R)$ is the set of invertible elements in R.

2. I-Semicommutative Rings

A ring R is defined to be *nil-semicommutative-II* in case for any $a, b \in R$, $ab \in \mathrm{Nil}(R)$ implies that $arb \in \mathrm{Nil}(R)$ whenever $r \in R$ (see for detail [**6**]). In [**33**], J-semicommutative rings are studied. A ring is *J-semicommutative* if $ab = 0$ implies $aRb \subseteq J(R)$. It is well known that $J(R)$ has some nice properties to help us to get good results. A natural question arises here when we replace $J(R)$ by an ideal of R. In the preceding definitions, we replace $\mathrm{Nil}(R)$ and $J(R)$ by an ideal I and we call the ring R *I-semicommutative*. A ring R is called *abelian* if every idempotent is central, that is, $ae = ea$ for any $e^2 = e$, $a \in R$. Every local ring is J-semicommutative. Every semicommutative ring and every central semicommutative ring is abelian [**24**]. However, there are I-semicommutative rings that are neither semicommutative nor abelian. Also both nil-semicommutative-I rings and nil-semicommutative-II rings need not be abelian.

Example 2.1. Let $\mathbb{Z}$ denote the ring of integers and consider the ring $R = T_2(\mathbb{Z})$.
(1) Let $I = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$. Then R is I-semicommutative. For if $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $B = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in R$ and $AB = 0$, we have $ax = 0$, $ay + bz = 0$, $cz = 0$. Then $a = 0$ or $x = 0$ and $c = 0$ or $z = 0$. Accordingly, $ARB \subseteq I$ and R is I-semicommutative.
(2) Let $I = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. Then if $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $B = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in R$ and $AB = 0$, we have $ax = 0$, $ay + bz = 0$, $cz = 0$. Then $a = 0$ or $x = 0$ and $c = 0$ or $z = 0$. We have $ARB \subseteq I$ and R is I-semicommutative.
(3) Let $I = \begin{bmatrix} 0 & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$. A simple argument reveals that R is again I-semicommutative.
(4) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \in R$. Then $AB = 0$ and $ARB \neq 0$. Hence R is not semicommutative.

Example 2.2. Let R be a commutative ring and consider the ring
$$S = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{bmatrix} \mid a, b, c, d, e \in R \right\}, \text{ ideal } I = \left\{ \begin{bmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid b, c \in R \right\} \text{ of}$$

S. Let $A = \begin{bmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{bmatrix}$, $B = \begin{bmatrix} x & y & z \\ 0 & t & 0 \\ 0 & 0 & u \end{bmatrix}$ with $AB = 0$. Then $ASB = \left\{ \begin{bmatrix} 0 & n & l \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid n, l \in R \right\} \subseteq I$. Hence S is I-semicommutative.

Lemma 2.3. *Let R be a ring with an ideal I.*
(1) If R/I is a semicommutative ring, then R is I-semicommutative.
(2) If I consists of all nilpotent elements, then R is I-semicommutative.

Proof. (1) Let $a, b \in R$ with $ab = 0$. Then in $R/I := \bar{R}$ we have $\bar{a}\bar{b} = \bar{0}$. By the semicommutativity of R/I, $\bar{a}\bar{R}\bar{b} = \bar{0}$ and so $aRb \subseteq I$.
(2) Note first that being I nil ideal, I is contained in $J(R)$. Let $a, b \in R$ with $ab = 0$. Then $(bRa)^2 = 0$. By hypothesis on I, $bRa \subseteq I$ and $RbRa \subseteq I$. Then $(aRbR)^2 = a(RbRa)RbR \subseteq I$. Hence $(aRbR)^2$ consists of nilpotent elements. So $aRbR \subseteq I$. Since R has an identity, $aRb \subseteq I$. $\square$

Recall that a ring R is called *semiprime* if R has no nonzero nilpotent ideals. An ideal I in a ring R is called *semiprime* if the ring R/I is a semiprime ring. Equivalently, the ideal I is semiprime in the ring R if and only if for any ideal J of R and any positive integer n, $J^n \subseteq I$ implies $J \subseteq I$ if and only if for any $a \in R$, $aRa \subseteq I$ implies $a \in I$.

Proposition 2.4. *Let R be a ring. If I is a semiprime ideal, then the following are equivalent.*
(1) R is I-semicommutative.
(2) For any $a \in R$, $a^2 = 0$ implies that $a \in I$.
(3) For any $a, b \in R$, $ab = 0$ implies that $bRa \subseteq I$.

Proof. $(1) \Rightarrow (2)$ If $a^2 = 0$, then $aRa \subseteq I$. Hence, $(RaR)^2 \subseteq I$. As I is semiprime, it follows that $RaR \subseteq I$. Hence $a \in I$.
$(2) \Rightarrow (3)$ If $ab = 0$, then $(bra)^2 = 0$ for any $r \in R$. By (2) $bra \in I$. Thus, $bRa \subseteq I$.
$(3) \Rightarrow (1)$ If $ab = 0$, then $bRa \subseteq I$. Hence, $(RaRbR)^2 \subseteq I$. As I is semiprime, we deduce that $RaRbR \subseteq I$. This implies that $aRb \subseteq I$ since R has an identity, as required. $\square$

Proposition 2.5. *Let R and S be rings, $R \times S$ their direct product, I an ideal of R and L an ideal of S. Then $R \times S$ is $I \times L$-semicommutative if and only if R is I-semicommutative and S is L-semicommutative.*

Proof. Assume that $R \times S$ is $I \times L$-semicommutative and $a, b \in R$ with $ab = 0$. Then $(a, 0)(b, 0) = (0, 0)$ in $R \times S$. By assumption $(a, 0)(R \times S)(b, 0) \subseteq I \times L$. This implies $aRb \subseteq I$. Hence R is I-semicommutative. A similar discussion reveals that S is L-semicommutative. Conversely, suppose that R is I-semicommutative and S is L-semicommutative. Let $(r, s), (r_1, s_1) \in R \times S$ with $(r, s)(r_1, s_1) = (0, 0)$. Then $rr_1 = 0$ and $ss_1 = 0$. By supposition $rRr_1 \subseteq I$ and $sSs_1 \subseteq L$. Hence $(r, s)(R \times S)(r_1, s_1) \subseteq I \times L$. $\square$

Corollary 2.6. *Let R be an abelian ring. Then R is I-semicommutative if and only if for any idempotent $e \in R$, eR and $(1 - e)R$ are both I-semicommutative.*

Lemma 2.7. *If I is a semiprime ideal of R, R is an I-semicommutative ring and idempotents lift modulo I, then R/I is abelian.*

Proof. Let $y \in R$ with $y^2 = 0$. Then $yRy \subseteq I$. Hence $y \in I$, since I is semiprime. Let $e^2 = e \in R$. For any $x \in R$, $(ex - exe)^2 = 0$ and $(xe - exe)^2 = 0$ which implies $ex - exe, xe - exe \in I$. Since idempotents lift modulo I, hence $ex - xe \in I$. □

Proposition 2.8. *Let R be an I-semicommutative ring and I be an ideal contained in $J(R)$. If R is a von Neumann regular ring, then R is semicommutative.*

Proof. Let $x, y \in R$ with $xy = 0$. By hypothesis $xry \in I$ for all $r \in R$. Since $I \subseteq J(R)$, for all $a \in R$, $1 - a(xry)$ is invertible. Since R is a von Neumann regular, there exists $u \in R$ such that $xry = (xry)u(xry)$. It implies $xry(1 - u(xry)) = 0$. By invertibility of $1 - u(xry)$ we have $xry = 0$ for all $r \in R$. This shows that R is semicommutative. □

Lemma 2.9. *Let R be an I-semicommutative ring. If I is an ideal contained in $J(R)$, then R is directly finite.*

Proof. Let R be an I-semicommutative ring and $a, b \in R$ with $ab = 1$, $e = ba$. Clearly, $ae = a, eb = b$. Since R is I-semicommutative and $e(1 - e) = 0$, we get $eR(1 - e) \subseteq I$. In particular, $eb(1 - e) \in I$. Since $eb = b$, $b(1 - e) \in I$. Multiplying the latter from left by a, we have $ab(1 - e) \in I$. Since $ab = 1$, $1 - e \in I$. Being $1 - e$ an idempotent and $I \subseteq J(R)$, we have $1 = e = ba$. □

Proposition 2.10. *Let $e \in R$ be an idempotent. If R is I-semicommutative, then eRe is eIe-semicommutative.*

Proof. Let $eae, ebe \in eRe$ with $(eae)(ebe) = 0$. Since R is I-semicommutative, $(eae)R(ebe) \subseteq I$, and so $(eae)R(ebe) \subseteq eIe$. That is, $(eae)(eRe)(ebe) \subseteq eIe$. Therefore eRe is eIe-semicommutative. □

Let R be a ring, n a positive integer and R_n denote the subring of the $n \times n$ upper triangular matrix ring $T_n(R)$ defined by

$$R_n = \left\{ \begin{bmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix} \mid a, a_{ij} \in R(i < j) \right\}$$

Let I be an ideal of R. Then $I_n = \left\{ \begin{bmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix} \in T_n(R) \mid a \in I \right\}$ is an ideal of R_n.

Proposition 2.11. *Let R be a ring and n a positive integer with $2 \leq n$. Then the following are equivalent.*
(1) R is I-semicommutative.
(2) R_n is I_n-semicommutative.

Proof. Assume that R is I-semicommutative. Let $A, B \in R_n$ with $AB = 0$, where

$$A = \begin{bmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix}, \; B = \begin{bmatrix} b & b_{12} & \cdots & b_{1n} \\ 0 & b & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{bmatrix}. \text{ Then } ab = 0. \text{ Since } R$$

is I-semicommutative, $aRb \subseteq I$. The diagonal entries of AR_nB consist of $arb \in I$ where $r \in R$ and other entries are in R. It follows that R_n is I_n-semicommutative.

Conversely, let $a, b \in R$ with $ab = 0$. Then $\begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix} \begin{bmatrix} b & 0 & \cdots & 0 \\ 0 & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{bmatrix} =$

0. For any $\begin{bmatrix} c & x & \cdots & y \\ 0 & c & \cdots & z \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{bmatrix} \in R_n$, we have

$$\begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix} \begin{bmatrix} c & x & \cdots & y \\ 0 & c & \cdots & z \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{bmatrix} \begin{bmatrix} b & 0 & \cdots & 0 \\ 0 & b & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b \end{bmatrix} \in I_n.$$

Therefore $acb \in I$ for all $c \in R$. So R is I-semicommutative. $\square$

Proposition 2.12. *Let R be a ring and $K \subseteq K_1 \subseteq R$ and $L \subseteq L_1 \subseteq R$ be ideals of R. If $(R/K) \times (R/L)$ is $(K_1/K) \times (L_1/L)$-semicommutative, then R is $K_1 \cap L_1$-semicommutative.*

Proof. Suppose that $(R/K) \times (R/L)$ is $(K_1/K) \times (L_1/L)$-semicommutative, and let $a, b \in R$ with $ab = 0$. Then $(a + K, 0 + L)(b + K, 0 + L) = (0 + K, 0 + L)$ and $(0 + K, a + L)(0 + K, b + L) = (0 + K, 0 + L)$ in $(R/K) \times (R/L)$. By hypothesis $(a+K, 0+L)(R/K \times R/L)(b+K, 0+L) \subseteq K_1/K \times L_1/L$ and $(0+K, a+L)(R/K \times R/L)(0 + K, b + L) \subseteq K_1/K \times L_1/L$ and so $(a + K)(R/K)(b + K) \subseteq K_1/K$ and $(a + L)(R/L)(b + L) \subseteq L_1/L$. Hence $aRb \subseteq K_1$ and $aRb \subseteq L_1$. Thus $aRb \subseteq K_1 \cap L_1$. $\square$

Proposition 2.13. *R is an I-semicommutative ring if and only if the ring $S = \{(x, y) \in R \times R \mid x - y \in I\}$ is $I \times I$-semicommutative.*

Proof. $S = \{(x, y) \in R \times R \mid x - y \in I\}$ forms a subring of $R \times R$. If R is an I-semicommutative ring and $(x, y), (a, b) \in S$ and $(a, b)(x, y) = 0$, then $ax = 0$ and $by = 0$ implies $aRx \subseteq I$ and $bRy \subseteq I$. Hence $(a, b)S(x, y) \subseteq I \times I$.
Conversely, suppose that S is $I \times I$-semicommutative and let $a, b \in R$ with $ab = 0$. Then $(a, a)(b, b) = 0$ in S. Since S is $I \times I$-semicommutative, $(a, a)(x, y)(b, b)$ lies in $I \times I$, for all $(x, y) \in S$. In particular for $(x, x) \in S$, we have $(a, a)(x, x)(b, b) \in I \times I$ for all $x \in R$. So $axb \in I$. Therefore R is I-semicommutative. $\square$

Definition 2.14. Let R be a ring with an ideal I. R is called I-*semipotent* if each left ideal (resp., right ideal) not contained in I contains a non-zero idempotent, and R is called I-*potent* if, in addition, idempotents lift modulo I.

Proposition 2.15. *Let R be an abelian I-semipotent ring. If I is a semiprime ideal, then R is I-semicommutative.*

Proof. Let $ab = 0$. Assume that $bra \notin I$ for some $r \in R$. Then there is a non-zero idempotent $e \in braR$. Write $e = brat$. Then $e = e^2 = (brat)(brat) = bra(brat)t = br(ab)rat^2 = 0$ since e is central. This is a contradiction. So $bRa \subseteq I$. This proves that $ab = 0$ implies $bRa \subseteq I$. By Proposition 2.4, R is I-semicommutative. $\square$

3. Certain Extensions and Applications to $J(R)$

The goal of this section is to consider some extensions of J-semicommutative rings. Let S and T be any rings, M an S-T-bimodule and R the formal triangular matrix ring $\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$. Let I_1 and I_2 be ideals of S and T respectively. Then $I = \begin{bmatrix} I_1 & M \\ 0 & I_2 \end{bmatrix}$ is an ideal of R.

Proposition 3.1. *Let S, T, M, R and I be as above. Then R is I-semicommutative if and only if S and T are I_1-semicommutative and I_2-semicommutative, respectively.*

Proof. Necessity: Assume that $R = \begin{bmatrix} S & M \\ 0 & T \end{bmatrix}$ is $I = \begin{bmatrix} I_1 & M \\ 0 & I_2 \end{bmatrix}$-semicommutative. By Proposition 2.10, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_1 & M \\ 0 & I_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$-semicommutative. Since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S & M \\ 0 & T \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is isomorphic to S and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I_1 & M \\ 0 & I_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is isomorphic to I_1, S is I_1-semicommutative. Replacing the idempotent $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ by $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, a similar discussion reveals that T is I_2-semicommutative.

Sufficiency: Suppose that S and T are I_1-semicommutative and I_2-semicommutative respectively, and $A = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix}$, $B = \begin{bmatrix} c & y \\ 0 & d \end{bmatrix} \in R$ with $AB = 0$. Then $ac = 0$ in S and $bd = 0$ in T. By supposition, $aSc \subseteq I_1$ and $bTd \subseteq I_2$. Thus $ARB \subseteq I$ and so R is I-semicommutative. $\square$

Let I be an ideal of R, for a positive integer n, and $T_n(I)$ denote the ring of all $n \times n$ upper triangular matrices $[a_{ij}]$ with $a_{ii} \in I$. Then $T_n(I)$ is an ideal of $T_n(R)$.

Theorem 3.2. *Let R be a ring, I an ideal of R and n a positive integer. Then the following are equivalent.*
(1) R is I-semicommutative.
(2) $T_n(R)$ is $T_n(I)$-semicommutative.

Proof. (1) $\Rightarrow$ (2) Let $[a_{ij}]$, $[b_{ij}] \in T_n(R)$ with $[a_{ij}][b_{ij}] = 0$. Then $a_{ii}b_{ii} = 0$ and so by (1), $a_{ii}Rb_{ii} \subseteq I$. Thus $[a_{ij}]T_n(R)[b_{ij}] \subseteq T_n(I)$ and so $T_n(R)$ is $T_n(I)$-semicommutative.

(2) $\Rightarrow$ (1) Let a, b $\in$ R with ab $=$ 0 and $A = \begin{bmatrix} a & 0 & 0 & ... & 0 \\ 0 & a & 0 & ... & 0 \\ . & . & ... & . & . \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & ... & a \end{bmatrix}$ and

$B = \begin{bmatrix} b & 0 & 0 & ... & 0 \\ 0 & b & 0 & ... & 0 \\ \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & ... & b \end{bmatrix}$ $\in$ $T_n(R)$. Then $AB = 0$. By (2),

$AT_n(R)B \subseteq T_n(I)$. It follows that $aRb \subseteq I$. $\qquad\square$

Corollary 3.3. *Let R be a ring. Then the following are equivalent.*
(1) *R is I-semicommutative.*
(2) *$R[x]/(x^n)$ is $\big(I[x]+(x^n)\big)/(x^n)$-semicommutative for all $n \geq 2$.*

Proof. Since

$$R[x]/(x^n) \cong \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \ddots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & a_1 \end{bmatrix} \mid a_i \in R \right\},$$

we get the result as in Theorem 3.2. $\qquad\square$

The ring R is called *Armendariz* if for any $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{s} b_j x^j \in R[x]$, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all i and j (See [**12**]). For example every reduced ring is Armendariz.

Proposition 3.4. *Let R be a ring. Let K be an ideal of the ring $R[x]$ of polynomials over R. Let I be the ideal of R generated by the coefficients of all polynomials in K. If $R[x]$ is K-semicommutative, then R is I-semicommutative. The converse holds if R is Armendariz.*

Proof. Assume that $R[x]$ is K-semicommutative. Let $a, b \in R$ and $ab = 0$. Then $aR[x]b \subseteq K$, so $aRb \subseteq I$. Conversely, suppose that R is an I-semicommutative Armendariz ring, $f(x) = \sum_{i=0}^{s} a_i x^i$, $g(x) = \sum_{j=0}^{t} b_j x^j \in R[x]$ and $f(x)g(x) = 0$. R being Armendariz, we have $a_i b_j = 0$, where $0 \leq i \leq s$, $0 \leq j \leq t$. By hypothesis $a_i R b_j \subseteq I$. Then for all i with $0 \leq i \leq s$ $a_i x^i R b_j \subseteq I[x]$ and $a_i x^i R[x] b_j \subseteq I[x]$. So $f(x)R[x]b_j \subseteq I[x]$. Similarly, $f(x)R[x]g(x) \subseteq I[x]$. Since all coefficients of the polynomials in $I[x]$ belong to the K, $I[x] \subseteq K$. Thus $f(x)R[x]g(x) \subseteq K$. $\qquad\square$

In general, J-semicommutative rings are not Armendariz.

Example 3.5. Let $\mathbb{R}$ be the field of real numbers, $R = T_3(\mathbb{R})$. By Theorem 3.2, R is a J-semicommutative ring, but it is not an abelian ring, so R is not an Armendariz ring by [**11**, Corollary 8].

Let R be a ring with an ideal I and V an R-R-bimodule which is a general ring (possibly with no unity) in which $(vw)r = v(wr)$, $(vr)w = v(rw)$ and $(rv)w = r(vw)$ hold for all $v, w \in V$ and $r \in I$. Then *the ideal extension* (it is also called *the Dorroh extension* (see [**7**])) $D(I;V)$ of I by V is defined to be the additive abelian group $I \oplus V$ with multiplication $(r, v)(s, w) = (rs, rw + vs + vw)$ where $(r, v), (s, w) \in I \oplus V$. Since R is an ideal of itself, $D(R;V)$ is a well defined ring and it is also called *the Dorroh extension* of R by V. Note that $D(I;V)$ is an ideal of $D(R;V)$ (see [**19**]).

Proposition 3.6. *Suppose that $V = J(V)$. Then the following are equivalent for a ring R.*
(1) *R is I-semicommutative.*
(2) *$D(R;V)$ is $D(I;V)$-semicommutative.*

Proof. (1) $\Rightarrow$ (2) Let $s = (r, v)$, $a = (b, c) \in D(R;V)$ with $sa = (0, 0)$. Then $rb = 0$ and so $rRb \subseteq I$ by (1). Hence $sD(R;V)a \subseteq D(I;V)$.
(2) $\Rightarrow$ (1) Suppose that $D(R;V)$ is $D(I;V)$-semicommutative. Let $a, b \in R$ with $ab = 0$. Then $(a, 0)(b, 0) = (0, 0)$ and so $(a, 0)D(R;V)(b, 0) \subseteq D(I;V)$ by (2). Hence $aRb \subseteq I$. $\qquad\square$

Corollary 3.7. *Suppose that $V = J(V)$. Then the following are equivalent for a ring R.*
(1) *R is $J(R)$-semicommutative.*
(2) *The ideal extension $D(R;V)$ is $J(D(R;V))$-semicommutative.*

Proof. Note that by the hypothesis and [**19**, Theorem 13], $(r, v) \in J(D(R;V))$ if and only if $r \in J(R)$. Assume that R is $J(R)$-semicommutative. Let (r, v), $(s, u) \in D(R;V)$ with $(r, v)(s, u) = 0$ in $D(R;V)$. Then $rs = 0$ and so $rRs \subseteq J(R)$. Hence $(r, v)D(R;V)(s, u) \subseteq J(D(R;V))$. Conversely, suppose that $D(R;V)$ is $J(D(R;V))$-semicommutative. Let $r, s \in R$ with $rs = 0$. Then $(r, 0)(s, 0) = (0, 0)$. By supposition $(r, 0)D(R;V)(s, 0) \subseteq J(D(R;V))$. It follows that $rRs \subseteq J(R)$. $\qquad\square$

Let R be a commutative ring, M be an R-module and σ be an endomorphism of R. The abelian group $R \oplus M$ has a ring structure with multiplication

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, \sigma(r_1)m_2 + r_2 m_1),$$

where $r_i \in R$, $m_i \in M$. This extension is constructed by Nagata in [**21**], is called *the Nagata extension of R by M and σ*, and is denoted by $N(R, M, \sigma)$.

Theorem 3.8. *Let R be a ring with an ideal I. Then, R is I-semicommutative if and only if $N(R, M, \sigma)$ is $I \oplus M$-semicommutative.*

Proof. Assume that R is I-semicommutative. Let $(r_1, m_1), (r_2, m_2) \in N(R, M, \sigma)$ with $(r_1, m_1)(r_2, m_2) = 0$; then $r_1 r_2 = 0$. By assumption, $r_1 R r_2 \subseteq I$. Then $r_1 r r_2 \in I$ for all $r \in R$. Hence, for any $(r, m) \in R \oplus M$, we have

$$(r_1, m_1)(r, m)(r_2, m_2) = (r_1 r r_2, x) \in I \oplus M,$$

since $x \in M$. This implies $N(R, M, \sigma)$ is an $I \oplus M$-semicommutative ring. The converse is clear. $\qquad\square$

Let R be a ring, and let σ be an endomorphism of R. Let $T_2(R,\sigma)$ be the set of all 2×2 upper triangular matrices over R. For $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in T_2(R,\sigma)$, we define the multiplication

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} ax & ay + b\sigma(z) \\ 0 & cz \end{bmatrix}.$$

This multiplication and usual matrix addition makes $T_2(R,\sigma)$ is a ring since σ is a ring homomorphism.

Theorem 3.9. *Let R be a ring with an ideal I. Then, R is I-semicommutative if and only if $T_2(R,\sigma)$ is $I(R,\sigma)$-semicommutative, where*

$$I(R,\sigma) = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a,\ c \in I,\ b \in R \right\}.$$

Proof. Assume that R is I-semicommutative and let $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, $\begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \in T_2(R,\sigma)$ with $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = 0$. Then $ad = 0$, $ae + b\sigma(f) = 0$ and $cf = 0$. By the I-semicommutativity of R, we have $aRd \subseteq I$ and $cRf \subseteq I$. This implies $T_2(R,\sigma)$ is $I(R,\sigma)$-semicommutative. Conversely, suppose that $T_2(R,\sigma)$ is $I(R,\sigma)$-semicommutative and let $a,b \in R$ with $ab = 0$. We have $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} = 0$. So for any $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in T_2(R,\sigma)$, we get $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \in I(R,\sigma)$. Hence we have $aRb \subseteq I$. Thus R is I-semicommutative. $\square$

Let R be a ring with a ring homomorphism $\sigma : R \to R$ and $R[[x,\sigma]]$ the ring of skew formal power series over R; that is, all formal power series in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$ for all $r \in R$. In particular, $R[[x]] = R[[x,1_R]]$ is the ring of formal power series over R. Note that $J(R[[x,\sigma]]) = J(R) + (x)$ (see [**27**, Lemma 16.10]).

Proposition 3.10. *Let R be a ring, $\sigma : R \to R$ a ring homomorphism and I an ideal of R. Then the following are equivalent.*
(1) R is I-semicommutative.
(2) $R[[x,\sigma]]$ is $I + (x)$-semicommutative.

Proof. (1)$\Rightarrow$(2) Assume that R is I-semicommutative. Let $f(x) = \sum a_i x^i$, $g(x) = \sum b_i x^i \in R[[x,\sigma]]$ with $f(x)g(x) = 0$. Then $a_0 b_0 = 0$. By assumption $a_0 R b_0 \subseteq I$. Then $f(x)R[[x,\sigma]]g(x) \subseteq a_0 R b_0 + (x) \subseteq I + (x)$.
(2)$\Rightarrow$(1) Let $a,\ b \in R$ with $ab = 0$. Then $aR[[x,\sigma]]b \subseteq I + (x)$. Hence we have $aRb \subseteq I$. $\square$

Let R be a ring and $a \in R$. a is called *a r.q.r element*(r.q.r for short) in R if there exists $a' \in R$ such that $a \circ a' = 0$, where $a \circ a' = a + a' - aa'$. In terms of r.q.r elements of R, $J(R) = \{a \in R \mid aR$ is r.q.r in $R\}$, see also [**18**, Definition 6.6 and Chapter 6] for details. Let A be a ring and B a subring of A and

$$R[A,B] = \{(a_1, a_2, \cdots, a_n, b, b, \cdots) : a_i \in A, b \in B, n \geq 1, 1 \leq i \leq n\}.$$

Then $R[A, B]$ is a ring under the componentwise addition and multiplication. Since the ring $R[A, B]$ need not have an identity, we use the preceding characterization of the Jacobson radical in the next lemma. Next we first show the equality $J(R[A, B]) = R[J(A), J(A) \cap J(B)]$ and then we prove necessary and sufficient conditions for $R[A, B]$ to be J-semicommutative.

Lemma 3.11. *Let A be a ring and B a subring of A. Then $J(R[A, B]) = R[J(A), J(A) \cap J(B)]$.*

Proof. Let $X = (a_1, a_2, \cdots, a_n, b, b, \cdots) \in J(R[A, B])$. Then for any $Y = (a'_1, a'_2, \cdots, a'_m, b', b', \cdots) \in R[A, B]$, XY is r.q.r for all $Y \in R[A, B]$. So for such X and Y, there exists $Z = (x_1, x_2, \cdots, x_t, y, y, \cdots) \in R[A, B]$ such that $(XY) \circ Z = 0$. We prove that the components $a_1, a_2, \cdots, a_n$ of X are in $J(A)$ and $b \in J(A) \cap J(B)$. For this it is enough to show $a_1 \in J(A)$. For any $a'_1 \in A$, set $Y = (a'_1, 0, \cdots, 0, 0, , \cdots) \in R[A, B]$. Then there exists $Z = (x_1, x_2, \cdots, x_n, y, y, \cdots) \in R[A, B]$ such that $(XY) \circ Z = 0$. This implies $(a_1 a'_1) \circ x_1 = 0$, that is, $a_1 \in J(A)$. In the same way, we may show that the other components $a_2, ..., a_n$ of X belong to $J(A)$. As for the other components b, let $b' \in B$ be an arbitrary element. Set $Y' = (0, 0, \cdots, 0, b', b', \cdots)$. Then XY' is r.q.r. Hence bb' is r.q.r for each $b' \in B$. Thus $b \in J(B)$. To prove $b \in J(A)$, let $a'_{n+1} \in A$ be an arbitrary element in A and set $Y'' = (a'_1, a'_2, \cdots, a'_{n+1}, b', \cdots) \in R[A, B]$. Then XY'' is r.q.r. Hence ba'_{n+1} is r.q.r in A for all $a'_{n+1} \in A$. So $b \in J(A)$. Thus $b \in J(A) \cap J(B)$. This implies that $J(R[A, B]) \subseteq R[J(A), J(A) \cap J(B)]$.

For the converse inclusion, let $X = (a_1, a_2, \cdots, a_n, b, b, \cdots) \in R[J(A), J(A) \cap J(B)]$. Then $a_i \in J(A)$ for each $1 \le i \le n$, and $b \in J(A) \cap J(B)$. Let $Y = (x_1, x_2, \cdots, x_m, y, y, \cdots)$ be an arbitrary element in $R[A, B]$. We prove that there exists $Z = (z_1, z_2, \cdots, z_t, z, z, \cdots)$ such that $(XY) \circ Z = 0$. We divide the proof into cases.

Case I: $n = m$. Since $a_i \in J(A)$ and $b \in J(A)$ and $b \in J(B)$, $a_i x_i$ are r.q.r in A for $1 \le i \le n$ and by is r.q.r in both A and B. So there exist $z_i \in A$ and $z \in B$ such that $(a_i x_i) \circ z_i = 0$ where $1 \le i \le n$ and $(by) \circ z = 0$. Set $Z = (z_1, z_2, \cdots, z_n, z, z, \cdots)$. Since the ring operations in $R[A, B]$ is componentwise, $(XY) \circ Z = 0$.

Case II: $n < m$. $a_i x_i$ are r.q.r for $1 \le i \le n$ and bx_i is r.q.r in A for $n+1 \le i \le m$ and by is r.q.r in B. There exist $z_i \in A$, $z_j \in A$ and $z \in B$ such that $(a_i x_i) \circ z_i = 0$ for $1 \le i \le n$ and $(bx_j) \circ z_j = 0$ for $n+1 \le j \le m$. Set $Z = (z_1, z_2, \cdots, z_m, z, z, \cdots)$. Hence $(XY) \circ Z = 0$ holds.

Case III: $n > m$. $a_i x_i$ is r.q.r for $1 \le i \le m$ and $a_i y$ is r.q.r for $m+1 \le i \le n$ in A and by is r.q.r in B. There exist $z_i \in A$ and $z \in B$ such that $(a_i x_i) \circ z_i = 0$ for $1 \le i \le m$ and $(a_i y) \circ z_i = 0$ for $m+1 \le i \le n$ and $(by) \circ z = 0$. Let $Z = (z_1, z_2, \cdots, z_n, z, z, \cdots)$. By using componentwise operations in $R[A, B]$, we have $(XY) \circ Z = 0$. So $R[J(A), J(A) \cap J(B)] \subseteq J(R[A, B])$ holds. This completes the proof. $\square$

Proposition 3.12. *Let A be a ring with B a subring of A. Then the following are equivalent.*
(1) A and B are J-semicommutative.
(2) $R[A, B]$ is J-semicommutative.

Proof. $(1) \Rightarrow (2)$ Let $a = (a_1, \cdots, a_n, b, b, \cdots)$, $c = (c_1, \cdots, c_m, d, d, \cdots) \in R[A, B]$ with $ac = 0$. Then $bd = 0$, so $bBd \subseteq J(A) \cap J(B)$ by (1). Let $k = min(n, m)$. Then for $1 \leq i \leq k$, $a_i c_i = 0$, so $a_i A c_i \subseteq J(A)$, also by (1). If $m < n$ then $a_i d = 0$ whenever $m < i \leq n$, so $a_i A d \subseteq J(A)$ for such i, by (1), while if $n < m$ then $bc_j = 0$ whenever $n < j \leq m$, so $bAc_j \subseteq J(A)$ for such j, again by (1). Hence $aR[A, B]c \subseteq J(R[A, B])$.

$(2) \Rightarrow (1)$ Let a, $b \in A$ with $ab = 0$ and let $x = (a, 0, 0, \cdots) \in R[A, B]$, $y = (b, 0, 0, \cdots) \in R[A, B]$. Then $xy = 0$. By (2), $xR[A, B]y \subseteq J(R[A, B])$. Hence $aAb \subseteq J(A)$. Therefore A is J-semicommutative. To prove that B is J-semicommutative, let b, $c \in B$ with $bc = 0$, and let $\alpha = (0, b, b, \cdots) \in R[A, B]$ and $\beta = (0, c, c, \cdots) \in R[A, B]$. Then $\alpha\beta = 0$. By (2), $\alpha R[A, B]\beta \subseteq J(R[A, B])$. It follows that $bBc \subseteq J(B)$, as desired. $\square$

4. Exchange Rings with $I = J(R)$

The class of exchange rings is very large. It includes all regular rings, all π-regular rings, all strongly π-regular rings, all semiperfect rings, all left or right continuous rings, all clean rings, all unit C^*-algebras of real rank zero and all right semi-artinian rings (see for detail [**3**],[**4**],[**5**],[**28**],[**30**]). The aim of this section is to consider J-semicommutativity of such rings.

Recall that a ring R is *exchange* provided that for any $a \in R$, there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$ (see for detail [**1**]). A ring R is *clean* in case every element in R is the sum of an idempotent and a unit. Clean rings always exchange. The converse holds if all idempotents of R are central. Also note that a ring R is an exchange ring if and only if idempotents can be lifted modulo every left (respectively right) ideal (see [**22**]).

Note that if u is an invertible element of a ring R and $r \in J(R)$, then $u + r$ is invertible. For if $a = u + r$, then $u^{-1}a - 1 = u^{-1}r \in J(R)$. As $1 + u^{-1}r$ is invertible, $u^{-1}a$ is invertible and so is a. Also for an element $a \in R$, a is invertible in R if and only if $\overline{a}$ is invertible in $R/J(R)$.

Theorem 4.1. *Let R be a J-semicommutative ring. Then the following are equivalent.*
(1) *R is an exchange ring.*
(2) *R is clean.*

Proof. $(2) \Rightarrow (1)$ is obvious from [**22**, Proposition 1.8] because [**22**, Proposition 1.8] says that every clean ring is a suitable ring. Note that suitable ring is another name of exchange ring.

$(1) \Rightarrow (2)$ Set $S = R/J(R)$. Then S is an exchange ring by [**22**, Proposition 1.5]. Let $f \in S$ be an idempotent. Then $f = \overline{e} \in S$ for some idempotent $e \in R$ where $\overline{e} \in S$ refers to $e + J(R) \in S$. By Lemma 2.7, $\overline{er} = \overline{re}$ for all $\overline{r} \in S$. So S is an exchange ring with all idempotents central. This yields that S is clean. For any $a \in R$, we have an idempotent $g \in R$ and an invertible element $\overline{u} \in S$ such that $\overline{a} = \overline{g} + \overline{u}$. Write $a - g = u + r$ for some $r \in J(R)$. Then $u + r$ is invertible as noted before, and so $a = g + (u + r)$, as required. $\square$

Theorem 4.2. *Let R be an exchange ring. Then the following are equivalent.*
(1) *R is J-semicommutative.*
(2) *$R/J(R)$ is semicommutative.*

Proof. $(2) \Rightarrow (1)$ is obvious from Lemma 2.3.

$(1) \Rightarrow (2)$ Suppose that $R/J(R)$ is not semicommutative. Then there exist $\bar{a}$, $\bar{b} \in R/J(R)$ and $r \in R$ such that $\bar{a}\bar{b} = \bar{0}$ and $\bar{a}\bar{r}\bar{b} \neq \bar{0}$. In view of [**16**, Theorem 2.1], the principal ideal generated by $\bar{a}$ contains a system $\{\bar{e}_{11}, \bar{e}_{12}, \bar{e}_{21}, \bar{e}_{22}\}$ of 2×2 matrix units. As $\bar{e}_{11}^2 = \bar{e}_{11} \in R/J(R)$, we can find an idempotent $f \in R$ such that $\bar{e}_{11} = \bar{f}$. For any $\bar{r} \in R/J(R)$, by Lemma 2.7, $\bar{f}\bar{r} = \bar{r}\bar{f}$. Hence $\overline{e_{11}r} = \bar{f}r = \overline{frf} = \bar{r}\bar{f} = \overline{re_{11}}$. Choose $\bar{r} = \overline{e_{12}}$. Then $\overline{e_{11}e_{12}} = \overline{e_{12}} = \overline{e_{12}e_{11}} = \bar{0}$, a contradiction. Therefore $R/J(R)$ is semicommutative. $\square$

Wei and Li, in [**32**], introduced quasi-normal rings. A ring R is called *quasi-normal* if $ae = 0$ implies $eaRe = 0$ for any nilpotent a and idempotent e in R, and R is *right(or left) quasi-duo* if every maximal right(or left) ideal is an ideal (see [**14**]). Wei and Li in [**32**, Lemma 3.5] proved that for an exchange ring R, the following conditions are equivalent:

(1) R is reduced;
(2) $R/J(R)$ is abelian;
(3) $R/J(R)$ is semiabelian;
(4) $R/J(R)$ is quasi-normal;
(5) R is quasi-duo;
(6) R is left quasi-duo.

Corollary 4.3. *Let R be an exchange ring. Consider the following conditions :*
(1) *R is J-semicommutative.*
(2) *R is right (left) quasi-duo.*
(3) *$R/J(R)$ is quasi-normal.*
(4) *R is clean.*
Then $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$.

Proof. $(2) \Rightarrow (1)$ Assume that R is right quasi-duo. By [**32**, Lemma 3.5], $R/J(R)$ is reduced, therefore semicommutative and by Theorem 4.2, R is J-semicommutative.

$(1) \Rightarrow (2)$ Let $\bar{a}, \bar{e} \in R/J(R)$ with $\bar{e}^2 = \bar{e}$ and $\overline{ae} = 0$. By Theorem 4.2, $R/J(R)$ is semicommutative, and so is abelian. Then $\overline{ea}(R/J(R))\bar{e} = 0$. Hence $R/J(R)$ is quasi-normal. By [**32**, Lemma 3.5], R is quasi-duo.

$(2) \Leftrightarrow (3)$ Clear from [**32**, Lemma 3.5].

$(1) \Rightarrow (4)$ Let R be a J-semicommutative ring. By Theorem 4.2, $R/J(R)$ is semicommutative, hence it is abelian. $R/J(R)$ is an exchange ring by hypothesis. So exchange and abelian ring $R/J(R)$ is clean by [**23**, Theorem 2] and idempotents are lifted modulo $J(R)$ by [**22**, Proposition 1.5]. We prove that R is clean. For if $a \in R$, there exists an idempotent $\bar{f} \in R/J(R)$ and an invertible $\bar{u} \in R/J(R)$ such that $\bar{a} = \bar{e} + \bar{u}$. Then there exists an idempotent $e \in R$ such that $\bar{f} = \bar{e}$. Note that u is invertible in R, and $a - e = u + r$ for some $r \in J(R)$. As noted before $u + r$ is invertible in R. Hence R is a clean ring. $\square$

The following example reveals that the implication $(4) \Rightarrow (1)$ in Corollary 4.3 need not hold in general.

Example 4.4. *There are clean rings which are not J-semicommutative.*

Proof. Let R be a unit regular ring, that is, for every element $a \in R$ there exists an invertible element $u \in R$ such that $a = aua$. Every unit regular ring is clean

by [**2**], and also every unit regular ring has Jacobson radical zero (see [**18**, Exercise 6.27 (4)]). Assume also that R is J-semicommutative. Then R is abelian. Hence $a = aua$ implies ua is central idempotent and so $a = a^2u$. It follows that R is strongly regular. This leads us to a contradiction since there are unit regular rings that are not strongly regular. Therefore a unit regular ring that is not strongly regular cannot be J-semicommutative. □

Let R be a ring. Then R is called *J-Armendariz* if whenever polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, we have $a_i b_j \in J(R)$ for every i and j. In case the ideal $J(R) = 0$, a J-Armendariz ring is Armendariz.

Theorem 4.5. *Let R be a ring.*
(1) *If $R[x]$ is $J(R)[x]$-semicommutative, then R is J-semicommutative. The converse holds if R is Armendariz.*
(2) *If $R[x]$ is $J(R[x])$-semicommutative, then R is J-semicommutative.*

Proof. (1) Assume that $R[x]$ is $J(R)[x]$-semicommutative. Let $a, b \in R$ and $ab = 0$. Then $aR[x]b \subseteq J(R)[x]$. Hence $aRb \subseteq J(R)$. Conversely, suppose that R is a J-semicommutative Armendariz ring. Let $f(x) = \sum_{i=0}^{s} a_i x^i$, $g(x) = \sum_{j=0}^{t} b_j x^j \in R[x]$ with $f(x)g(x) = 0$. Since R is Armendariz, we have $a_i b_j = 0$ where $0 \leq i \leq s$, $0 \leq j \leq t$. By hypothesis $a_i R b_j \subseteq J(R)$. Hence $a_i R[x] b_j \subseteq J(R)[x]$. Thus $f(x)R[x]g(x) \subseteq J(R)[x]$.
(2) Let $a, b \in R$ and $ab = 0$. Then $aR[x]b \subseteq J(R[x])$. Since $J(R[x]) \subseteq J(R) + xR[x]$, we have $aRb \subseteq J(R)$. □

Theorem 4.6. *Every J-semicommutative exchange ring is J-Armendariz.*

Proof. Let R be a J-semicommutative exchange ring. Then $R/J(R)$ is semicommutative by Theorem 4.2. For any $a, b \in R$, $ab \in J(R)$ implies $ba \in J(R)$. For if $ab \in J(R)$, then $\overline{a}\overline{b} = \overline{0}$ in $\overline{R} = R/J(R)$. Since $\overline{R}$ is semicommutative, $\overline{a}\overline{R}\overline{b} = \overline{0}$ and $(\overline{b}\overline{R}\overline{a})^2 = \overline{0}$. Hence $(\overline{b}\overline{R}\overline{a})\overline{R}(\overline{b}\overline{R}\overline{a}) = \overline{0}$, and $(\overline{b}\overline{R}\overline{a}\overline{R})^2 = \overline{0}$. Hence $bRaR \subseteq J(R)$ and so $bRa \subseteq J(R)$. In particular, for any $a \in R$ with $a^2 = 0$ implies $aRa \subseteq J(R)$. Hence $a \in J(R)$ since $J(R)$ is a semiprime ideal.

Let $f(x) = a_0 + a_1 x + ... + a_n x^n$, $g(x) = b_0 + b_1 x + ... + b_m x^m \in R[x]$ with $f(x)g(x) = 0$. Without loss of generality, we may assume that $n = m$. Then we have

$$a_0 b_0 = 0 \qquad (1)$$
$$a_0 b_1 + a_1 b_0 = 0 \qquad (2)$$
$$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \qquad (3)$$
$$... \qquad ...$$
$$a_{n-1} b_0 + a_{n-2} b_1 + ... + a_1 b_{n-2} + a_0 b_{n-1} = 0 \qquad (n-1)$$
$$a_n b_0 + a_{n-1} b_1 + ... + a_1 b_{n-1} + a_0 b_n = 0 \qquad (n)$$
$$a_n b_1 + a_{n-1} b_2 + ... + a_2 b_{n-1} + a_1 b_n = 0 \qquad (n+1)$$

Since $a_0 b_0 = 0 \in J(R)$, $b_0 a_0 \in J(R)$. Multiplying (2) by b_0 from the left we have $b_0 a_0 b_1 + b_0 a_1 b_0 = 0$. Since $b_0 a_0 \in J(R)$, $b_0 a_1 b_0 \in J(R)$. So $b_0 a_1 \in J(R)$ and

$a_1b_0 \in J(R)$. Left multiplying (3) by b_0 yields $b_0a_0b_2 + b_0a_1b_1 + b_0a_2b_0 = 0$. Since $b_0a_1 \in J(R)$ and $b_0a_0 \in J(R)$, $b_0a_2b_0 \in J(R)$ and so $a_2b_0 \in J(R)$ and $b_0a_2 \in J(R)$. By (3) we have $a_1b_1 + a_0b_2 \in J(R)$. Left multiplying the latter by b_1 and using $b_1a_0 \in J(R)$ we have $b_1a_1b_1 \in J(R)$ and so $b_1a_1 \in J(R)$ and $a_1b_1 \in J(R)$. By (3), $a_0b_2 \in J(R)$. We prove by induction on n and assume that $a_ib_j \in J(R)$ for $0 \le i, j \le n-1$. Multiplying the equation (n) by b_0 from the left we have

$$b_0a_nb_0 + b_0a_{n-1}b_1 + \ldots + b_0a_1b_{n-1} + b_0a_0b_n = 0.$$

By assumption, $b_0a_{n-1}b_1 + \ldots + b_0a_1b_{n-1} + b_0a_0b_n \in J(R)$. So $b_0a_nb_0 \in J(R)$. Then $b_0a_n \in J(R)$ and $a_nb_0 \in J(R)$. To have $a_nb_1 \in J(R)$, we multiply (n+1) by b_1 from the left and using the induction assumption, we have $b_1a_nb_1 \in J(R)$. Hence $b_1a_n \in J(R)$ and $a_nb_1 \in J(R)$. To complete the proof, we use induction on n and assume that $b_ja_n \in J(R)$ for $0 \le j \le n-1$, we show that $b_na_n \in J(R)$. This is clear since $a_nb_n = 0$. This completes the proof. □

Theorem 4.6 is very useful for supplying examples for J-Armendariz rings.

As is well known, the transpose of an invertible matrix over a noncommutative ring may be not invertible. In [**8**, Theorem 2.3], Gupta et al. proved that every transpose of an invertible matrix over a ring R is invertible if and only if $R/J(R)$ is commutative. In the next, we will characterize the J-semicommutative exchange rings over which the transpose of every invertible square matrix is invertible in terms of commutators of invertible elements, and that give several further explicit results.

Let R be a ring. By $C(R)$ we denote the center of R, that is, $C(R) = \{a \in R \mid ar = ra \text{ for all } r \in R\}$. It is believed that Lemma 4.7 is in the literature. However, we can not reach the title that contains its proof or to refer to. Therefore for the sake of completeness we give a proof here.

Lemma 4.7. *Let R be a semiprimitive exchange ring with all idempotents central. Then the following are equivalent.*
(1) R is commutative.
(2) For any $u, v \in U(R)$, $[u, v] \in C(R)$.

Proof. $(1) \Rightarrow (2)$ is trivial.
$(2) \Rightarrow (1)$ By [**22**, Proposition 1.8(2)], R is a clean ring. Let $a, b \in R$. Then there exist idempotents $e, f \in R$ and units $u, v \in U(R)$ such that $a = e + u$ and $b = f + v$. Thus, $[a, b] = (e + u)(f + v) - (f + v)(e + u) = ef + ev + uf + uv - fe - fu - ve - vu = [u, v]$. By hypothesis, $[a, b] \in C(R)$. According to [**8**, Theorem 2.2], R is commutative. □

Recall that a ring R is said to have *stable range one* if for any $a, b \in R$ satisfying $aR + bR = R$, there exists $y \in R$ such that $a + by$ is right invertible (cf.[**29**]). Note that commutative exchange rings have stable range one (cf.[**28**, Theorem 6]).

Let R be a ring and $a, b, c \in R$. Following Herstein (cf. [**10**]), an element of the form $[a, b, c] = abc - cba$ in the ring R is called *a generalized commutator*.

From [**15**] it is known that if R has stable range one, then $J(R) = \{x \in R \mid x - u \in U(R) \text{ for any } u \in U(R)\}$. We give and prove necessary and sufficient conditions for a J-semicommutative semiprimitive exchange ring to be commutative.

Theorem 4.8. *If R is a J-semicommutative exchange ring, then the following are equivalent.*

(1) *$R/J(R)$ is commutative.*

(2) *For all n, $GL_n(R)$ is closed under transposition.*

(3) *For all $a, b, c \in U(R)$, $c + [a, b] \in U(R)$.*

(4) *For all $a, b \in U(R)$, $[a, b] \in J(R)$.*

(5) *For all $a, b, c \in U(R)$, $1 + [a, b, c] \in U(R)$.*

Proof. $(1) \Leftrightarrow (2)$ is obvious from [**8**, Theorem 2.3].

$(2) \Rightarrow (3)$ For all $a, b, c \in U(R)$, $c - ba + ba = c \in U(R)$. We infer that $\begin{pmatrix} b & c - ba \\ -1 & a \end{pmatrix} \in GL_2(R)$; hence, $\begin{pmatrix} b & -1 \\ c - ba & a \end{pmatrix} \in GL_2(R)$. This implies that $c - ba + ab \in U(R)$, and so $c + [a, b] \in U(R)$.

$(3) \Rightarrow (4)$ Let $a, b \in U(R)$. Then $c + [a, b] \in U(R)$ for all $c \in U(R)$. Being R an exchange ring, [**22**, Proposition 1.5] implies $R/J(R)$ is an exchange ring and idempotents lift modulo $J(R)$. By Lemma 2.7, $R/J(R)$ is abelian. Therefore it has stable range one by [**28**, Theorem 6]. This yields that R has stable range one. Having R stable range one, as it is mentioned before, $J(R) = \{x \in R \mid x - u \in U(R)$ for every $u \in U(R)\}$. So for any $a, b \in U(R)$, by (3) $[a, b] - (-c) \in U(R)$ for any $c \in U(R)$. Hence $[a, b] \in J(R)$.

$(4) \Rightarrow (1)$ Let $S = R/J(R)$. Then S is an exchange ring with all idempotents central by Lemma 2.7. Moreover, $J(S) = 0$, i.e., S is semiprimitive. For any $\bar{a}, \bar{b} \in U(S)$, we have $a, b \in U(R)$. By hypothesis, $1 + [a, b] \in U(R)$ as $[a, b] \in J(R)$. Set $w = 1 + [a, b]$. For any $r \in R$, in light of Lemma 4.7, it will suffice to show that $\overline{wr} = \overline{rw}$. It is clear from the fact that for any $r \in R$, $[w, r] = wr - rw = [a, b]r - r[a, b] \in J(R)$ since $[a, b] \in J(R)$. Therefore $\overline{w} \in C(S)$, and then $[\bar{a}, \bar{b}] \in C(S)$. According to Lemma 4.7, S is commutative.

$(1) \Rightarrow (5)$ For all $a, b, c \in U(R)$, $\overline{1 + [a, b, c]} = \overline{1 + abc - cba} = \bar{1} \in R/J(R)$. Therefore $1 + [a, b, c] \in U(R)$.

$(5) \Rightarrow (1)$ For all $a, b, c \in U(R)$, $c + [a, b] = c + ab - ba = c(1 + c^{-1}(ab - ba)) = c(1 + (c^{-1}a)c(c^{-1}b) - (c^{-1}b)c(c^{-1}a)) = c(1 + [c^{-1}a, c, c^{-1}b]) \in U(R)$. Thus, the result is clear. $\square$

Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$. Then according to [**33**, Example 3] R is a J-semicommutative exchange ring. For all $a, b \in U(R)$, it is easy to verify that $ab = ba$. That is, all units in R commute. Thus, $[a, b] = 0 \in J(R)$. By Theorem 4.8, $R/J(R)$ is commutative. Choose $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in U(R)$. Then $u \in R$ is not central. In this case, R is not commutative.

Corollary 4.9. *Let R be an exchange ring with all idempotents central. Then the following are equivalent.*

(1) *$R/J(R)$ is commutative.*

(2) *For all n, $GL_n(R)$ is closed under transposition.*

(3) *For all $a, b, c \in U(R)$, $c + [a, b] \in U(R)$.*

(4) *For all $a, b \in U(R)$, $[a, b] \in J(R)$.*

(5) *For all $a, b, c \in U(R)$, $1 + [a, b, c] \in U(R)$.*

Proof. It is proven by [**23**, Proposition 5] that every exchange general ring is J-semipotent. Then R is a J-semicommutative exchange ring by Corollary 2.15, and so the result follows. $\square$

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