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ON THE GRACEFULNESS OF THE GRAPH $P_{2m,2n}$

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Let $P_{a,b}$ denotes a graph obtained by identifying the end vertices of b internally disjoint paths each of length a. Kathiresan conjectured that graph $P_{a,b}$ is graceful except when a is odd and $b \equiv 2 \pmod 4$. In this paper we show that the graph $P_{a,b}$ is graceful when both a and b are even.

1. INTRODUCTION

The following notations are used frequently. The symbol Z_n denotes the residue ring of integers modulo n. Let Z be the set of all integers. Then the symbol $[a, b]$ is defined by $\{x \mid x \in Z, a \leq x \leq b\}$, $[a, b]_k$ is defined by $\{x \mid x \in Z, a \leq x \leq b, x \equiv a \pmod k\}$, and the symbol $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the greatest (least) integer y such that $y \leq x$ ($y \geq x$). When S is a set, let $f(S)$ denotes the set $\{f(x) \mid x \in S\}$ and $S + a$=$\{s + a \mid s \in S\}$.

Graphs labeling, where the vertices are assigned values subject to certain conditions, have often been motivated by practical problems. Labeled graphs serves as useful mathematical models for a broad range of applications such as coding theory, including the design of good radar type codes, synch-set codes, missile guidance codes and convolution codes with optimal autocorrelation properties. They facilitate the optimal nonstandard encoding of integers. A systematic presentation of diverse applications of graph labelings is presented in [**1**]. A function f is called a graceful labeling of a (p, q)-graph G=(V, E) if f is an injection from the vertices of G to the set $[0, q]$, such that the induced mapping $f^*(uv)$=$|f(u) - f(v)|$ is a bijection from $E(G)$ onto $[1, q]$. Rosa [**5**] introduced this concept in 1967 and also defined an α-labeling of a graph G as a graceful labeling f of G such that for each edge uv of G either $f(u) \leq c < f(v)$ or $f(v) \leq c < f(u)$ for some integer c, called characteristic of f.

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Let $P_{a,b}$ denotes a graph obtained by identifying the end vertices of b internally disjoint paths each of length a. KATHIRESAN [**4**] conjectured that graph $P_{a,b}$ is graceful except when a is odd and $b \equiv 2 \bmod 4$. He showed that the conjecture is true for the case a even and b odd. SEKER [**7**] has shown that $P_{a,b}$ is graceful when $a \neq 4r+1$, $r > 1$; b=$4m$, $m > r$. YANG et al. [**6**, **8**] showed that $P_{2r+1,2m-1}$ is graceful for any positive integer r and any positive integer m, and $P_{2r,2m}$ is graceful for any positive integer m and r=1, 2, 3, 4, 5, 6, 7, and 9. DAI et al. [**2**] showed that $P_{3,4m}$ is graceful for any positive integer m. GALLIAN [**3**] surveyed the results on graceful labeling of graphs. In this paper, we investigate the gracefulness of graph $P_{a,b}$ when both a and b are even.

2. MAIN RESULTS

For a $(k+1)$-dimensional vector A=$(a_1, a_2, ..., a_{k+1})$, the k-dimensional vector DA=$(|a_1 - a_2|, |a_2 - a_3|, ..., |a_{k-1} - a_k|, |a_k - a_{k+1}|)$ is called a difference vector of A. A vector (or matrix) is called an EPD vector (or matrix) if its elements are pairwise distinct. The symbol $< A >$ denotes the set of all elements of vector (or matrix) A. We obtain the following two lemmas.

Lemma 2.1. *Let* A=$(a_1, a_2, ..., a_{k+1})$, *where*

$$a_i = \begin{cases} s + (i-1)/2, & i \in [1, k+1]_2 \\ t - (i-2)/2, & i \in [2, k+1]_2. \end{cases}$$

Then $< DA >$=$[t-s-k+1, t-s]$ *when* $t-s \geq k$ *and* $< DA >$=$[s-t, s-t+k-1]$ *when* $s > t$.

Proof. When $t - s \geq k$, we have A_1=$\{|a_{2i} - a_{2i-1}| \mid i \in [1, \lfloor (k+1)/2 \rfloor]\}$=$\{t - s - 2i + 2 \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = [t - s - 2\lfloor (k+1)/2 \rfloor + 2, t-s]_2$,
A_2=$\{|a_{2i+1} - a_{2i}| \mid i \in [1, \lfloor k/2 \rfloor]\}$=$\{t-s-2i+1 \mid i \in [1, \lfloor k/2 \rfloor]\} = [t-s-2\lfloor k/2 \rfloor + 1, t-s-1]_2$. Therefore, $< DA >$=$A_1 \cup A_2$=$[t-s-k+1, t-s]$.

When $s > t$, we have A_1=$\{|a_{2i} - a_{2i-1}| \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = \{s-t+2i-2 \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = [s-t, s-t+2\lfloor (k+1)/2 \rfloor - 2]_2$,
A_2=$\{|a_{2i+1} - a_{2i}| \mid i \in [1, \lfloor k/2 \rfloor]\}$=$\{s-t+2i-1 \mid i \in [1, \lfloor k/2 \rfloor]\} = [s-t+1, s-t+2\lfloor k/2 \rfloor - 1]_2$. Therefore, $< DA >$=$A_1 \cup A_2$=$[s-t, s-t+k-1]$. □

Lemma 2.2. *Let* A=$(a_1, a_2, ..., a_{k+1})$, *where*

$$a_i = \begin{cases} s - (i-1)/2, & i \in [1, k+1]_2 \\ t + (i-2)/2, & i \in [2, k+1]_2. \end{cases}$$

Then $< DA >$=$[s-t-k+1, s-t]$ *when* $s-t \geq k$ *and* $< DA >$=$[t-s, t-s+k-1]$ *when* $s < t$.

Proof. When $s - t \geq k$, we have A_1=$\{|a_{2i} - a_{2i-1}| \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = \{s-t-2i+2 \mid i \in [1, \lfloor (k+1)/2 \rfloor]\} = [s-t-2\lfloor (k+1)/2 \rfloor + 2, s-t]_2$,

A_2=$\{|a_{2i+1}-a_{2i}| \mid i \in [1,\lfloor k/2 \rfloor]\}$=$\{s-t-2i+1 \mid i \in [1,\lfloor k/2 \rfloor]\} = [s-t-2\lfloor k/2 \rfloor+1, s-t-1]_2$. Therefore, $< DA >$=$A_1 \cup A_2$=$[s-t-k+1, s-t]$.

When $s < t$, we have A_1=$\{|a_{2i}-a_{2i-1}| \mid i \in [1,\lfloor (k+1)/2 \rfloor]\} = \{t-s+2i-2 \mid i \in [1,\lfloor (k+1)/2 \rfloor]\} = [t-s, t-s+2\lfloor (k+1)/2 \rfloor -2]_2$,
A_2=$\{|a_{2i+1}-a_{2i}| \mid i \in [1,\lfloor k/2 \rfloor]\}$=$\{t-s+2i-1 \mid i \in [1,\lfloor k/2 \rfloor]\} = [t-s+1, t-s+2\lfloor k/2 \rfloor -1]_2$. Therefore, $< DA >$=$A_1 \cup A_2$=$[t-s, t-s+k-1]$. □

Let A_i=$(a_{i,1}, a_{i,2}, ..., a_{i,2m-1})$, $i \in [1, 2n]$ and

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{2n} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,2m-1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,2m-1} \\ & \cdots & \cdots & \\ a_{2n,1} & a_{2n,2} & \cdots & a_{2n,2m-1} \end{pmatrix}. \tag{1}$$

The difference matrix of the matrix A, denoted by DA, is a matrix whose row vectors are $DA_1, DA_2, ..., DA_{2n}$. That is

$$DA = \begin{pmatrix} DA_1 \\ DA_2 \\ \vdots \\ DA_{2n} \end{pmatrix}. \tag{2}$$

Lemma 2.3. *In the matrix* (1), *let*

$$\begin{aligned} a_{2i-1,2j-1} &= 4m(n-i)+4m-j+1 \text{ if } i \in [1,n] \text{ and } j \in [1,m], \\ a_{2i,2j-1} &= 4m(n-i)+2m+j \text{ if } i \in [1,n] \text{ and } j \in [1,m], \\ a_{2i-1,2j} &= 4m(i-1)+j \text{ if } i \in [1,n] \text{ and } j \in [1,m-1], \\ a_{2i,2j} &= 4m(i-1)+2m-j+1 \text{ if } i \in [1,n] \text{ and } j \in [1,m-1]. \end{aligned}$$

Then the matrix A is an EPD matrix and $\{< DA >, B\}$ is a partition of set $[1, 4mn]$, *where* B=$[2m+1, 4mn-2m+1]_{4m} \bigcup [4m, 4mn]_{4m} \bigcup [2m, 4mn-2m]_{4m} \bigcup [2m-1, 4mn-2m-1]_{4m}$.

Proof. It is easy to see that the matrix A and the matrix DA have $2n(2m-1)$ and $4n(m-1)$ elements, respectively. We have
A_1=$\{a_{2i-1,2j-1}, a_{2i,2j-1} \mid i \in [1,n], j \in [1,m]\} = \bigcup_{i=1}^{n}[4m(n-i)+2m+1, 4m(n-i)+4m]$, and
A_2=$\{a_{2i-1,2j}, a_{2i,2j} \mid i \in [1,n], j \in [1,m-1]\} = \bigcup_{i=1}^{n} 4m(i-1)+([1,2m] \setminus \{m, m+1\})$. Therefore, $< A >$=$[1, 4mn] \setminus (\bigcup_{i=1}^{n}\{4m(i-1)+m, 4m(i-1)+m+1\})$, and there are $2n(2m-1)$ elements in $< A >$. This implies the matrix A is an EPD matrix.

Let s=$4mn-4mi+2m+1$, t=$4mi-2m$, k=$2m-2$. When $1 \leq i \leq \lfloor (n+1)/2 \rfloor$, there is $s > t$. By Lemma 2.1, we obtain
$< DA_{2i} >$=$[s-t, s-t+k-1]$=$[4mn-8mi+4m+1, 4mn-8mi+6m-2]$.

When $\lfloor (n+1)/2 \rfloor < i \leq n$, there is $s < t$. By Lemma 2.1, we obtain
$< DA_{2i} >$=$[t-s-k+1, t-s]$=$[8mi-4mn-6m+2, 8mi-4mn-4m-1]$.

Let s=$4mn-4mi+4m$, t=$4mi-4m+1$, k=$2m-2$. When $1 \leq i < \lceil (n+2)/2 \rceil$, there is $s > t$. By Lemma 2.2, we obtain
$< DA_{2i-1} >$=$[s-t-k+1, s-t]$=$[4mn-8mi+6m+2, 4mn-8mi+8m-1]$.

When $\lceil (n+2)/2 \rceil \leq i \leq n$, there is $s < t$. By Lemma 2.2, we obtain
$< DA_{2i-1} >$=$[t-s, t-s+k-1]$=$[8mi-4mn-8m+1, 8mi-4mn-6m-2]$.

When n is odd, we have

$$
\begin{aligned}
&\cup_{i=1}^{2n} < DA_i > \\
&=(\cup_{i=1}^{(n+1)/2}[4mn-8mi+4m+1, 4mn-8mi+6m-2]) \\
&\quad \bigcup(\bigcup_{i=(n+3)/2}^{n}[8mi-4mn-6m+2, 8mi-4mn-4m-1]) \\
&\quad \bigcup(\bigcup_{i=1}^{(n+1)/2}[4mn-8mi+6m+2, 4mn-8mi+8m-1]) \\
&\quad \bigcup(\bigcup_{i=(n+3)/2}^{n}[8mi-4mn-8m+1, 8mi-4mn-6m-2]) \\
&=(\cup_{i=0}^{(n-1)/2}(8mi+[1, 2m-2]))\bigcup(\bigcup_{i=0}^{(n-3)/2}(8mi+[6m+2, 8m-1])) \\
&\quad \bigcup(\bigcup_{i=0}^{(n-1)/2}(8mi+[2m+2, 4m-1]))\bigcup(\bigcup_{i=0}^{(n-3)/2}(8mi+[4m+1, 6m-2])) \\
&=[1, 4mn] \setminus B.
\end{aligned}
$$

When n is even, we have

$$
\begin{aligned}
&\cup_{i=1}^{2n} < DA_i > \\
&=(\cup_{i=1}^{n/2}[4mn-8mi+4m+1, 4mn-8mi+6m-2]) \\
&\quad \bigcup(\bigcup_{i=(n+2)/2}^{n}[8mi-4mn-6m+2, 8mi-4mn-4m-1]) \\
&\quad \bigcup(\bigcup_{i=1}^{n/2}[4mn-8mi+6m+2, 4mn-8mi+8m-1]) \\
&\quad \bigcup(\bigcup_{i=(n+2)/2}^{n}[8mi-4mn-8m+1, 8mi-4mn-6m-2]) \\
&=(\cup_{i=0}^{(n-2)/2}(8mi+[4m+1, 6m-2]))\bigcup(\bigcup_{i=0}^{(n-2)/2}(8mi+[2m+2, 4m-1])) \\
&\quad \bigcup(\bigcup_{i=0}^{(n-2)/2}(8mi+[6m+2, 8m-1]))\bigcup(\bigcup_{i=0}^{(n-2)/2}(8mi+[1, 2m-2])) \\
&=[1, 4mn] \setminus B.
\end{aligned}
$$

Therefore, we have $B \cup (\cup_{i=1}^{2n} < DA_i >)$=$[1, 4mn]$ and $B \bigcap(\cup_{i=1}^{2n} < DA_i >)$=$\phi$. This completes the proof. □

For an integer i satisfying $1 \leq i \leq b$, let P_a^i be ith path of $P_{a,b}$, and the successive vertices of P_a^i be $x, x_{i,1}, x_{i,2}, \cdots, x_{i,a-1}, y$, where x and y are two common end vertices of b internally disjoint paths each of length a. We use the symbol P_{a-2}^i to denote the subpath of P_a^i with consecutive vertices $x_{i,1}, x_{i,2}, \cdots, x_{i,a-1}$.

Example 2.4. Figure 1 displays a graph $P_{m,4}$ and Figure 2 shows a graceful labeling of

$P_{4,2}$.

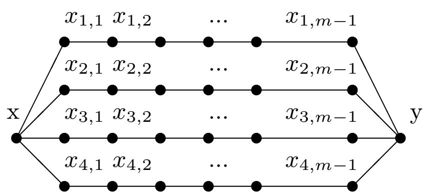

Figure 1

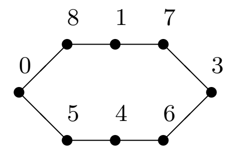

Figure 2

Theorem 2.5. *Graph $P_{2m,2n}$ is graceful for any positive integers m and n.*

Proof. It is easy to see that the graph $P_{2m,2n}$ has $4mn$ edges and $2n(2m-1)+2$ vertices. Let $a_{i,j}, i \in [1,2n], j \in [1,2m-1]$ be the same as in Lemma 2.3. We define a function g on the point set $V(P_{2m,2n})$ as follows: $g(x)$=0, $g(y)$=$m+1$, $g(x_{i,j})$=$a_{i,j}$ if $i \in [1,2n]$ and $j \in [1,2m-1]$.

From Lemma 2.3, we obtain that the g is an injection from $V(P_{2m,2n})$ to $[0,4mn]$. By the above definition, we obtain the following results.

$$\begin{aligned}
E_1 &= \{g^*(xx_{2i,1}) \mid i \in [1,n]\} = \{4m(n-i)+2m+1 \mid i \in [1,n]\} \\
&= [2m+1, 4mn-2m+1]_{4m}, \\
E_2 &= \{g^*(xx_{2i-1,1}) \mid i \in [1,n]\} = \{4m(n-i+1) \mid i \in [1,n]\} = [4m, 4mn]_{4m}, \\
E_3 &= \{g^*(yx_{2i-1,2m-1}) \mid i \in [1,n]\} = \{4m(n-i)+2m \mid i \in [1,n]\} \\
&= [2m, 4mn-2m]_{4m}, \\
E_4 &= \{g^*(yx_{2i,2m-1}) \mid i \in [1,n]\} = \{4m(n-i)+2m-1 \mid i \in [1,n]\} \\
&= [2m-1, 4mn-2m-1]_{4m}.
\end{aligned}$$

$g^*(E(P^i_{2m-2}))$=$< DA_i >$ for $i \in [1,2n]$. Let B=$E_1 \bigcup E_2 \bigcup E_3 \bigcup E_4$. Applying Lemma 2.3, we obtain $g^*(E(P_{2m,2n})) =< DA > \bigcup B = [1,4mn]$. This implies that g^* is a bijection from $E(P_{2m,2n})$ onto $[1,4mn]$. □

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