

## Strongly Zeroed Distance Spaces\*

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It is quite common to motivate the study of general topology as a generalization of the study of metric spaces, and, thus, to consider the concept of a neighborhood as being related to, or motivated by, the idea of “closeness”. This relationship, however, between “closeness” and the open sets of a general topological space is not a particularly obvious one.

In [1] the concepts of distance spaces and zeroed distance spaces were introduced and it was shown that any topological space can be derived from a distance space, in much the same manner that (metrizable) topological spaces are derived from metric spaces. In all of the constructions in [1], the distance functions  $\delta$  lack the (clearly desirable) characteristic that  $\delta(x, y) > \delta(x, x)$ . Noting this, it is natural to raise the question of whether the addition of this property yields a distinct class of spaces, or whether we could assume this additional property of distances without losing any generality.

In this paper we show that this property does, in fact, describe a distinctly different category of distance spaces (with a very different category of associated topological spaces.) We also give an example which tends to indicate that this property is a very strong one, satisfied by relatively few distance spaces.

Recall from [1] the following definitions and results:

**Definition 1.** *By a distance space we will mean a set  $Y$  together with a function  $\delta$  from  $Y \times Y$  to a partially ordered set  $P$  such that:*

*$D_1$ . for any  $x, y \in Y$ , if  $\delta(x, y) < p \in P$ , then  $\delta(x, x) < p$  and  $\delta(y, y) < p$ .*

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- $D_2$ .  $\delta(x, y) = \delta(y, x)$  for all  $x, y \in Y$
- $D_3$ . if  $\delta(x, y) < \sigma$ , then there exists some  $\mu \in P$  such that  $\delta(y, y) < \mu$  and such that  $\delta(y, z) < \mu$  implies  $\delta(x, z) < \sigma$ .
- $D_4$ . If  $\delta(x, y) < \mu$  and  $\delta(x, y) < \nu$ , then there exists some  $\sigma \in P$  such that  $\delta(x, y) < \sigma$ ,  $\sigma \leq \mu$  and  $\sigma \leq \nu$ .
- $D_5$ . For any  $x, y \in Y$ , there exists some  $p \in P$  such that  $\delta(x, y) < p$ .

The partially ordered set  $P$  is called a **distance set** for  $Y$  and the function  $\delta$  is called a **distance function**. We denote by  $N_\epsilon(x)$  the collection  $\{y \in Y : \delta(x, y) < \epsilon\}$ . A set  $N_\epsilon(x)$  is said to be a **distance neighborhood** (or a  $\delta$  neighborhood) of  $x$ . Please note that distance neighborhoods may be empty.

A distance space is a triple  $\langle X, \delta, P \rangle$  where  $P$  is a distance set for  $X$  and  $\delta : X \times X \rightarrow P$  is a distance function. If there exists an element  $O_\delta \in P$  such that  $\delta(x, x) = O_\delta$  for all  $x \in X$ , then  $\langle X, \delta, P \rangle$  is called a **zeroed distance space**.

If  $\langle X, \delta, P \rangle$  and  $\langle Y, \gamma, Q \rangle$  are distance spaces and if  $f$  is a function from  $X$  to  $Y$ , then  $f$  is said to be (distance) continuous provided that for any  $\epsilon \in Q$  and any  $x \in X$ , if  $\gamma(f(x), f(x)) < \epsilon$ , then there exists some  $\sigma \in P$  such that  $\delta(x, x) < \sigma$  and such that  $\delta(x, z) < \sigma$  implies that  $\gamma(f(x), f(z)) < \epsilon$ . The collection of all distance spaces and all distance continuous functions forms a category  $DST$ . The full subcategory whose objects are zeroed distance spaces is called  $ZDST$ .

For any distance space  $\langle X, \delta, P \rangle$ , the collection

$$\{N_\epsilon(x) : x \in X, \epsilon \in P\}$$

is a base for a topology  $\mathcal{T}_\delta$  on  $X$ .

The association

$$\langle X, \delta, P \rangle \rightarrow \langle X, \mathcal{T}_\delta \rangle$$

induces a functor  $F_{DT}$  from  $DST$  onto the category  $TOP$  of all topological spaces and all continuous functions. The image under  $F_{DT}$  of  $ZDST$  is the category of  $R_0$  spaces (which includes the category of  $T_1$  spaces.) Any two distance spaces with the same image (or homeomorphic images) under  $F_{DT}$  are isomorphic. Thus the isomorphism equivalence classes of  $DST$  form a category equivalent to  $TOP$  and the isomorphism equivalence classes of  $ZDST$  form a category equivalent to  $RZERO$ .

In showing that every  $R_0$  space (and thus any  $T_1$  space) is the image under  $F_{DT}$  of some zeroed distance space, we find the following construction:

Given any topological space  $\langle X, \mathcal{T} \rangle$ , denote by  $\mathcal{P}(X \times X)$  the collection of all subsets of the product  $X \times X$ . The collection  $\mathcal{P}(X \times X)$  can be partially ordered by requiring that  $A \leq B$  only if  $A \subseteq B$ , that  $B$  be symmetric and open in  $X \times X$  and that the diagonal  $\Delta_X = \{(x, x) : x \in X\}$  be contained in  $B$ . Denote this partially ordered set as  $\mathcal{Z}_{\mathcal{T}}$  and define a function  $\zeta_{\mathcal{T}}$  from  $X \times X$  to  $\mathcal{Z}_{\mathcal{T}}$  by  $\zeta_{\mathcal{T}}(x, y) = \Delta_X \cup \{(x, y), (y, x)\}$ . Then  $\langle X, \zeta_{\mathcal{T}}, \mathcal{Z}_{\mathcal{T}} \rangle$  is a zeroed distance space, the association  $\langle X, \mathcal{T} \rangle \rightarrow \langle X, \zeta_{\mathcal{T}}, \mathcal{Z}_{\mathcal{T}} \rangle$  induces a functor  $Z_{TD}$  from  $TOP$  into  $ZDST$  and the composition  $F_{DT} \circ Z_{TD}$  is the identity on the category of  $R_0$  spaces.

Note that with this construction, it is NEVER the case that  $\zeta_{\mathcal{T}}(x, x) \leq \zeta_{\mathcal{T}}(x, y)$  when  $x \neq y$ . We now consider the case where these related distances ARE comparable.

**Definition 2.** A zeroed distance space  $\langle X, \delta, P \rangle$  will be called a *strongly zeroed distance space* provided that  $\delta(x, x) < \delta(x, y)$  for all pairs of distinct points  $x, y \in X$ .

Metric spaces are obviously strongly zeroed distance spaces, so we are not talking about the empty category. Our first project, then, will be to demonstrate that the category of strongly zeroed distance spaces generates a category of topological spaces distinct from that generated by the category of zeroed distance spaces. Specifically, we will show that certain  $T_1$  spaces cannot be generated by strongly zeroed distance spaces.

**Theorem 1.** If  $\langle X, \delta, P \rangle$  is a strongly zeroed distance space then  $F_{DT}(\langle X, \delta, P \rangle)$  is not an uncountable cofinite space.

**Proof.** Suppose that  $F_{DT}(\langle X, \delta, P \rangle)$  is an uncountable cofinite space. Select a point  $z_0 \in X$  and an infinite sequence:

$$\{z_1, z_2, z_3, \dots\}$$

in  $X$ . Let  $\mathcal{S}$  denote the set  $\{\delta(z_0, z_i) : i \in \mathbb{N}\}$ . Then the set

$$U = \cup\{X \setminus N_{\epsilon}(z_0) : \epsilon \in \mathcal{S}\}$$

is a countable union of finite sets, and, hence, countable. Since  $U$  is countable, it cannot be equal to  $X$  and so there is some point  $w \in X \setminus U$ . Since  $X \setminus U$

is equal to  $\bigcap\{N_\epsilon(z_0) : \epsilon \in S\}$ , the neighborhood  $N_\gamma(z_0)$ , where  $\gamma$  denotes  $\delta(z_0, w)$ , cannot contain any of the points  $x_i$ . Since the complement of the open set  $N_\gamma(z_0)$  is an infinite set, this contradicts the assumption that the space  $F_{DT}(\langle X, \delta, P \rangle)$  has the cofinite topology. ■

We will now present an example to demonstrate that spaces very similar to those described above can be derived from strongly zeroed distance spaces.

**Theorem 2.** *Suppose  $\mathbb{N}$  denotes the natural numbers and that the function  $\delta$  is defined as  $\delta(m, n) = 1/(m+n)$  if  $m$  and  $n$  are distinct and  $\delta(m, m) = 0$ . Then  $\langle \mathbb{N}, \delta, \mathbb{R} \rangle$  is a strongly zeroed distance space and the associated topological space is a countable cofinite space.*

**Proof.** It is immediate that the function  $\delta$  satisfies conditions  $D_1, D_2, D_4$  and  $D_5$ . If  $\delta(m, n) < \epsilon$  then  $m + n > 1/\epsilon$ . Let  $M$  be  $\max(m, n)$ . Then  $\delta(n, p) < 1/(2M)$  implies  $p > M$  which implies that  $m + p > m + n$ , and so  $\delta(m, p) < \delta(m, n)$ . Thus  $\delta$  satisfies condition  $D_3$  and  $\langle \mathbb{N}, \delta, \mathbb{R} \rangle$  is a strongly zeroed distance space. For any  $\epsilon > 0$  and any  $m \in \mathbb{N}$ , if  $\delta(m, n) \geq \epsilon$ , then  $m + n \leq 1/\epsilon$  and so there are only a finite collection of numbers  $n$  for which  $\delta(m, n) \geq \epsilon$ . Thus, basic open sets are complements of finite sets, and so the topology generated is the cofinite topology on  $\mathbb{N}$ . ■

Please note that the example of theorem 2 has several characteristics in addition to those of being a strongly zeroed distance space. The distance set is totally ordered, among other things. Note also that our topological examples (uncountable cofinite spaces and countable cofinite spaces) are very similar. This leads us to conclude that the property of being the image of a strongly zeroed distance space would be, at best, difficult to characterize topologically.

## References

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