NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 43 (2013), 37-42

A SEQUENCE CONSIDERED BY SHAUN COOPER

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(Received 12 June, 2013)

Abstract. We find the asymptotic behaviour of a sequence considered by Shaun Cooper.

1. Introduction

Shaun Cooper recently [**1**] brought to our attention the sequence s_n of integers which satisfies the recurrence

$$(n+1)^3 s_{n+1} - (26n^3 + 39n^2 + 21n + 4)s_n - (27n^3 - 3n)s_{n-1} = 0,$$

together with $s_0 = 1$, $s_1 = 4$. It was pointed out to Cooper by Wadim Zudilin that

$$s_n = \sum_{k=\lfloor \frac{n+1}{2} \rfloor}^{n} \binom{n}{k}^2 \binom{2k}{n} \binom{n+k}{n},$$

or, if we reverse the sum,

$$s_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k}^2 \binom{2n-k}{n} \binom{2n-2k}{n}.$$

The object of this note is to determine the asymptotic behaviour of s_n. Indeed, we show that

$$s_n \sim \frac{1}{4}\left(\frac{3}{\pi n}\right)^{\frac{3}{2}} 27^n \left(1 - \frac{65}{144n} + \frac{3865}{41472n^2} + \frac{111727}{17915904n^3} - \frac{74703503}{10319560704n^4} \right.$$
$$\left. - \frac{4147191181}{1486016741376n^5} + \frac{7487947116533}{1283918464548864n^6} + \cdots \right)$$

as $n \to \infty$.

2. The Dominant Term

The first step is to find the value of k for which the term $\binom{n}{k}^2 \binom{2n-k}{n} \binom{2n-2k}{n}$ is a maximum. We do this by setting

$$\binom{n}{k}^2 \binom{2n-k}{n} \binom{2n-2k}{n} = \binom{n}{k+1}^2 \binom{2n-k-1}{n} \binom{2n-2k-2}{n}.$$

This yields

$$\frac{(2n-k)!(2n-2k)!}{k!^2(n-k)!^3(n-2k)!} = \frac{(2n-k-1)!(2n-2k-2)!}{(k+1)!^2(n-k-1)!^3(n-2k-2)!},$$

2010 *Mathematics Subject Classification* 41A60.
Key words and phrases: Integer sequence; asymptotic behaviour.

or,

$$(k+1)^2(2n-k)(2n-2k)(2n-2k-1) = (n-2k)(n-2k-1)(n-k)^3$$

If we suppose $k = \theta n$, where $\theta < \dfrac{1}{2}$ is to be determined, and divide by n^5, we find

$$\left(\theta + \frac{1}{n}\right)^2 (2-\theta)(2-2\theta)\left(2 - 2\theta - \frac{1}{n}\right) = (1-2\theta)\left(1 - 2\theta - \frac{1}{n}\right)(1-\theta)^3.$$

If we let $n \to \infty$, this becomes

$$\theta^2(2-\theta)(2-2\theta)^2 = (1-2\theta)^2(1-\theta)^3,$$

or, on simplification, remarkably,

$$\theta = \frac{1}{5}.$$

At $k \approx \dfrac{n}{5}$, the value of the term is

$$\begin{aligned}
H &= \binom{n}{\frac{n}{5}}^2 \binom{\frac{9n}{5}}{n} \binom{\frac{8n}{5}}{n} \\
&= \frac{\left(\frac{9n}{5}\right)! \left(\frac{8n}{5}\right)!}{\left(\frac{n}{5}\right)!^2 \left(\frac{4n}{5}\right)!^3 \left(\frac{3n}{5}\right)!} \\
&\approx \frac{\sqrt{2\pi\left(\frac{9n}{5}\right)}\left(\frac{9n}{5e}\right)^{\frac{9n}{5}} \sqrt{2\pi\left(\frac{8n}{5}\right)}\left(\frac{8n}{5e}\right)^{\frac{8n}{5}}}{\left(\sqrt{2\pi\left(\frac{n}{5}\right)}\left(\frac{n}{5e}\right)^{\frac{n}{5}}\right)^2 \sqrt{2\pi\left(\frac{3n}{5}\right)}\left(\frac{3n}{5e}\right)^{\frac{3n}{5}} \left(\sqrt{2\pi\left(\frac{4n}{5}\right)}\left(\frac{4n}{5e}\right)^{\frac{4n}{5}}\right)^3} \\
&= \left(\frac{1}{\sqrt{2\pi n}}\right)^4 \frac{\sqrt{\frac{9}{5}}\sqrt{\frac{8}{5}}}{\left(\sqrt{\frac{1}{5}}\right)^2 \sqrt{\frac{3}{5}} \left(\sqrt{\frac{4}{5}}\right)^3} \left(\frac{9^{\frac{9}{5}} \times 8^{\frac{8}{5}}}{1^{\frac{2}{5}} \times 3^{\frac{3}{5}} \times 4^{\frac{12}{5}}}\right)^n \\
&= \frac{25\sqrt{6}}{16\pi^2 n^2} 27^n
\end{aligned}$$

after considerable simplification. (See fig. 1.)

At points near $\dfrac{n}{5}$, the terms of the sum are given by

$$\begin{aligned}
&\binom{n}{\frac{n}{5}+k}^2 \binom{\frac{9n}{5}-k}{n} \binom{\frac{8n}{5}-2k}{n} \\
&\quad = H \cdot \binom{n}{\frac{n}{5}+k}^2 \binom{\frac{9n}{5}-k}{n} \binom{\frac{8n}{5}-2k}{n} \Big/ \binom{n}{\frac{n}{5}}^2 \binom{\frac{9n}{5}}{n} \binom{\frac{8n}{5}}{n}
\end{aligned}$$

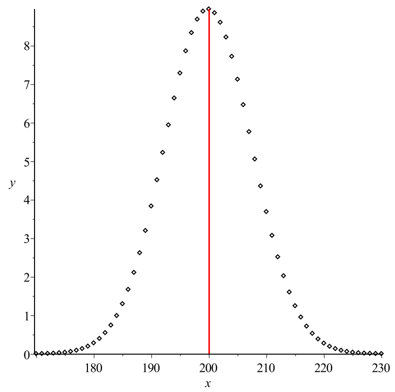

FIGURE 1. The case $n = 1000$, showing the points $\left(k, \binom{n}{k}^2\binom{2n-k}{n}\binom{2n-2k}{n}\right)$ for $170 \le k \le 230$, together with the vertical $x = \dfrac{n}{5}$, $0 \le y \le \dfrac{25\sqrt{6}}{16\pi^2 n^2} 27^n$.

$$
\begin{aligned}
&= H \cdot \left(\frac{\left(\frac{n}{5}\right)!\left(\frac{4n}{5}\right)!}{\left(\frac{n}{5}+k\right)!\left(\frac{4n}{5}-k\right)!} \right)^2 \frac{\left(\frac{9n}{5}-k\right)!\left(\frac{4n}{5}\right)!}{\left(\frac{9n}{5}\right)!\left(\frac{4n}{5}-k\right)!} \frac{\left(\frac{8n}{5}-2k\right)!\left(\frac{3n}{5}\right)!}{\left(\frac{8n}{5}\right)!\left(\frac{3n}{5}-2k\right)!} \\
&= H \cdot \left(\frac{\left(\frac{4n}{5}\right)\cdots\left(\frac{4n}{5}-k+1\right)}{\left(\frac{n}{5}+1\right)\cdots\left(\frac{n}{5}+k\right)} \right)^2 \frac{\left(\frac{4n}{5}\right)\cdots\left(\frac{4n}{5}-k+1\right)}{\left(\frac{9n}{5}\right)\cdots\left(\frac{9n}{5}-k+1\right)} \frac{\left(\frac{3n}{5}\right)\cdots\left(\frac{3n}{5}-2k+1\right)}{\left(\frac{8n}{5}\right)\cdots\left(\frac{8n}{5}-2k+1\right)} \\
&= H \cdot \left(\frac{\left(1-\frac{5}{4n}\right)\cdots\left(1-\frac{5(k-1)}{4n}\right)}{\left(1+\frac{5}{n}\right)\cdots\left(1+\frac{5k}{n}\right)} \right)^2 \frac{\left(1-\frac{5}{4n}\right)\cdots\left(1-\frac{5(k-1)}{4n}\right)}{\left(1-\frac{5}{9n}\right)\cdots\left(1-\frac{5(k-1)}{9n}\right)} \\
&\qquad \cdot \frac{\left(1-\frac{5}{3n}\right)\cdots\left(1-\frac{5(2k-1)}{3n}\right)}{\left(1-\frac{5}{8n}\right)\cdots\left(1-\frac{5(2k-1)}{8n}\right)} \\
&\approx H \exp\left\{ \left(-\frac{5}{4n}\frac{k^2}{2} - \frac{5}{n}\frac{k^2}{2} \right) \times 2 - \frac{5}{4n}\frac{k^2}{2} + \frac{5}{9n}\frac{k^2}{2} - \frac{5}{3n} \times 2k^2 + \frac{5}{8n} \times 2k^2 \right\} \\
&= H \exp\left\{ -\frac{k^2}{72n}\left(90 + 360 + 45 - 20 + 240 - 90\right) \right\} \\
&= H \exp\left\{ -\frac{625k^2}{72n} \right\}.
\end{aligned}
$$

Thus, the terms are essentially distributed normally, with $\sigma^2 = \dfrac{36n}{625}$, and the sum is given by

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k}^2 \binom{2n-k}{n} \binom{2n-2k}{n} \approx H \int_{-\infty}^{\infty} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} dx = H \cdot \sigma\sqrt{2\pi}$$

$$\approx \frac{25\sqrt{6}}{16\pi^2 n^2} 27^n \cdot \frac{6\sqrt{n}}{25}\sqrt{2\pi}$$

$$= \frac{3\sqrt{3}}{4\pi\sqrt{\pi n}\sqrt{n}} 27^n$$

$$= \frac{1}{4}\left(\frac{3}{\pi n}\right)^{\frac{3}{2}} 27^n.$$

as claimed. (See fig. 2.)

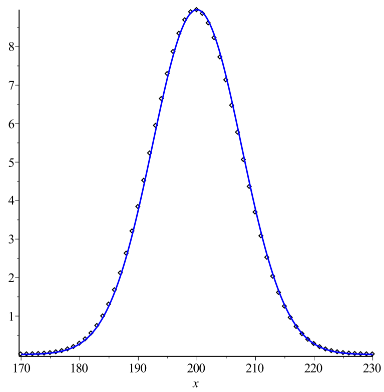

FIGURE 2. The casae $n = 1000$, showing the points $(k, \binom{n}{k}^2 \binom{2n-k}{n} \binom{2n-2k}{n})$ for $170 \le k \le 230$, together with the normal $y = \dfrac{25\sqrt{6}}{16\pi^2 n^2} \exp\left\{-\dfrac{625}{72n}\left(x - \dfrac{n}{5}\right)^2\right\}$.

3. The Correction Term

Now s_n satisfies the recurrence

$$(n+1)^3 s_{n+1} - (26n^3 + 39n^2 + 21n + 4)s_n - (27n^3 - 3n)s_{n-1} = 0,$$

or,

$$\left(1 + \frac{1}{n}\right)^3 s_{n+1} - \left(26 + \frac{39}{n} + \frac{21}{n^2} + \frac{4}{n^3}\right) s_n - \left(27 - \frac{3}{n^2}\right) s_{n-1} = 0.$$

We now suppose that

$$s_n = Cn^{-\frac{3}{2}} 27^n \left(1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \frac{a_3}{n^3} + \cdots\right),$$

where $C = \frac{1}{4}\left(\frac{3}{\pi}\right)^{\frac{3}{2}}$, and substitute into the recurrence, to obtain

$$\begin{aligned}
&\left(1+\frac{1}{n}\right)^3 C(n+1)^{-\frac{3}{2}}27^{n+1}\left(1+\frac{a_1}{n+1}+\frac{a_2}{(n+1)^2}+\frac{a_3}{(n+1)^3}+\cdots\right)\\
&\quad-\left(26+\frac{39}{n}+\frac{21}{n^2}+\frac{4}{n^3}\right)Cn^{-\frac{3}{2}}27^n\left(1+\frac{a_1}{n}+\frac{a_2}{n^2}+\frac{a_3}{n^3}+\ \cdots\right)\\
&\qquad-\left(27-\frac{3}{n^2}\right)C(n-1)^{-\frac{3}{2}}27^{n-1}\left(1+\frac{a_1}{n-1}+\frac{a_2}{(n-1)^2}+\frac{a_3}{(n-1)^3}+\ \cdots\right)\\
&\qquad=0.
\end{aligned}$$

If we now divide by $C27^n$, and multiply by $n^{\frac{3}{2}}$, we find

$$\begin{aligned}
&\left(1+\frac{1}{n}\right)^{\frac{3}{2}} 27\left(1+\frac{a_1}{n+1}+\frac{a_2}{(n+1)^2}+\frac{a_3}{(n+1)^3}+\ \cdots\right)\\
&\quad-\left(26+\frac{39}{n}+\frac{21}{n^3}+\frac{4}{n^4}\right)\left(1+\frac{a_1}{n}+\frac{a_2}{n^2}+\frac{a_3}{n^3}+\ \cdots\right)\\
&\qquad-\left(1-\frac{1}{n}\right)^{-\frac{3}{2}}\left(27-\frac{3}{n^2}\right)\frac{1}{27}\left(1+\frac{a_1}{n-1}+\frac{a_2}{(n-1)^2}+\frac{a_3}{(n-1)^3}+\ \cdots\right)\\
&\qquad=0.
\end{aligned}$$

If we set $\frac{1}{n}=u, \quad \frac{1}{n+1}=\frac{u}{1+u}, \quad \frac{1}{n-1}=\frac{u}{1-u}$, and expand in powers of u, we find

$$\begin{aligned}
&27\left(1+\frac{3u}{2}+\frac{3u^2}{8}-\frac{u^3}{16}+\ \cdots\right)(1+a_1u+(a_2-a_1)u^2+(a_3-2a_2+a_1)u^3+\ \cdots)\\
&\quad-(26+39u+21u^2+4u^3)(1+a_1u+a_2u^2+a_3u^3+\ \cdots)\\
&\qquad-\left(1+\frac{3u}{2}+\frac{15u^2}{8}+\frac{35u^3}{16}+\ \cdots\right)\left(1-\frac{u^2}{9}\right)\\
&\qquad\cdot(1+a_1u+(a_2+a_1)u^2+(a_3+2a_2+a_1)u^3+\ \cdots)\\
&\qquad=0.
\end{aligned}$$

We now set the coefficients of the powers of u equal to zero, and solve for $a_1,\ a_2,\ a_3$ and so on. The constant term and the coefficient of u are automatically zero, because we had the base 27 correct and the factor $n^{-\frac{3}{2}}$ correct. The coefficient of u^2 is

$$-28a_1-\frac{455}{36}=0,$$

so

$$a_1=-\frac{65}{144}.$$

If we continue in the same way, we find

$$a_2=\frac{3865}{41472},\quad a_3=\frac{111727}{17915904},\quad a_4=-\frac{74703503}{10319560704},\quad a_5=-\frac{4147191181}{1486016741376}$$

and so on.

This completes the proof.

References

[1] S. Cooper, *Sporadic sequences, modular forms and new series for* $1/\pi$, Ramanujan Journal, to appear.

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