

ON A SET-THEORETIC EQUALITY FOR MULTIFUNCTIONS

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Abstract. Let $F : X \rightarrow Y$ be a multifunction between two topological spaces X and Y . This paper presents conditions which are sufficient for the inclusion $F^+(\bigcup_{\alpha \in \Lambda} V_\alpha) \subseteq \bigcup_{\alpha \in \Lambda} F^+(V_\alpha)$, where $\{V_\alpha : \alpha \in \Lambda\}$ is a collection of subsets of Y .

1. Introduction

By a multifunction $F : (X, \sigma) \rightarrow (Y, \tau)$, we mean a point-to-set correspondence from (X, σ) into (Y, τ) , and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For each $B \subseteq Y$, $F^+(B) = \{x \in X : F(x) \subseteq B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. $F^+(B)$ is called the *upper pre-image* of B , and $F^-(B)$ is called the *lower pre-image* of B . For each $A \subseteq X$, $F(A) = \bigcup_{x \in A} F(x)$. Then F is said to be a *surjection* if $F(X) = Y$, or equivalently, if for each $y \in Y$, there exists an $x \in X$ such that $y \in F(x)$. Moreover, $F : (X, \sigma) \rightarrow (Y, \tau)$ is called *upper semicontinuous*, abbreviated as *u.s.c.* (resp. *lower semicontinuous*, abbreviated as *l.s.c.*) at a point $x \in X$ if for each open set V in Y with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$), there exists an open set U containing x such that $F(U) \subseteq V$ (resp. $F(z) \cap V \neq \emptyset$ for every $z \in U$), or equivalently, $F^+(V)$ (resp. $F^-(V)$) is a neighborhood of x in (X, σ) for every open set V in Y with $F(x) \subseteq V$ (resp. $F(x) \cap V \neq \emptyset$).

In general, for any index set Λ , we have the set equality $F^-(\bigcup_{\alpha \in \Lambda} V_\alpha) = \bigcup_{\alpha \in \Lambda} F^-(V_\alpha)$. The authors pointed out in [2] that the “lower” multifunction continuities behave well from the point of view of change of topology. This set equality between lower pre-images is a main reason for this behaviour. On the other hand, for any index set Λ , we have the set inclusion

$$\bigcup_{\alpha \in \Lambda} F^+(V_\alpha) \subseteq F^+(\bigcup_{\alpha \in \Lambda} V_\alpha) \quad (1)$$

for upper pre-images. In general, this is a proper set inclusion, and we do not have a set equality, see the simple Example 1.1 below.

Example 1.1. Let $X = Y = \{1, 2, 3, 4\}$ and define the multifunction $F : X \rightarrow Y$ by $F(1) = \{1, 3\}$, $F(2) = \{1, 2\}$, $F(3) = \{3\}$, $F(4) = \{4\}$. Furthermore, consider the following collection of subsets of Y : $V_{\alpha_1} = \{1, 4\}$, $V_{\alpha_2} = \{2\}$, $V_{\alpha_3} = \{1, 2\}$ and $V_{\alpha_4} = \{3, 4\}$. Then we have that $F^+(V_{\alpha_1}) = \{4\}$, $F^+(V_{\alpha_2}) = \emptyset$, $F^+(V_{\alpha_3}) = \{2\}$ and $F^+(V_{\alpha_4}) = \{3, 4\}$. Hence $\bigcup_{i=1}^4 F^+(V_{\alpha_i}) = \{2, 3, 4\}$. But we note that $\bigcup_{i=1}^4 V_{\alpha_i} = \{1, 2, 3, 4\}$, and $F^+(\bigcup_{i=1}^4 V_{\alpha_i}) = \{1, 2, 3, 4\}$. Thus, $F^+(\bigcup_{i=1}^4 V_{\alpha_i})$ contains $\bigcup_{i=1}^4 F^+(V_{\alpha_i})$ as a proper subset.

The previously mentioned fact is a major reason for the (mis)behaviour of “upper” multifunction continuities with respect to change of topology noted in [2]. The question arises as to whether there are conditions on the family of subsets $\{V_\alpha : \alpha \in \Lambda\}$ which are sufficient to make (1) a set equality. In the next section, we give two propositions which provide such conditions.

2. Main Results

In his study of relative topological properties, Arhangel'skii [1] has introduced some concepts that we can use in this context.

Definition 2.1 ([1]). Let Y be a subspace of a topological space X , and let $\mathcal{P}$ be a topological property. If there is a subspace Z of X such that Y is contained in Z , and the space Z has the property $\mathcal{P}$ in itself, we shall say that Y has the *property $\mathcal{P}$ in X from outside*.

In particular, if $F : X \rightarrow Y$ is a multifunction, and $x \in X$, then $F(x)$ is called *open in X from the outside with V* , if V is an open set in Y with $F(x) \subseteq V$.

Proposition 2.2. *Let $F : (X, \sigma) \rightarrow (Y, \tau)$ be a multifunction, and $\{V_\alpha : \alpha \in \Lambda\}$ a collection of open subsets of Y . If for each point $x \in F^+(\bigcup_{\alpha \in \Lambda} V_\alpha)$, $F(x)$ is open in X from outside with some V_β ($\beta \in \Lambda$), then $\bigcup_{\alpha \in \Lambda} F^+(V_\alpha) = F^+(\bigcup_{\alpha \in \Lambda} V_\alpha)$.*

Proof. We have $\bigcup_{\alpha \in \Lambda} F^+(V_\alpha) \subseteq F^+(\bigcup_{\alpha \in \Lambda} V_\alpha)$ in general. To show the opposite inclusion, we take $x \in F^+(\bigcup_{\alpha \in \Lambda} V_\alpha)$. There is a $\beta \in \Lambda$ such that $F(x)$ is open in Y from the outside with V_β . Hence, $F(x) \subseteq V_\beta$, and so $x \in F^+(V_\beta)$. Therefore, $x \in \bigcup_{\alpha \in \Lambda} F^+(V_\alpha)$. Thus, we have the inclusion $F^+(\bigcup_{\alpha \in \Lambda} V_\alpha) \subseteq \bigcup_{\alpha \in \Lambda} F^+(V_\alpha)$, as desired. $\square$

Definition 2.3 ([1]). Let Y be a subspace of a topological space X . A subset A of X is said to be *concentrated on Y* if A is contained in the closure of $A \cap Y$ in X .

Proposition 2.4. *Let $F : (X, \sigma) \rightarrow (Y, \tau)$ be a multifunction, and let $\{V_\alpha : \alpha \in \Lambda\}$ be a collection of closed subsets of Y . If for each point $x \in F^+(\bigcup_{\alpha \in \Lambda} V_\alpha)$, $F(x)$ is concentrated on some V_β ($\beta \in \Lambda$) then $\bigcup_{\alpha \in \Lambda} F^+(V_\alpha) = F^+(\bigcup_{\alpha \in \Lambda} V_\alpha)$.*

Proof. We have $\bigcup_{\alpha \in \Lambda} F^+(V_\alpha) \subseteq F^+(\bigcup_{\alpha \in \Lambda} V_\alpha)$ in general. To show the opposite inclusion, take $x \in F^+(\bigcup_{\alpha \in \Lambda} V_\alpha)$. Since there is a $\beta \in \Lambda$ such that $F(x)$ is concentrated on V_β , we have that $F(x) \subseteq \text{cl}(F(x) \cap V_\beta) \subseteq \text{cl}V_\beta = V_\beta$, as V_β is closed. Hence $x \in F^+(V_\beta)$, and therefore $x \in \bigcup_{\alpha \in \Lambda} F^+(V_\alpha)$. Thus, we have the inclusion $F^+(\bigcup_{\alpha \in \Lambda} V_\alpha) \subseteq \bigcup_{\alpha \in \Lambda} F^+(V_\alpha)$, as desired. $\square$

Question. What other (weaker) conditions on a collection $\{V_\alpha : \alpha \in \Lambda\}$ of subsets of Y are sufficient for the inclusion $F^+(\bigcup_{\alpha \in \Lambda} V_\alpha) \subseteq \bigcup_{\alpha \in \Lambda} F^+(V_\alpha)$ to hold, where $F : (X, \sigma) \rightarrow (Y, \tau)$ is a multifunction?

References

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