NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 42 (2012), 115-120

LINDELOF WITH RESPECT TO AN IDEAL

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(Received 1 March, 2011)

Abstract. We define Lindelof with respect to an ideal and investigate basic properties of the concept, its relation to known concepts,and its preservation by functions, subspaces, pre-images, and products.

1. Introduction and Preliminaries.

An *ideal* on a set X is a nonempty collection of subsets of X closed under the operations of subset ("heredity") and finite union ("finite additivity"). If in addition the ideal is closed under the operation of countable unions, it is called a $\sigma - ideal$. We denote a topological space (X, τ) with an ideal $\mathbf{I}$ defined on X as $(X, \tau, \mathbf{I})$ and call $(X, \tau, \mathbf{I})$ an *ideal topological space*. Given a space (X, τ) and $A \subseteq X$, we denote by $Int_\tau(A)$ and $Cl_\tau(A)$ the interior and closure of A, respectively, with respect to τ. When no ambiguity is present we write simply $Int(A)$ and $Cl(A)$. We abbreviate "if and only if" with "iff" and "neighborhood" with "nbd." The conclusion or omission of a proof is indicated by the symbol $\|$. No separation properties are assumed unless explicitly stated. We denote the natural numbers by N.

2. Basic Results.

We begin with the following definition.

Definition. A space $(X, \tau, \mathbf{I})$ is said to be $\mathbf{I}$-*Lindelof* or *Lindelof with respect to* $\mathbf{I}$, if every open cover $\mathbf{U}$ of X has a countable subcollection $\mathbf{V}$ such that $X - \cup\mathbf{V} \in \mathbf{I}$.

Obviously, a space is Lindelof iff it is $\{\emptyset\}$-Lindelof. Frolik **[4]** defines a space to be *weakly Lindelof* if every open cover $\mathbf{U}$ of the space has a countable subcollection $\mathbf{V}$ such that $X = Cl(\cup\mathbf{V})$. We now show that weakly Lindelof spaces are a special case of Lindelof with respect to an ideal. If (X, τ) is a space, we denote the ideal of nowhere dense sets by $N(\tau)$ and the σ-ideal of meager (first category) subsets by $M(\tau)$. An ideal $\mathbf{I}$ on (X, τ) is said to be $\tau - codense$ if $\mathbf{I} \cap \tau = \{\emptyset\}$.

Theorem 2.1. *Let (X, τ) be a space.*
(1) *(X, τ) is weakly Lindelof iff (X, τ) is $N(\tau)$-Lindelof.*

2010 *Mathematics Subject Classification* 54D20, 54D30.
Key words and phrases: Lindelof, weakly Lindelof, ideal, codense ideal, continuous function, open map, closed map, perfect map.
The author gratefully acknowledges the financial support of Oklahoma Christian University as the 2010 recipient of the Rowe Distinguished Scholar Award.

(2) (X, τ) *is weakly Lindelof iff* (X, τ) *is Lindelof with respect to some* τ*-codense ideal.*

(3) *If* (X, τ) *is a Baire space, then* (X, τ) *is weakly Lindelof iff* (X, τ) *is* $M(\tau)$*-Lindelof.*

Proof. (1) *Necessity.* Assume (X, τ) is weakly Lindelof and let $\mathbf{U}$ be an open cover of X. Then by assumption there exists a countable subcollection $\mathbf{V}$ of $\mathbf{U}$ such that $X = Cl(\cup \mathbf{V})$. $X - \cup \mathbf{V}$ is then a closed set with empty interior; i.e., $X - \cup \mathbf{V} \in N(\tau)$.

Sufficiency. Assume (X, τ) is $N(\tau)$-Lindelof and let $\mathbf{U}$ be an open cover of X. By assumption, there exists a countable subcollection $\mathbf{V}$ of $\mathbf{U}$ such that $X - \cup \mathbf{V} \in N(\tau)$. This implies that $X - \cup \mathbf{V}$ has empty interior and hence $X = Cl(\cup \mathbf{V})$.

(2) *Necessity.* $N(\tau)$ is τ-codense.

Sufficiency. Assume that $\mathbf{I}$ is a τ-codense ideal on X and that (X, τ) is $\mathbf{I}$-Lindelof. Let $\mathbf{U}$ be an open cover of X. Then, by assumption, there exists a countable subcollection $\mathbf{V}$ of $\mathbf{U}$ such that $X - \cup \mathbf{V} \in \mathbf{I}$. Since $\mathbf{I}$ is τ-codense, $X - \cup \mathbf{V}$ has empty interior and hence $X = Cl(\cup \mathbf{V})$.

(3) (X, τ) is a Baire space iff $M(\tau)$ is τ-codense, The result now follows from (2). $\|$

Willard [**11**] defined a space to be *almost Lindelof* if for every open cover $\mathbf{U}$ of X there exists a countable subcollection $\mathbf{V}$ of $\mathbf{U}$ such that $X = \cup \{Cl(V) : V \in \mathbf{V}\}$.

Theorem 2.2. *If a space* (X, τ) *is almost Lindelof, then it is* $M(\tau)$*-Lindelof.*

Proof. Let $\mathbf{U}$ be an open cover of X and let N denote the natural numbers. By assumption, there exists a countable subcollection $\mathbf{V} = \{V_i : i \in N\}$ of $\mathbf{U}$ such that $X = \cup \{Cl(V_i) : i \in N\}$. Now observe that

$$X - \cup \mathbf{V} \subseteq \cup \{Cl(V_i) - V_i : i \in N\}$$

and each set $Cl(V_i) - V_i \in N(\tau)$. Thus $\cup \{Cl(V_i) - V_i : i \in \mathbf{N}\} \in M(\tau)$. $\|$

The next example is of an $M(\tau)$-Lindelof space which is not almost Lindelof. Observe that if $\mathbf{I}$ and $\mathbf{J}$ are ideals on a space (X, τ) with $\mathbf{I} \subseteq \mathbf{J}$ and (X, τ) is $\mathbf{I}$-Lindelof, then (X, τ) is $\mathbf{J}$-Lindelof.

Example 1. *Let* $\Re$ *denote the real numbers and let* $X = \Re \times [0, +\infty)$ *denote the upper half plane together with* $L = \{(x, 0) : x \in \Re\}$. *Equip* X *with the Niemytzki tangent disk topology, denoted by* τ. *Points in* $\Re \times (0, +\infty)$ *have the usual Euclidean nbds while points on the line* L *have nbds of the form* $D \cup \{(x, 0)\}$ *where* D *is the interior of a disk (circle) lying in* X *tangent to* L *at the point* $(x, 0)$. *Let* Q *denote the rational numbers and let* Q^+ *denote the positive rational numbers. Since* $Q \times Q^+$ *is countable and dense in* X, X *is separable and hence weakly Lindelof* [**8**], *hence* $N(\tau)$*-Lindelof, hence* $M(\tau)$*-Lindelof (since* $N(\tau) \subseteq M(\tau)$ *). However, this space is not almost Lindelof. To see this, for each* $(x, 0) \in L$, *let* D_x *be a basic nbd of* $(x, 0)$; *i.e., each* D_x *is the interior of a disk lying in* X *tangent to* L *at the point* $(x, 0)$ *together with the point* $(x, 0)$. *Let* $\mathbf{U}$ *be an open cover of* $\Re \times (0, +\infty)$ *consisting of open sets whose closures do not intersect* L. *For each* $(x, 0) \in L$, *let* D_x *be a basic nbd of* $(x, 0)$ *as described above. Now* $\mathbf{V} = \mathbf{U} \cup \{D_x : x \in \Re\}$ *is an open cover of* X *and each point* $(x, 0)$ *is an element of the closure of exactly one* *set in* $\mathbf{V}$*; i.e.,* $(x, 0) \in Cl(D_x)$*. Hence there can be no countable subcollection of* $\mathbf{V}$ *whose closures cover* X*, and therefore* (X, τ) *is not almost Lindelof.*

3. Subspaces.

Let $(X, \tau, \mathbf{I})$ be an ideal topological space and let $A \subseteq X$, $A \neq \emptyset$. We denote by $\mathbf{I}_A$ the collection $\{I \cap A : I \in \mathbf{I}\}$. We denote by τ_A the subspace topology on A. Then we have the following theorem.

Theorem 3.1. *Let* $(X, \tau, \mathbf{I})$ *be an* $\mathbf{I}$*-Lindelof space and let* A *be a closed subset of* X*. Then* $(A, \tau_A, \mathbf{I}_A)$ *is* $\mathbf{I}_A$*-Lindelof.*

Proof. Let $\{U \cap A : U \in \mathbf{U} \subseteq \tau\}$ be a τ_A-open cover of A. Then $\mathbf{U} \cup \{X - A\}$ is an open cover of X and hence there is a countable subcollection $\mathbf{V} = \{U_i : i \in N\} \cup \{X - A\}$ such that $X - \cup\mathbf{V} = I \in \mathbf{I}$. Now we must have $A \subseteq \cup\{U_i : i \in N\} \cup I$ and $A = \cup\{U_i \cap A : i \in N\} \cup \{I \cap A\}$. Hence

$$A - \cup\{U_i \cap A : i \in N\} \subseteq I \cap A \in \mathbf{I}_A. \; \|$$

The well known result that a closed subspace of a Lindelof space is Lindelof is a special case by taking $\mathbf{I} = \{\emptyset\}$. Closed subspaces of weakly Lindelof spaces are not necessarily weakly Lindelof (as the subspace L of Example 1 shows), however we obtain the following known result as a corollary.

Corollary 3.2. **[9]** *If* $A \subseteq (X, \tau)$ *is a clopen (closed and open) subset of a weakly Lindelof space* (X, τ)*, then* (A, τ_A) *is weakly Lindelof.*

Proof. By Theorem 3.1, (A, τ_A) is $N(\tau)_A$-Lindelof. Since A is open $N(\tau)_A = \{I \cap A : I \in N(\tau)\}$ is τ_A-codense; i.e., if $Int_{\tau_A}(I \cap A) \neq \emptyset$, then $Int_{\tau}(I \cap A) \neq \emptyset$. Now apply Theorem 2.1, (2). $\|$

4. Preservation by Functions and Products.

Lindelof with respect to an ideal is preserved by continuous functions in the following manner. First note that if $f : X \to Y$ and $\mathbf{I}$ is an ideal on X, then $f(\mathbf{I}) = \{f(I) : I \in \mathbf{I}\}$ is an ideal on Y. Also if $\mathbf{J}$ is an ideal on Y, then $f^{-1}(\mathbf{J}) = \{f^{-1}(J) : J \in \mathbf{J}\}$ is an ideal on X.

Theorem 4.1. *Let* $f : (X, \tau, \mathbf{I}) \to (Y, \sigma)$ *be a continuous surjection. If* (X, τ) *is* $\mathbf{I}$*-Lindelof, then* (Y, σ) *is* $f(\mathbf{I})$*-Lindelof.*

Proof. Let $\mathbf{V}$ be an open cover of Y. Then $f^{-1}(\mathbf{V}) = \{f^{-1}(V) : V \in \mathbf{V}\}$ is an open cover of X and hence there exists a countable subcollection $\mathbf{U} = \{U_i = f^{-1}(V_i) : i \in N\}$ of $f^{-1}(\mathbf{V})$ and an I $\in \mathbf{I}$ such that $X = \cup\, \mathbf{U} \cup \{\text{ I }\}$. Now we have

$$Y = f(X) = \cup\{V_i : i \in N\} \cup \{f(I)\}. \; \|$$

By taking $\mathbf{I} = \{\emptyset\}$ in the above theorem, we get the well known result that Lindelof is preserved by continuous surjections. We also have the following results concerning pre-images.

Theorem 4.2. *Let* $f : X \to (Y, \sigma, \mathbf{J})$ *be a surjection onto a* $\mathbf{J}$*-Lindelof space. If* $f^{-1}(\sigma)$ *is the weak topology on* X *induced by* f *and* σ*, then* $(X, f^{-1}(\sigma))$ *is* $f^{-1}(\mathbf{J})$*-Lindelof.*

Proof. Let $\mathbf{U} = \{f^{-1}(V) : V \in \mathbf{V}\}$ be an open cover of X. Then $\mathbf{V}$ is an open cover of Y and hence there exists a countable subcollection $\{V_i : i \in N\}$ of $\mathbf{V}$ and $J \in \mathbf{J}$ such that

$$Y = \cup\{V_i : i \in N\} \cup \{J\}.$$

Now we have

$$X = \cup\{f^{-1}(V_i) : i \in N\} \cup \{f^{-1}(J)\}. \parallel$$

The following Lemma is used in the proof of the Corollary that follows.

Lemma 4.3. *If $f : X \to (Y, \sigma, \mathbf{J})$ is a surjection and $\mathbf{J}$ is σ-codense, then $f^{-1}(\mathbf{J})$ is $f^{-1}(\sigma)$-codense.*

Proof. (*Contrapositive*). Assume $f : X \to (Y, \sigma, \mathbf{J})$ is a surjection and $f^{-1}(\mathbf{J})$ is not $f^{-1}(\sigma)$-codense, then there exists a J $\in \mathbf{J}$ such that $f^{-1}(J) \in f^{-1}(\sigma) - \{\emptyset\}$, say $f^{-1}(J) = f^{-1}(V)$ where $V \in \sigma - \{\emptyset\}$. Then $J = V \in \sigma - \{\emptyset\}$ and $\mathbf{J}$ is not σ-codense. $\parallel$

Corollary III.4. Let $f : X \to (Y, \sigma)$ be a surjection and let $f^{-1}(\sigma)$ denote the weak topology on X induced by f and σ.

(1) If (Y, σ) is Lindelof, then $(X, f^{-1}(\sigma))$ is Lindelof.

(2) If (Y, σ) is weakly Lindelof, then $(X, f^{-1}(\sigma))$ is weakly Lindelof.

Proof. (1) Let $\mathbf{J} = \{\emptyset\}$ and apply Theorem 4.2.

(2) If (Y, σ) is weakly Lindelof then there exists a σ-codense ideal $\mathbf{J}$ such that (Y, σ) is $\mathbf{J}$-Lindelof. By Lemma 4.3, $f^{-1}(\mathbf{J})$ is $f^{-1}(\sigma)$-codense and apply Theorem 4.2. $\parallel$

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be *open* if f preserves open subsets and is said to be *perfect* if f is continuous, closed (preserves closed subsets), and has compact fibers ($f^{-1}(y)$ is compact for every $y \in Y$). The following Theorem and Lemma are useful for our consideration of products.

Theorem 4.4. *Let $f : (X, \tau) \to (Y, \sigma, \mathbf{J})$ be a perfect open surjection. If (Y, σ) is $\mathbf{J}$-Lindelof, then (X, τ) is $f^{-1}(\mathbf{J})$-Lindelof.*

Proof. Let $\mathbf{U}$ be an open cover of X. For each $y \in Y$ there exists a finite subcollection $\{U_{y_i} : i = 1, 2, 3, ..., n_y\}$ of $\mathbf{U}$ which covers $f^{-1}(y)$. Let $U_y = \cup\{U_{y_i} : i = 1, 2, 3, ..., n_y\}$. Each U_y is an open set in (X, τ) and $f^{-1}(y) \subseteq U_y$. Now each set $f(X - U_y)$ is closed in Y and $y \notin f(X - U_y)$, hence $V_y = Y - f(X - U_y)$ is a nbd of y. Note that $f^{-1}(V_y) \subseteq U_y$. The collection $\{V_y : y \in Y\}$ is an open cover of Y, hence there exists a countable subcollection $\{V_{y_i} : i \in N\}$ and $J \in \mathbf{J}$ such that

$$Y = \cup\{V_{y_i} : i \in N\} \cup \{J\}.$$

We claim that $X = \cup\{U_{y_i} : i \in N\} \cup \{f^{-1}(J)\}$. Indeed, let $x \in X$; then $y = f(x) \in V_{y_i} \cup J$ for some y_i. Then

$$x \in f^{-1}(y) \subseteq f^{-1}(V_{y_i}) \cup f^{-1}(J) \subseteq U_{y_i} \cup f^{-1}(J).$$

Since x was arbitrary, the claim is established and the theorem is proved. $\parallel$

The following Lemma is used to prove the corollary which follows it.

Lemma 4.5. *If $f : (X, \tau) \to (Y, \sigma, \mathbf{J})$ is an open surjection and $\mathbf{J}$ is σ-codense, then $f^{-1}(\mathbf{J})$ is τ-codense.*

Proof. (*contrapositive*). Suppose that $f^{-1}(\mathbf{J})$ is not τ-codense. Then there exists $J \in \mathbf{J}$ such that $f^{-1}(J) \in \tau - \{\emptyset\}$. Since f is open and surjective, $f(f^{-1}(J)) = J \in \sigma - \{\emptyset\}$ and $\mathbf{J}$ is not σ-codense. ‖

The following known results are now obtained as corollaries.

Corollary 4.6. *Let $f : (X, \tau) \to (Y, \sigma)$ be a perfect open surjection.*
(1) *If (Y, σ) is Lindelof, then (X, τ) is Lindelof.*
(2) **[9]** *If (Y, σ) is weakly Lindelof, then (X, τ) is weakly Lindelof.*

Proof. (1) Let $\mathbf{J} = \{\emptyset\}$ and apply Theorem 4.4.

(2) Assume that (Y, σ) is weakly Lindelof. Then (Y, σ) is $\mathbf{J}$-Lindelof with respect to some σ-codense ideal $\mathbf{J}$. Then by Lemma 4.5, $f^{-1}(\mathbf{J})$ is τ-codense and apply Theorem 4.4. ‖

Utilizing Theorem 4.4, we obtain the following theorem concerning products and some known results as corollaries.

Theorem 4.7. *If $(X, \tau, \mathbf{I})$ is $\mathbf{I}$-Lindelof and (Y, σ) is compact, then$(X \times Y, \tau \times \sigma)$ is $p^{-1}(\mathbf{I})$-Lindelof where $\tau \times \sigma$ is the usual product topology and $p : X \times Y \to X$ is the projection map onto X defined by $p(x, y) = x$.*

Proof. The projection $p : (X \times Y, \tau \times \sigma) \to (X, \tau)$ is a perfect open surjection. Now apply Theorem 4.4. ‖

The following results are known.

Corollary 4.8. (1) *If (X, τ) is Lindelof and (Y, σ) is compact, then $(X \times Y, \tau \times \sigma)$ is Lindelof.*
(2) **[9]** *If (X, τ) is weakly Lindelof and (Y, σ) is compact, then $(X \times Y, \tau \times \sigma)$ is weakly Lindelof.*

Proof. (1) Let $\mathbf{I} = \{\emptyset\}$ and apply Theorem 4.7.

(2) If (X, τ) is weakly Lindelof, then by Theorem 2.1, (2), there exists some τ-codense ideal $\mathbf{I}$ such that (X, τ) is $\mathbf{I}$-Lindelof. By Lemma 4.5, $p^{-1}(\mathbf{I})$ is $\tau \times \sigma$-codense where $p : X \times Y \to X$ is the projection map onto X. Now apply Theorem 4.7. ‖

References

[1] M. Bonanzinga, M. V. Matveev and C. M. Pareek, *Some remarks on generalizations of countably compact spaces and Lindelof spaces*, Rend. Circ. Mat. Palermo, **51** (1) (2002), 163-174.

[2] F. Cammaroto and G. Santoro, *Some counterexamples and properties on generalizations of Lindelof spaces*, Internat. J. Math. Math. Sci., **19** (4) (1996), 737-746.

[3] A. J. Fawakhreh and A. Kilicman, *Mappings and decompositions of continuity on almost Lindelof spaces*, Internat. J. Math. Math. Sci., **2006** (2006), 1-7.

[4] Z. Frolik, *Generalizations of compact and Lindelof spaces*, Czech. Math. J., **9** (84) (1959), 172-217.

[5] A. Hajnal and I. Juhasz, *On the products of weakly Lindelof spaces*, Proc. Amer. Math. Soc., **48** (2) (1975), 454-456.

[6] H. Z. Hdeib and M. S. Sarsak, *On almost Lindelof spaces*, Q & A in Gen. Top, **19** (2001), 17-25.

[7] D. Jankovic and T. R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, **97** (1990), 293-310.

[8] C. M. Pareek, *Some generalizations of Lindelof spaces and hereditarily Lindelof spaces*, Q & A in Gen. Top, **2** (1984), 131-142.

[9] Y. K. Song and Y. Y. Zhang, *Some remarks on almost Lindelof spaces and weakly Lindelof spaces*, Math. Vesnik, **62** (1) (2010), 77-83.

[10] M. Ulmer, *Products of weakly-$\aleph$-compact spaces*, Trans. Amer. Math. Soc., **170** (1972), 279-284.

[11] S. Willard and U. N. B. Dissanayake, *The almost Lindelof degree*, Canad. Math. Bull., **27** (4) (1984), 452-455.

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