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VOTING "AGAINST" IN REGULAR AND NEARLY REGULAR GRAPHS

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Let $G = (V, E)$ be a graph. A function $f : V(G) \to \{-1, 1\}$ is called *negative* if $\sum_{v \in N[v]} f(v) \leq 1$ for every $v \in V(G)$. A negative function f of a graph G is maximal if there exists no negative function g such that $g \neq f$ and $g(v) \geq f(v)$ for every $v \in V$. The minimum of the values of $\sum_{v \in V} f(v)$, taken over all maximal negative functions f, is called the *lower against number* and is denoted by $\beta^*_{\mathbb{N}}(G)$. In this paper, we present lower bounds on this number for regular graphs and nearly regular graphs, and we characterize the graphs attaining those bounds.

1. INTRODUCTION

Let $G = (V, E)$ be a graph with vertex set V and edge set E. For a vertex $v \in V$, we denote by $N(v)$ the *open neighbourhood* of v, and by $N[v] = N(v) \cup \{v\}$ its *closed neighbourhood*. For a subset $S \subseteq V$ and a vertex $v \in V$, we let $d_S(v)$ denote the number of vertices in S that are joined to v. In particular, $d_V(v) = \deg(v)$, the *degree* of v in G. If $f : V \to \mathbb{R}$ is a real-valued function, then we denote $f(S) = \sum_{v \in S} f(v)$ and $f[v] = f(N[v])$. A graph G is called *r-regular* if $\deg(v) = r$ for every $v \in V$, and *nearly r-regular* if $\deg(v) \in \{r - 1, r\}$ for every $v \in V$. For two disjoint subsets S and T of vertices, $e(S, T)$ denotes the number of edges with one endvertex in S and the other in T.

A function $f : V \to \{-1, 1\}$ is called *negative* if $\sum_{u \in N[v]} f(u) \leq 1$ for every $v \in V$. The maximum of the values of $f(V)$, taken over all negative functions f, is called the *against number* $\beta_{\mathbb{N}}(G)$.

The motivation for studying this parameter may be varied from a modelling perspective. For instance, in a social network (a network of people), if we assign

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the values -1 or $+1$ to the vertices, then we may model such a network in which global decisions have to be made (e.g. positive or negative responses). In certain circumstances, a negative decision should be made if there are not significantly more people voting for than those voting against. We assume that each individual has one vote, and each has an initial opinion. We assign $+1$ to individuals (vertices) who have a positive opinion and -1 to individuals who have a negative opinion. A voter votes '*for*' if there are at least two or more vertices in its closed neighborhood with positive opinions than with negative opinions, otherwise the voter votes '*against*'. We seek an assignment of opinions that guarantee a unanimous decision; that is, for which every individual votes 'against'. We say such an assignment of opinions *a uniformly negative assignment*. Among all uniformly negative assignments of opinions, we are interested in the minimum number of individuals (vertices) who have a negative opinion. The against number is the maximum possible sum of all opinions ($+1$ for a positive opinion and -1 for a negative opinion) in a uniformly negative assignment of opinions. The against number corresponds to the minimum number of individuals who can have negative opinions and in doing so force every individual to vote against.

The against number has been studied in [**5**]. If the "open neighbourhood" is used in the above definition, then $\beta_{\mathbb{N}}(G)$ is the *negative decision number* investigated in [**4**].

We shall mention that a similarly defined but well-studied graph parameter is the signed domination number. A function $f : V \to \{-1, 1\}$ is called a *signed dominating function* if $\sum\limits_{u \in N[v]} f(u) \geq 1$ for every $v \in V$. The minimum of the values of $f(V)$, taken over all signed dominating functions f, is called the *signed domination number* $\gamma_s(G)$. The signed domination and its variants have been extensively studied. For a comprehensive introduction to theoretical and applied facets of domination in graphs the reader is directed to the books [**2**, **3**].

In this paper, we continue the investigation of the against number in a graph. We say that a negative function f of a graph G *maximal* if there exists no negative function g such that $g \neq f$ and $g(v) \geq f(v)$ for every $v \in V$. The *lower against number*, written $\beta^*_{\mathbb{N}}(G)$, is the minimum of the values of $f(V)$, taken over all maximal negative functions f of G.

All graphs considered in this paper are simple, finite and undirected. For a general reference on graph theory, the reader is referred to [**1**, **6**]. Note that $\beta^*_{\mathbb{N}}(G \cup H) = \beta^*_{\mathbb{N}}(G) + \beta^*_{\mathbb{N}}(H)$ for disjoint union of two graphs G and H. Hence, we may assume that all graphs in this paper are connected.

2. MAIN RESULTS

2.1. Regular graphs

In this subsection, we present a lower bound on $\beta^*_{\mathbb{N}}$ for an r-regular graph, and we characterize the graphs attaining this bound. We first define two families $\mathcal{F}_0(r)$ and $\mathcal{F}_1(r)$ of graphs as follows. Fix an integer $r \geq 3$.

(i) When r is even, p and $a_{(r+2)/2} \geq r$ are two integers such that $rp = \frac{r+2}{2} a_{(r+2)/2}$.

We construct a set of graphs G by adding some edges among vertices of three vertex-disjoint subgraphs A_0, $A_{(r+2)/2}$ and P, where A_0 is an $(r-1)$-regular graph of order $a_0 = \frac{r-2}{2} a_{(r+2)/2}$, $A_{(r+2)/2}$ is an independent set of order $a_{(r+2)/2}$, and P is an independent set of order p, such that

1. there are no edges between vertices of A_0 and P;
2. each vertex in $A_{(r+2)/2}$ is joined to exactly $(r-2)/2$ vertices in A_0, and each vertex of A_0 has exactly one neighbor in $A_{(r+2)/2}$;
3. each vertex in P is joined to exactly r vertices in $A_{(r+2)/2}$, and each vertex in $A_{(r+2)/2}$ has exactly $(r+2)/2$ neighbours in P.

Therefore, the order n of these graphs G is

$$n = a_0 + a_{(r+2)/2} + p = \left(\frac{r-2}{2} + 1 + \frac{r+2}{2r} \right) a_{(r+2)/2},$$

and we define $\mathcal{F}_0(r)$ to be a set of such graphs G for all possible choices of p and $a_{(r+2)/2}$ such that $a_{(r+2)/2} \geq r$ and $rp = \frac{r+2}{2} a_{(r+2)/2}$.

(ii) When r is odd, p and $a_{(r+1)/2} \geq r$ are two integers such that $rp = \frac{r+1}{2} a_{(r+1)/2}$.

We construct a set of graphs G by adding some edges among vertices of three vertex-disjoint subgraphs A_0, $A_{(r+1)/2}$ and P, where A_0 is an $(r-1)$-regular graph of order $a_0 = \frac{r-1}{2} a_{(r+1)/2}$, $A_{(r+1)/2}$ is an independent set of order $a_{(r+1)/2}$, and P is an independent set of order p, such that

1. there are no edges between vertices of A_0 and P;
2. each vertex in $A_{(r+1)/2}$ is joined to exactly $(r-1)/2$ vertices in A_0, and each vertex of A_0 has exactly one neighbor in $A_{(r+1)/2}$;
3. each vertex in P is joined to exactly r vertices in $A_{(r+1)/2}$, and each vertex in $A_{(r+1)/2}$ has exactly $(r+1)/2$ neighbours in P.

We define $\mathcal{F}_1(r)$ to be a set of such graphs G for all possible choices of p and $a_{(r+1)/2}$ such that $a_{(r+1)/2} \geq r$ and $rp = \frac{r+1}{2} a_{(r+1)/2}$.

Theorem 1. *Let G be an r-regular graph of order n. Then*

$$\beta^*_{\mathbb{N}}(G) \geq \begin{cases} \dfrac{(-r^2+r+2)n}{r^2+r+2}, & r \equiv 0 \pmod 2; \\ \dfrac{(1-r)n}{1+r}, & r \equiv 1 \pmod 2. \end{cases}$$

This bound is best possible.

Our key lemma is the following result, which is useful in proving both Theorem 1 and Theorem 5.

Lemma 2. *A negative function f of a graph G is maximal if and only if for every $v \in V$ with $f(v) = -1$, there exists at least one vertex $u \in N[v]$ such that $f[u] = 0$ or* 1.

Proof. Necessity. Let f be a maximal negative function of G. To the contrary, suppose that there exists a vertex $v \in V$ with $f(v) = -1$ such that for every $u \in N[v]$ with $f[u] \leq -1$. Define $g : V \to \{-1, 1\}$ as follows:

$$g(w) = \begin{cases} f(w), & w \neq v; \\ f(w) + 2, & w = v. \end{cases}$$

It is not hard to see that g is a negative function of G such that $g \neq f$ and $g(v) \geq f(v)$ for every $v \in V$. This contradicts that f is a maximal negative function of G.

Sufficiency. We will prove it by contradiction. Suppose that for every $v \in V$ with $f(v) = -1$, there exists at least one vertex $u \in N[v]$ such that $f[u] = 0$ or 1, but f is not maximal. Hence, there is a negative function g of G such that $g \neq f$ and $g(v) \geq f(v)$ for every $v \in V$. Take a vertex $u \in V$ such that $g(u) > f(u)$. Since both $g(u)$ and $f(u)$ equal to either 1 or -1, we obtain that $g(u) = +1$ and $f(u) = -1$. As g is a negative function of G, $g[w] \leq 1$ for every $w \in N[u]$.

Hence, we have that $f[w] \leq g[w] - 2 \leq -1$ for every $w \in N[u]$, which is a contradiction. □

Remark 3. For any integer $r \geq 3$, the following statements are true.

1. For any graph $G \in \mathcal{F}_0(r)$ of order n, $\beta^*_{\mathbb{N}}(G) = \dfrac{(-r^2 + r + 2)n}{r^2 + r + 2}$.
2. For any graph $G \in \mathcal{F}_1(r)$ of order n, $\beta^*_{\mathbb{N}}(G) = \dfrac{(1 - r)n}{1 + r}$.

Proof. We only prove item (1), as the proof of item (2) is similar. If $G \in \mathcal{F}_0(r)$, then by the above construction, we can produce a maximal negative function f of G by assigning -1 to every vertex in $A_0 \cup A_{(r+2)/2}$ and $+1$ to every vertex in P. Note that $n = \left(\dfrac{r-2}{2} + 1 + \dfrac{r+2}{2r}\right) a_{(r+2)/2}$. Clearly,

$$\begin{aligned} f(V) &= -a_0 - a_{(r+2)/2} + p \\ &= \left(-\frac{r-2}{2} - 1 + \frac{r+2}{2r}\right) a_{(r+2)/2} \\ &= \frac{-\dfrac{r-2}{2} - 1 + \dfrac{r+2}{2r}}{\dfrac{r-2}{2} + 1 + \dfrac{r+2}{2r}} n \\ &= \frac{(-r^2 + r + 2)n}{r^2 + r + 2}. \end{aligned}$$

So, $\beta^*_{\mathbb{N}}(G) \le \dfrac{(-r^2+r+2)n}{r^2+r+2}$, and thus $\beta^*_{\mathbb{N}}(G) = \dfrac{(-r^2+r+2)n}{r^2+r+2}$ by Theorem 1. $\square$

Proof of Theorem 1. It is not hard to see that Theorem 1 holds when $r = 1$ and $r = 2$. In what follows we may assume that $r \ge 3$.

Let f be a maximal negative function such that $\beta^*_{\mathbb{N}}(G) = f(V)$. Let P and Q denote the sets of those vertices of G which are assigned (under f) the values 1 and -1, respectively, and we let $p = |P|$ and $q = |Q|$. Therefore, $\beta^*_{\mathbb{N}}(G) = f(V) = |P| - |Q| = p - q = 2p - n$.

We now claim that $P \ne \emptyset$ for otherwise $f[v] = -r - 1 \le -4$ for every $v \in V$, and so f is not maximal by Lemma 2, a contradiction.

Obviously, $f[v] = -r+1+2d_P(v)$ for every $v \in P$ and $f[v] = -r-1+2d_P(v)$ for every $v \in Q$. As f is a maximal negative function, we have that for every $v \in Q$

$$2d_P(v) \le r + 2.$$

For each $0 \le i \le \lfloor (r+2)/2 \rfloor$, we set $A_i = \{v \in Q | d_P(v) = i\}$ and $a_i = |A_i|$. Clearly, the sets $A_0, \ldots, A_{\lfloor (r+2)/2 \rfloor}$ form a partition of Q. Moreover, $f[v_i] = -r - 1 + 2i$ for every $v_i \in A_i$ ($i = 0, 1, \ldots, \lfloor (r+2)/2 \rfloor$). In particular, $f[v] = 0$ or 1 for every $v \in A_{\lfloor (r+2)/2 \rfloor}$. It is clear that

$$n = p + \sum_{i=0}^{\lfloor (r+2)/2 \rfloor} a_i. \tag{2.1}$$

As every $v \in P$ has degree r, we have $e(P, Q) \le rp$. Hence,

$$a_1 + 2a_2 + \ldots + \lfloor (r+2)/2 \rfloor a_{\lfloor (r+2)/2 \rfloor} \le rp. \tag{2.2}$$

Case 1. $r \equiv 0 \pmod 2$.

Obviously, $\lfloor (r+2)/2 \rfloor = (r+2)/2$. As f is a maximal negative function, by Lemma 2, every $v \in Q$ has at least one $u \in N[v]$ such that $f[u] = 0$ or 1. Denote by A'_1 ($\subseteq A_1$) the set of vertices having no neighbours u in $A_{(r+2)/2}$ such that $f[u] = 0$ or 1. Set $|A'_1| = a'_1$, and so $0 \le a'_1 \le a_1$. Hence, each $v \in A'_1$ shall have at least one neighbour $u \in P$ such that $f[u] = 0$ or 1. Consequently, (2.2) will be corrected to be the following.

$$a_1 + 2a_2 + \ldots + \lfloor (r+2)/2 \rfloor a_{\lfloor (r+2)/2 \rfloor} \le rp - a'_1. \tag{2.3}$$

As every vertex $v \in A_0$ is joined to no vertex in P, v must have at least one neighbour in $A_{(r+2)/2}$. Hence,

$$e(A_0 \cup A_1, A_{(r+2)/2}) \ge a_0 + a_1 - a'_1.$$

On the other hand, every $v \in A_{(r+2)/2}$ is joined to exactly $(r+2)/2$ vertices in P, and so is joined at most $(r-2)/2$ vertices in $A_0 \cup A_1$. Hence,

$$e(A_0 \cup A_1, A_{(r+2)/2}) \le \frac{r-2}{2} a_{(r+2)/2}.$$

Thus, combining with the last two inequalities, we obtain that

$$a_0 + a_1 - a_1' \le \frac{r-2}{2} a_{(r+2)/2}. \tag{2.4}$$

By (2.1), (2.3) and (2.4), noting that $0 \le a_1' \le a_1$, we have that

$$\begin{aligned} n &= p + \sum_{i=0}^{(r+2)/2} a_i \\ &\le p + a_1' + a_2 + \ldots + a_{r/2} + \frac{r}{2} a_{(r+2)/2} \\ &\le p + \frac{r}{r+2}\left(a_1 + 2a_2 + \ldots + \frac{r+2}{2} a_{(r+2)/2}\right) + a_1' - \frac{ra_1}{r+2} \\ &\le p + \frac{r}{r+2}(rp - a_1') + a_1' - \frac{ra_1}{r+2} \\ &\le p + \frac{r}{r+2} rp. \end{aligned}$$

Solving the above inequality for p, we obtain that

$$p \ge \frac{(r+2)n}{r^2+r+2}.$$

Hence,

$$\beta_{\mathbb{N}}^*(G) = 2p - n \ge \frac{(-r^2+r+2)n}{r^2+r+2}.$$

The equality $\beta_{\mathbb{N}}^*(G) = \dfrac{(-r^2+r+2)n}{r^2+r+2}$ holds if and only if all the inequalities occurring in the previous proof are equalities. That means that $A_1' = A_1 = A_2 = \ldots = A_{r/2} = \emptyset$, the set P is independent (since $e(A_{(r+2)/2}, P) = rp$) and each of its vertices has exactly r neighbors in $A_{(r+2)/2}$. This implies that $a_{(r+2)/2} \ge r$. Moreover $A_{(r+2)/2}$ is an independent set (since $e(A_0, A_{(r+2)/2}) = \frac{r-2}{2} a_{(r+2)/2}$) of order $a_{(r+2)/2}$, where $a_{(r+2)/2} \ge r$ is an integer such that the equation $rp = \frac{r+2}{2} a_{(r+2)/2}$ has an integer solution p. Each vertex in A_0 has exactly one neighbour in $A_{(r+2)/2}$, and so A_0 induces an $(r-1)$-regular graph of order $a_0 = \frac{r-2}{2} a_{(r+2)/2}$ by the equality in (2.4). Consequently, each vertex in $A_{(r+2)/2}$ is joined to exactly $(r-2)/2$ vertices in A_0. Thus, $G \in \mathcal{F}_0(r)$. It follows by Remark 3 that, $\beta_{\mathbb{N}}^*(G) = \dfrac{(-r^2+r+2)n}{r^2+r+2}$ if and only if $G \in \mathcal{F}_0(r)$.

Case 2. $r \equiv 1 \pmod 2$.

In this case, $\lfloor (r+2)/2 \rfloor = (r+1)/2$. Assuming that exactly a_1' ($0 \le a_1' \le a_1$) vertices in A_1 have no neighbours u in $A_{(r+2)/2}$ such that $f[u] = 0$ or 1, and using a similar argument to that in Case 1, we can claim that

$$a_1 + 2a_2 + \ldots + \lfloor (r+2)/2 \rfloor a_{\lfloor (r+2)/2 \rfloor} \le rp - a_1', \tag{2.5}$$

and

$$e(A_0 \cup A_1, A_{(r+1)/2}) \geq a_0 + a_1 - a_1'.$$

On the other hand, every $v \in A_{(r+1)/2}$ is joined to $(r+1)/2$ vertices in P, and so is joined at most $(r-1)/2$ vertices in $A_0 \cup A_1$. Hence,

$$e(A_0 \cup A_1, A_{(r+1)/2}) \leq \frac{r-1}{2} a_{(r+1)/2}.$$

Therefore, we obtain that

$$a_0 + a_1 - a_1' \leq \frac{r-1}{2} a_{(r+1)/2}. \tag{2.6}$$

By (2.1), (2.5) and (2.6), noting that $0 \leq a_1' \leq a_1$, we have that

$$\begin{aligned} n &= p + \sum_{i=0}^{(r+1)/2} a_i \\ &\leq p + a_1' + a_2 + \ldots + a_{(r-1)/2} + \frac{r+1}{2} a_{(r+1)/2} \\ &\leq p + \left(a_1 + 2a_2 + \ldots + \frac{r-1}{2} a_{(r-1)/2} + \frac{r+1}{2} a_{(r+1)/2} \right) \\ &\leq p + (rp - a_1') \\ &\leq p + rp. \end{aligned}$$

Solving the above inequality for p, we obtain that

$$p \geq \frac{n}{r+1}.$$

Hence,

$$\beta^*_{\mathbb{N}}(G) = 2p - n \geq \frac{(1-r)n}{1+r}.$$

By a similar argument to that in Case 1, one can show that $\beta^*_{\mathbb{N}}(G) = \dfrac{(1-r)n}{1+r}$ if and only if $G \in \mathcal{F}_1(r)$. □

2.2. Nearly regular graphs

In this subsection, we present a lower bound on $\beta^*_{\mathbb{N}}$ for a nearly r-regular graph, and we characterize the graphs attaining this bound.

For this purpose, we define another two families $\mathcal{F}_0'(r)$ and $\mathcal{F}_1'(r)$ of graphs as follows. Let $r \geq 3$ be a fixed integer.

(i) When r is even, $p \geq r/2$ is an integer, and $b_{r/2} = 2p$.

We obtain a set of graphs G by adding some edges among vertices of three vertex-disjoint subgraphs B_0, $B_{r/2}$ and P, where B_0 is a nearly $(r-1)$-regular graph of order $b_0 = \dfrac{r-2}{2} b_{r/2}$, $B_{r/2}$ is an independent set of order $b_{r/2}$, and P is an independent set of order p, such that

1. there are no edges between vertices of B_0 and P;

2. each vertex in $B_{r/2}$ is joined to exactly $(r-2)/2$ vertices in B_0, and each vertex of B_0 has exactly one neighbor in $B_{r/2}$;

3. each vertex in P is joined to exactly r vertices in $B_{r/2}$, and each vertex in $B_{r/2}$ has exactly $r/2$ neighbours in P.

We define $\mathcal{F}_0'(r)$ to be a set of such graphs G for all possible choices of p and $b_{r/2} \geq r$ such that $b_{r/2} = 2p$.

(ii) When r is odd, p and $a_{(r+1)/2} \geq r$ are two integers such that $\frac{r+1}{2}a_{(r+1)/2} = rp$.

We obtain a set of graphs G by adding some edges among vertices of three vertex-disjoint subgraphs A_0, $A_{(r+1)/2}$ and P, where A_0 is a nearly $(r-1)$-regular graph of order $a_0 = \frac{r-1}{2}a_{(r+1)/2}$, $A_{(r+1)/2}$ is an independent set of order $a_{(r+1)/2}$, and P is an independent set of order p, such that

1. there are no edges between vertices of A_0 and P;

2. each vertex in $A_{(r+1)/2}$ is joined to exactly $(r-1)/2$ vertices in A_0, and each vertex of A_0 has exactly one neighbor in $A_{(r+1)/2}$;

3. each vertex in P is joined to exactly r vertices in $A_{(r+1)/2}$, and each vertex in $A_{(r+1)/2}$ has exactly $(r+1)/2$ neighbours in P.

We define $\mathcal{F}_1'(r)$ to be a set of such graphs G for all possible choices of p and $a_{(r+1)/2} \geq r$ such that $\frac{r+1}{2}a_{(r+1)/2} = rp$.

Remark 4. Let $r \geq 3$ be an integer. For any graph $G \in \mathcal{F}_0'(r) \cup \mathcal{F}_1'(r)$ of order n,

$$\beta_{\mathbb{N}}^*(G) = \frac{(1-r)n}{1+r}.$$

Proof. The proof is similar to that of Remark 3, and so is omitted. □

Theorem 5. *For any nearly r-regular graph G of order n,*

$$\beta_{\mathbb{N}}^*(G) \geq \frac{(1-r)n}{1+r}.$$

For $r \geq 3$ even, the equality holds if and only if $G \in \mathcal{F}_0'(r)$. For $r \geq 3$ odd, the equality holds if and only if $G \in \mathcal{F}_1'(r)$.

Proof. It is not difficult to see that the assertion holds for $r = 1$ and $r = 2$. Hence, we may assume that $r \geq 3$ in the following.

Let f be a maximal negative function such that $\beta_{\mathbb{N}}^*(G) = f(V)$. Let P and Q denote the sets of those vertices of G which are assigned (under f) the values 1 and -1, respectively.

We shall claim that $P \neq \emptyset$ for otherwise $f[v] \leq -r \leq -3$ for every $v \in V$, and so f is not maximal, a contradiction.

Let $M = \{v \in V : \deg(v) = r-1\}$ and $N = \{v \in V : \deg(v) = r\}$. Denote by $|P \cap M| = p_1$ and $|P \cap N| = p_2$. Therefore, $\beta^*_{\mathbb{N}}(G) = |P| - |Q| = 2(p_1 + p_2) - n$.

Obviously, $f[v] = 2d_P(v) - r$ for every $v \in Q \cap M$, and $f[v] = 2d_P(v) - r - 1$ for every $v \in Q \cap N$. As f is a maximal negative function, we have that for every $v \in Q \cap N$

$$2d_P(v) \leq r + 2,$$

and for every $v \in Q \cap M$

$$2d_P(v) \leq r + 1.$$

For each $0 \leq i \leq \lfloor (r+2)/2 \rfloor$ and $0 \leq j \leq \lfloor (r+1)/2 \rfloor$, we set $A_i = \{v \in Q \cap N | d_P(v) = i\}$ and $B_j = \{v \in Q \cap M | d_P(v) = j\}$, and let $a_i = |A_i|$ and $b_j = |B_j|$. Obviously, the sets $A_0, \ldots, A_{\lfloor (r+2)/2 \rfloor}, B_0, \ldots, B_{\lfloor (r+1)/2 \rfloor}$ form a partition of Q. Moreover, $f[v_i] = -r - 1 + 2i$ for every $v_i \in A_i$ $(i = 0, 1, \ldots, \lfloor (r+2)/2 \rfloor)$, and $f[u_j] = -r + 2j$ for every $u_j \in B_j$ $(j = 0, \ldots, \lfloor (r+1)/2 \rfloor)$. In particular, $f[v] = 0$ or 1 for every $v \in A_{\lfloor (r+2)/2 \rfloor} \cup B_{\lfloor (r+1)/2 \rfloor}$. It is clear that

$$n = p_1 + p_2 + \sum_{i=0}^{\lfloor (r+2)/2 \rfloor} a_i + \sum_{j=0}^{\lfloor (r+1)/2 \rfloor} b_j. \tag{2.7}$$

Since every $v \in P \cap M$ has degree $r - 1$, and every $v \in P \cap N$ has degree r, we have $e(P, Q) \leq (r-1)p_1 + rp_2$. Hence,

$$\sum_{i=1}^{\lfloor (r+2)/2 \rfloor} ia_i + \sum_{j=1}^{\lfloor (r+1)/2 \rfloor} jb_j \leq (r-1)p_1 + rp_2. \tag{2.8}$$

Case 1. $r \equiv 1 \pmod 2$.

Note that $\lfloor (r+2)/2 \rfloor = \lfloor (r+1)/2 \rfloor = (r+1)/2$. As f is a maximal negative function, by Lemma 2, every $v \in Q$ has at least one $u \in N[v]$ such that $f[u] = 0$ or 1. Note that every vertex $v \in A_0 \cup B_0$ is joined to no vertex in P. Hence, for every $v \in A_0 \cup B_0$, there exists at least one neighbour in $A_{(r+1)/2} \cup B_{(r+1)/2}$. Thus,

$$e(A_0 \cup B_0, A_{(r+1)/2} \cup B_{(r+1)/2}) \geq a_0 + b_0.$$

On the other hand, every $v \in A_{(r+1)/2}$ is joined to $(r+1)/2$ vertices in P, and so is joined at most $(r-1)/2$ vertices in $A_0 \cup B_0$; every $v \in B_{(r+1)/2}$ is joined to $(r+1)/2$ vertices in P, and so is joined at most $(r-3)/2$ vertices in $A_0 \cup B_0$. Hence,

$$e(A_0 \cup B_0, A_{(r+1)/2} \cup B_{(r+1)/2}) \leq \frac{r-1}{2} a_{(r+1)/2} + \frac{r-3}{2} b_{(r+1)/2}.$$

Combining the last two inequalities, we have that

$$a_0 + b_0 \leq \frac{r-1}{2} a_{(r+1)/2} + \frac{r-3}{2} b_{(r+1)/2}. \tag{2.9}$$

By (2.7), (2.8) and (2.9), we have that

$$\begin{aligned}
n &= p_1 + p_2 + \sum_{i=0}^{(r+1)/2} a_i + \sum_{j=0}^{(r+1)/2} b_j \\
&\le p_1 + p_2 + \left(a_1 + a_2 + \ldots + a_{(r-1)/2} + \frac{r+1}{2} a_{(r+1)/2}\right) \\
&\quad + \left(b_1 + b_2 + \ldots + b_{(r-1)/2} + \frac{r-1}{2} b_{(r+1)/2}\right) \\
&\le p_1 + p_2 + \left(a_1 + 2a_2 + \ldots + \frac{r-1}{2} a_{(r-1)/2} + \frac{r+1}{2} a_{(r+1)/2}\right) \\
&\quad + \left(b_1 + 2b_2 + \ldots + \frac{r-1}{2} b_{(r-1)/2} + \frac{r+1}{2} b_{(r+1)/2}\right) \\
&\le p_1 + p_2 + ((r-1)p_1 + rp_2) \\
&\le (p_1 + p_2)(1 + r).
\end{aligned}$$

Solving the above inequality for $p_1 + p_2$, we obtain that

$$p_1 + p_2 \ge \frac{n}{1+r}.$$

Hence,

$$\beta^*_{\mathbb{N}}(G) = 2(p_1 + p_2) - n \ge \frac{(1-r)n}{1+r}.$$

The equality $\beta^*_{\mathbb{N}}(G) = \frac{(1-r)n}{1+r}$ holds if and only if all the inequalities occurring in the previous proof are equalities. That means that $p_1 = 0$, $A_1 = \ldots = A_{(r-1)/2} = B_1 = \ldots = B_{(r+1)/2} = \emptyset$, the set P is independent (since $e(A_{(r+1)/2}, P) = rp_2$) and each of its vertices has exactly r neighbors in $A_{(r+1)/2}$. This implies that $a_{(r+1)/2} \ge r$. Moreover $A_{(r+1)/2}$ is an independent set (since $e(A_0 \cup B_0, A_{(r+1)/2}) = \frac{r-1}{2} a_{(r+1)/2}$) of order $a_{(r+1)/2}$, where $a_{(r+1)/2} \ge r$ is an integer such that $\frac{r+1}{2} a_{(r+1)/2} = rp_2$ has an integer solution p_2. Each vertex in $A_0 \cup B_0$ has exactly one neighbour in $A_{(r+1)/2}$, and so $A_0 \cup B_0$ induces a nearly $(r-1)$-regular graph of order $a_0 + b_0 = \frac{r-1}{2} a_{(r+1)/2}$ by the equality in (2.9). Consequently, each vertex in $A_{(r+1)/2}$ is joined to exactly $(r-1)/2$ vertices in $A_0 \cup B_0$. Thus, $G \in \mathcal{F}'_1(r)$. It follows by Remark 4 that $\beta^*_{\mathbb{N}}(G) = \frac{(1-r)n}{1+r}$ if and only if $G \in \mathcal{F}'_1(r)$.

Case 2. $r \equiv 0 \pmod 2$.

Note that $\lfloor (r+2)/2 \rfloor = (r+2)/2$ and $\lfloor (r+1)/2 \rfloor = r/2$. As f is a maximal negative function, by Lemma 2, every $v \in Q$ has at least one $u \in N[v]$ such that $f[u] = 0$ or 1. Hence, for every $v \in A_0 \cup B_0$, there exists at least one neighbour in

$A_{(r+2)/2} \cup B_{r/2}$. Thus,

$$e(A_0 \cup B_0, A_{(r+2)/2} \cup B_{r/2}) \geq a_0 + b_0.$$

Note that every $v \in A_{(r+2)/2}$ is joined to $(r+2)/2$ vertices in P, and so is joined at most $(r-2)/2$ vertices in $A_0 \cup B_0$; every $v \in B_{r/2}$ is joined to $r/2$ vertices in P, and so is joined at most $(r-2)/2$ vertices in $A_0 \cup B_0$. Hence,

$$e(A_0 \cup B_0, A_{(r+2)/2} \cup B_{r/2}) \leq \frac{r-2}{2} a_{(r+2)/2} + \frac{r-2}{2} b_{r/2}.$$

Thus, we can obtain that

$$a_0 + b_0 \leq \frac{r-2}{2} a_{(r+2)/2} + \frac{r-2}{2} b_{r/2}. \tag{2.10}$$

By (2.7), (2.8) and (2.10), we have that

$$\begin{aligned}
n &= p_1 + p_2 + \sum_{i=0}^{(r+2)/2} a_i + \sum_{j=0}^{r/2} b_j \\
&\leq p_1 + p_2 + \left(a_1 + \ldots + a_{r/2} + \frac{r}{2} a_{(r+2)/2}\right) \\
&\quad + \left(b_1 + \ldots + b_{(r-2)/2} + \frac{r}{2} b_{r/2}\right) \\
&\leq p_1 + p_2 + \left(a_1 + 2a_2 + \ldots + \frac{r}{2} a_{r/2} + \frac{r+2}{2} a_{(r+2)/2}\right) \\
&\quad + \left(b_1 + 2b_2 + \ldots + \frac{r-2}{2} b_{(r-2)/2} + \frac{r}{2} b_{r/2}\right) \\
&\leq p_1 + p_2 + ((r-1)p_1 + rp_2) \\
&\leq (p_1 + p_2)(1 + r).
\end{aligned}$$

Solving the above inequality for $p_1 + p_2$, we obtain that

$$p_1 + p_2 \geq \frac{n}{1+r}.$$

Hence,

$$\beta^*_{\mathbb{N}}(G) = 2(p_1 + p_2) - n \geq \frac{(1-r)n}{1+r}.$$

By a similar argument to that in Case 1, one can show that $\beta^*_{\mathbb{N}}(G) = \dfrac{(1-r)n}{1+r}$ if and only if $G \in \mathcal{F}'_0(r)$. $\square$

We now conclude the paper with an open problem: *What's a sharp lower bound on $\beta^*_{\mathbb{N}}(G)$ for a general graph G?*

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