NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 42 (2012), 9-16

ON THE ZERO DISTRIBUTION OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS IN THE COMPLEX DOMAIN

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(Received 11 November, 2011)

Abstract. Let $f(z) \not\equiv 0$ be a solution of $f'' + P(z)f = 0$, where $P(z)$ is a polynomial. Then the set of accumulation lines of zero-sequence is a subset of the Borel directions of $f(z)$. Let f_1 and f_2 be two linearly independent solutions of $f'' + P(z)f = 0$, where $P(z)$ is a polynomial of degree n and set $E = f_1 f_2$. Then, for every accumulation line $\arg z = \theta$ of zero-sequence of E, there is another accumulation line $\arg z = \phi$ of zero-sequence of E such that $|\phi - \theta| = \frac{2\pi}{n+2}$.

1. Introduction

We shall consider the differential equation

$$f'' + P(z)f = 0, \tag{1.1}$$

where $P(z)$ is a polynomial of degree $n \geq 1$. It is well known that every solution $f \not\equiv 0$ of (1.1) is an entire function of order $\frac{n+2}{2}$. In past years, there have been a lot of work done on the oscillation properties of the solutions of (1.1)(see [1], [3], [5], [7], [14]). In this paper, we shall study the angular distributions of the solutions of (1.1).

We shall use the standard notations of Nevanlinna theory of meromorphic function (see [9], [11] or [16]). Especially, for a meromorphic function $f(z)$ in the finite complex plane $\mathbb{C}$, we use the notations $\rho(f)$ and $\lambda(f)$ to denote the order and the exponent of convergence of zero-sequence of $f(z)$, respectively.

Let $f(z)$ be a transcendental meromorphic function with $0 < \rho(f) < \infty$ in the finite complex plane $\mathbb{C}$. Recall that a ray $\arg z = \theta$ from the origin is called a Borel direction of $f(z)$, if for any $\varepsilon > 0$ and for any complex number $a \in \overline{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$, possibly with two exceptions, the following equality holds

$$\limsup_{r\to\infty} \frac{\log n(S(\theta-\varepsilon, \theta+\varepsilon, r), f = a)}{\log r} = \rho(f), \tag{1.2}$$

where $n(S(\theta-\varepsilon, \theta+\varepsilon, r), f = a)$ denotes the number of zeros, counting multiplicities, of $f - a$ in the region $S(\theta - \varepsilon, \theta + \varepsilon, r) = \{z : \theta - \varepsilon \leq \arg z \leq \theta + \varepsilon, 0 < |z| \leq r\}$.

2010 *Mathematics Subject Classification* : 34M10; 30D35.
Key words and phrases: Differential equation, accumulation line of zero-sequence, exponent of convergence of zero-sequence, angular distribution.
The research was supported by the National Natural Science Foundation of China(Grant No.11171080) and the Foundation of Science and Technology Department of Guizhou Province (Grant No.[2010]07).
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The fundamental result in angular distribution of meromorphic functions is due to Valiron which says that a meromorphic function of order $\rho > 0$ must have at least one Borel direction of order ρ (see [16]).

In order to state our results, we note that in [13], [14], [15], the authors introduce the accumulation lines of zero-sequence to study the angular distributions of solutions of the equation (1.1). The following is their definition.

Definition 1. Let $g(z)$ be an meromorphic function in the finite complex plane and let $\arg z = \theta \in R$ be a ray. We denote, for each $\varepsilon > 0$, the exponent of convergence of zero-sequence of $g(z)$ in the angular region $S(\theta-\varepsilon, \theta+\varepsilon) = \{z|\theta-\varepsilon \leq \arg z \leq \theta+\varepsilon, |z| > 0\}$ by $\lambda_{\theta,\varepsilon}(g)$ and by $\lambda_\theta(g) = \lim_{\varepsilon\to 0}\lambda_{\theta,\varepsilon}(g)$.

We call the ray $\arg z = \theta$ which has the property $\lambda_\theta(g) = \rho(g)$ a accumulation line of the zero-sequence of g.

In this paper, we shall be continuing to study the accumulation line of zero-sequence of the solutions of the equation (1.1). Our first result is the following:

Theorem 1. Let $f(z)$ be a nontrivial solution of equation $f'' + P(z)f = 0$, where $P(z) = a_n z^n + \cdots + a_0$ is a polynomial of degree $n \geq 1$. Then the set of accumulation lines of zero-sequence is a subset of the Borel directions of $f(z)$.

Let f_1 and f_2 be two linearly independent solutions of (1.1) and set $E = f_1 f_2$. Since 1982, there have also been a lot of work on the distributions of the zeros of E. In [1], Bank and Laine proved that $\lambda(E) = \frac{n+2}{2}$, where n is the degree of the polynomial $P(z)$ in (1.1). Later Gundersen proved in [3] that the exponent of convergence of the nonreal zero sequence of E is also $\frac{n+2}{2}$. In [14], Wu studied the distribution zeros of solutions of (1.1) and, by using angular Nevanlinna characteristics, obtained the following result.

Theorem A([14]). Let $P(z)$ be a polynomial of degree $n \geq 1$ and let f_1 and f_1 be two linearly independent solutions of $f'' + P(z)f = 0$. If for some real number θ_0

$$\limsup_{r\to\infty} \frac{\log^+ \log^+ |E(re^{i\theta_0})|}{\log r} = \frac{n+2}{2}, \tag{1.3}$$

where $E = f_1 f_2$, then there exist θ_1 and θ_2 with $\theta_1 \leq \theta_0 \leq \theta_2$ such that $\theta_2 - \theta_1 = \frac{2\pi}{n+2}$ and $\lambda_{\theta_1}(E) = \lambda_{\theta_2}(E) = \frac{n+2}{2}$.

It is obvious that Theorem A implies Gundersen's result in [3]. In this paper, we shall give a simple proof, which avoids the use of the complicated angular Nevanlinna characteristics, of a generalization of Theorem A.

Theorem 2. Let $P(z)$ be a polynomial of degree $n \geq 1$ and let f_1 and f_1 be two linearly independent solutions of $f'' + P(z)f = 0$. Set $E = f_1 f_2$. If the ray $\arg z = \theta \in [0, 2\pi)$ satisfies $\lambda_\theta(E) = \frac{n+2}{2}$, then there exists another ray $\arg z = \phi$ with $|\theta - \phi| = \frac{2\pi}{n+2}$, such that $\lambda_\phi(E) = \frac{n+2}{2}$.

This paper is organized as follows. In Section 2, we consider the distributions of the accumulative lines of zero-sequence of a single solution of the equation (1.1), prove Theorem 1 as well as discuss some further related results. In Section 3, we investigate the complex oscillations of the product of two linearly independent solutions of the equation (1.1) and prove Theorem 2.

2. Angular Distributions of a Solution of (1.1)

In this section, we will investigate the distribution of zeros of a single solution of the equation (1.1). In order to prove Theorem 1, we need more notations. Let $\alpha < \beta$ such that $\beta - \alpha < 2\pi$ and let $r > 0$. We denote $S(\alpha, \beta) = \{z : \alpha \leq \arg z \leq \beta\}$, $S(\alpha, \beta, r) = \{z : \alpha \leq \arg z \leq \beta\} \bigcap \{z : |z| \leq r\}$. Let $f(z)$ be an entire function of order $0 < \rho < \infty$ and let $S = S(\alpha, \beta)$ be a sector. For simplicity, set $\rho(f) = \rho$, we shall say that $f(z)$ blows up exponentially in S if for any $\theta(\alpha < \theta < \beta)$

$$\lim_{r\to\infty} \frac{\log\log|f(re^{i\theta})|}{\log r} = \rho \tag{2.1}$$

holds. We shall also say that $f(z)$ decays to zero exponentially in S if for any $\theta(\alpha < \theta < \beta)$

$$\lim_{r\to\infty} \frac{\log\log|f(re^{i\theta})|^{-1}}{\log r} = \rho \tag{2.2}$$

holds.

The following lemma, which is due to Hille, plays an important role in the proof of our results.

Lemma 1([6]). Let $f(z) \not\equiv 0$ be a solution of (1.1), where $P(z) = a_n z^n + \cdots + a_0 (a_n \neq 0)$. Set $\theta_j = \frac{2j\pi - \arg(a_n)}{n+2}$ and $S_j = \{z : \theta_j < \arg z < \theta_{j+1}\}$ $(j = 0, 1, 2, \cdots, n+1)$. Then $f(z)$ has the following properties:

(1)In each sector S_j, f either blows up or decays to zero exponentially in it;

(2)If, for some j, f decays to zero in S_j, then it must blow up in S_{j-1} and S_{j+1} (if $j = n+1$, set $S_{j+1} = S_0$). However, it is possible for f to blow up in many adjacent sectors;

(3)If f decays to zero in S_j, then f has at most finitely many zeros in any closed sub-sector within $S_{j-1} \cup \bar{S}_j \cup S_{j+1}$;

(4)If f blows up in S_{j-1} and S_j, then for each $\varepsilon > 0$, f has infinitely many zeros in each sector $\theta_j - \varepsilon \leq \arg z \leq \theta_j + \varepsilon$, and furthermore, as $r \to \infty$,

$$n(S(\theta_j - \varepsilon, \theta_j + \varepsilon, r), f = 0) = (1 + o(1)) \frac{4\sqrt{|a_n|}}{\pi(n+2)} r^{\frac{n+2}{2}}, \tag{2.3}$$

where $n(S(\theta_j - \varepsilon, \theta_j + \varepsilon, r), f = 0)$ is the number of zeros, counting multiplicities, of $f(z)$ in the region $S(\theta_j - \varepsilon, \theta_j + \varepsilon, r)$.

We also need the following lemma to prove Theorem 1.

Lemma 2([**10**, P.193]). Suppose that $S(\alpha, \beta)$ and $S(\alpha', \beta')$ are two sectors such that $\alpha < \alpha' < \beta' < \beta$ and that $g(z)$ is analytic on $S(\alpha, \beta)$. If

$$\limsup_{r\to\infty} \frac{\log\log M(r, S(\alpha', \beta'), g)}{\log r} \equiv \rho(S(\alpha', \beta'), g) > \frac{\pi}{\beta - \alpha},$$

where $M(r, S(\alpha', \beta'), g) = \max_{z \in S(\alpha', \beta', r)} |g(z)|$, then we have for every $a \in \mathbb{C}$ with at most one exception

$$\limsup_{r\to\infty} \frac{\log n(S(\alpha, \beta, r), g = a)}{\log r} \geq \rho(S(\alpha', \beta'), g),$$

where $n(S(\alpha, \beta, r), g = a)$ denotes the roots of the equation $g(z) = a$, counting multiplicities, in $S(\alpha, \beta, r)$.

Proof of Theorem 1. Let $f(z)(\not\equiv 0)$ be a solution of (1.1), where $P(z) = a_n z^n + \cdots + a_0$. Suppose that $\arg z = \theta_j$, which is defined in Lemma 1, is a accumulation line of zero-sequence of $f(z)$. We shall prove that $\arg z = \theta_j$ is a Borel direction of $f(z)$.

For any $0 < \eta < \frac{\pi}{n+2}$, set

$$\alpha = \theta_{j-1} + \eta, \beta = \theta_{j+1} - \eta,$$

$$\alpha' = \theta_{j-1} + 2\eta, \beta' = \theta_{j+1} - 2\eta.$$

Obviously, we have that $\alpha < \alpha' < \theta_j < \beta' < \beta$. It follows from Lemma 1 that there certainly exists $\theta \in (\alpha', \beta')$ such that

$$\limsup_{r\to\infty} \frac{\log\log M(r, S(\alpha', \beta'), f)}{\log r} \geq \limsup_{r\to\infty} \frac{\log\log |f(re^{i\theta})|}{\log r} = \frac{n+2}{2} > \frac{\pi}{\beta - \alpha}.$$

It follows from Lemma 2 that, for every $a \in \mathbb{C}$ with at most one exception,

$$\limsup_{r\to\infty} \frac{\log n(S(\alpha, \beta, r), f = a)}{\log r} \geq \frac{n+2}{2}. \tag{2.4}$$

For any $\varepsilon > 0 (< \eta)$, we have

$$\alpha - \frac{\varepsilon}{2} < \alpha < \theta_j - \varepsilon < \theta_j - \frac{\varepsilon}{2} < \theta_j < \theta_j + \frac{\varepsilon}{2} < \theta_j + \varepsilon < \beta < \beta + \frac{\varepsilon}{2}.$$

Thus

$$S(\alpha, \beta) \subset \{S(\alpha - \frac{\varepsilon}{2}, \theta_j - \frac{\varepsilon}{2}) \cup S(\theta_j - \varepsilon, \theta_j + \varepsilon) \cup S(\theta_j + \frac{\varepsilon}{2}, \beta + \frac{\varepsilon}{2})\}. \tag{2.5}$$

For any $a \in \mathbb{C}$ with at most one exception, we deduce from Lemma 1 that

$$\limsup_{r\to\infty} \frac{\log n(S(\alpha - \frac{\varepsilon}{2}, \theta_j - \frac{\varepsilon}{2}, r), f = a)}{\log r} < \frac{n+2}{2}, \tag{2.6}$$

and

$$\limsup_{r\to\infty} \frac{\log n(S(\theta_j + \frac{\varepsilon}{2}, \beta + \frac{\varepsilon}{2}, r), f = a)}{\log r} < \frac{n+2}{2}. \tag{2.7}$$

Therefore, combining (2.4), (2.5), (2.6) and (2.7) we have

$$\limsup_{r\to\infty} \frac{\log n(S(\theta_j - \varepsilon, \theta_j + \varepsilon, r), f = a)}{\log r} \geq \frac{n+2}{2}.$$

Thus it follows from the definition of Borel direction that $\arg z = \theta_j$ must be a Borel direction of $f(z)$. It is not hard to see from Lemma 1 (4) that the accumulation lines of zero-sequence of $f(z)$ only come from these rays $\arg z = \theta_j$, $j = 0, 1, 2, \cdots, n+1$. This implies that the set of accumulation lines of zero-sequence is a subset of the Borel directions of $f(z)$. The proof of Theorem 1 is completed.

Remark 1. However, it is possible that a Borel direction of solution of the equation (1.1) is not its accumulation line of zero-sequence. The following example shows that fact.

The well-known Airy differential equation

$$f'' - zf = 0 \tag{2.8}$$

possesses a solution f_0 such that the zeros of f_0 are all real and negative(see [12], p.413–415). It is easy to see that $\arg z = 0$, $\arg z = \frac{2\pi}{3}$ and $\arg z = \frac{4\pi}{3}$ are all the Borel directions of f_0.

In order to state our second result on the angular distributions of the zeros of solutions of (1.1), we need some more preparations.

For the solutions of the equation (1.1), where $P(z) = a_n z^n + \cdots + a_0$. In [3], Gundersen proved that either f has only finitely many zeros, or the exponent of convergence of the nonreal zero-sequence of f equal to $\frac{n+2}{2}$ when $n = 4k + 2$ for some non-negative integer k. We next discuss the exponent of convergence of the nonreal zero-sequence of a solution of (1.1). In order to state our results, we need give another definition. It follows Lemma 1 that the set of the accumulation lines of zero-sequence of a solution $f \not\equiv 0$ is the subset of $\{\theta_j, 0 \le j \le n+1\}$.

Definition 2. Let $f(z)$ be a nontrivial solution of the equation $f'' + P(z)f = 0$, where $P(z) = a_n z^n + \cdots + a_0$ is a polynomial of degree n. We denote by $p(f)$ the number of the rays $\arg z = \theta_j (j = 0, 1..., n+1)$ which are not accumulation lines of zero-sequence of $f(z)$.

Remark 2. It follows from Lemma 1(2) that $p(f)$ must be an even number.

Theorem 3. Let $f(z)$ be a nontrivial solution of equation $f'' + P(z)f = 0$, where $P(z) = a_n z^n + \cdots + a_0$ is a polynomial of degree n with Im $(a_n) \neq 0$. Then either $f(z)$ has only finite zeros, or the exponent of convergence of the nonreal zero-sequence of $f(z)$ is $\frac{n+2}{2}$. In particularly, when $n = 2k$ for some non-negative integer k, then either $f(z)$ has only finite zeros, or $f(z)$ has at least two accumulation lines of nonreal zero-sequence.

Proof. Suppose that $f(z)$ is a nontrivial solution of equation (1.1). Note that the accumulation lines of zero-sequence of $f(z)$ come only from those rays $\arg z = \theta_j = \frac{2j\pi - \arg(a_n)}{n+2} (j = 0, 1, ..., n+1)$. Since $\mathrm{Im}(a_n) \neq 0$ and

$$\arg(a_n) = 2j\pi - (n+2)\theta_j,$$

for $j = 0, 1, 2, \cdots, n+1$. It is easy to deduce that none of these rays $\arg z = \theta_j$ is the positive or the negative real axis. Now suppose that $f(z)$ has infinitely many zeros in the complex plane. For any $\varepsilon (0 < \varepsilon < c)$, where $c = \min_{0 \le j \le n+1}\{|\theta_j|, |\theta_j - \pi|\}$, it follows from Lemma 1 that there exists at least one ray $\arg z = \theta_{j_0}$ such that, as $r \to \infty$,

$$\begin{aligned} &n_{(NR)}(S(\theta_{j_0} - \varepsilon, \theta_{j_0} + \varepsilon, r), f = 0) \\ &= n(S(\theta_{j_0} - \varepsilon, \theta_{j_0} + \varepsilon, r), f = 0) \\ &= (1 + o(1)) \frac{4\sqrt{|a_n|}}{\pi(n+2)} r^{\frac{n+2}{2}} \end{aligned}$$

holds, where $n_{(NR)}(S(\theta_{j_0} - \varepsilon, \theta_{j_0} + \varepsilon, r), f = 0)$ is the number of nonreal zeros, counting multiplicities, of $f(z)$ in the region $S(\theta_{j_0} - \varepsilon, \theta_{j_0} + \varepsilon, r) = \{z : \theta_{j_0} - \varepsilon \le \arg z \le \theta_{j_0} + \varepsilon, 0 < z \le r\}$. From the definition of exponent of convergence, we have that

$$\lambda_{(NR)}(f) = \rho(f) = \frac{n+2}{2},$$

where $\lambda_{(NR)}(f)$ is the exponent of convergence of nonreal zero-sequence of $f(z)$.

Now assume that $n = 2k$ and that $f(z)$ has infinitely many zeros in the complex plane. Since both $p(f)$ and $n+2$ are even numbers, we see easily that f has at least two accumulation lines of nonreal zero-sequence. The proof of Theorem 3 is completed.

Remark 3. There are many examples which show that the condition $\mathrm{Im}(a_n) \neq 0$ in Theorem 3 is necessary. For any given real constant $a > 0$ and $b \geq 0$, Gundersen in [5] proved that there exists an infinite sequence of real constants λ_k satisfying $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$, where $\lambda_k \to \infty$ as $k \to \infty$, such that for each λ_k, the equation

$$f'' + (az^4 + bz^2 - \lambda_k)f = 0 \tag{2.9}$$

possesses a solution $f = f_k(z)$ that has an infinite number of real zeros and at most a finite number of nonreal zeros.

3. Complex Oscillations of Two Linearly Independent Solutions of (1.1)

In this section, we prove Theorem 2. Let f_1 and f_1 be two linearly independent solutions of (1.1) and $P(z) = a_n z^n + \cdots + a_0$ is a polynomial of degree n. Set $E = f_1 f_2$. Suppose that $\theta \in [0, 2\pi)$ such that $\lambda_\theta(E) = \frac{n+2}{2}$. It follows from Theorem 1 that there must exist some integer $j_0 : 0 \leq j_0 \leq n+1$ such that $\theta = \theta_{j_0} = \frac{2j_0\pi - \arg(a_n)}{n+2}$. By using Lemma 1, we see that f_1 or f_2 must blow up exponentially in S_{j_0-1} and S_{j_0}. Without loss of generality, we assume that f_1 blows up exponentially in the sectors S_{j_0-1} and S_{j_0}. Now we treat the following three cases.

Case 1. The solution f_2 blows up exponentially in S_{j_0} and decays to zero exponentially in S_{j_0-1}. In this case, both f_1 and f_2 blow up exponentially in the sector S_{j_0}, so we need only to consider the behaviors of f_1 and f_2 in S_{j_0+1}. Now we claim that it is impossible that both f_1 and f_2 decay to zero exponentially in common sector. To prove our claim, without loss of generality, we suppose that f_1 and f_2 decay to zero exponentially in S_0. Set $h = \frac{f_2}{f_1}$. We arbitrarily choose a constant $b \in \bar{\mathbb{C}}$, such that b is not a deficient value of h. Set $f = f_2 - bf_1$. It is easy to see that f is a solution of the equation (1.1), and 0 is not a deficient value of f. Therefore, f blows up exponentially in every sector S_j $(j = 0, 1, 2, \cdots, n+1)$(see, [3, Lemma 3]). This contradicts the fact that f decay to zero exponentially in S_0. The claim is proved.

Now we return to the proof of the theorem. It follows from Lemma 1(2) that one of the functions f_1 and f_2 must have $\lambda_{\theta_{j_0+1}}(f_i) = \frac{n+2}{2}$ if f_i(i=1 or 2) blows up exponentially in S_{j_0+1}. This implies that

$$\lambda_{\theta_{j_0}}(E) = \lambda_{\theta_{j_0+1}}(E) = \frac{n+2}{2}.$$

Set $\phi = \theta_{j_0+1}$(if $j_0 = n+1$, set $\theta_{j_0+1} = \theta_0$), we have

$$|\theta - \phi| = |\theta_{j_0+1} - \theta_{j_0}| = \frac{2\pi}{n+2}.$$

Case 2. The solution f_2 decays to zero exponentially in S_{j_0} and blows up exponentially in S_{j_0-1}. In this case both f_1 and f_2 blow up exponentially in the angular region S_{j_0-1}. As we did in case (1), we have $\lambda_{\theta_{j_0-1}}(f_i) = \frac{n+2}{2}$ if $f_i(i{=}1$ or $2)$ blows up exponentially in S_{j_0-2}.

Therefore it is easy to see that

$$\lambda_{\theta_{j_0}}(E) = \lambda_{\theta_{j_0-1}}(E) = \frac{n+2}{2}.$$

Set $\phi = \theta_{j_0-1}$(if $j_0 = 0$, set $\theta_{j_0-1} = \theta_{n+1}$), we have

$$|\theta - \phi| = |\theta_{j_0} - \theta_{j_0-1}| = \frac{2\pi}{n+2}.$$

Case 3. The solution f_2 blows up exponentially in S_{j_0-1} and S_{j_0}. In this case, we have

$$\lambda_{\theta_{j_0}}(E) = \lambda_{\theta_{j_0-1}}(E) = \lambda_{\theta_{j_0+1}}(E) = \frac{n+2}{2}.$$

The proof of Theorem 2 is completed.

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