NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 36 (2007), 93–99

A CHARACTERIZATION OF SPHERES IN A EUCLIDEAN SPACE

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(Received October 2004)

Abstract. For an orientable compact and connected hypersurface in the Euclidean space R^4 with positive scalar curvature S, the shape operator A and the mean curvature α, it is shown that the inequality

$$6\alpha \det A \geq \alpha^2 S + 4 \|\nabla\alpha\|^2$$

implies that the hypersurface is a sphere, where $\nabla\alpha$ is the gradient of α. A similar characterization is also obtained for spheres the Euclidean space R^3 (cf. Theorem 2).

1. Introduction

The class of compact hypersurfaces with positive scalar curvature in a Euclidean space R^n is quite large and therefore it is an interesting question in Geometry to obtain conditions which characterize spheres in this class. It is known that compact positively curved hypersurfaces with constant mean curvature in R^n are spheres (cf. [7] p.375) as well as that embedded compact hypersurfaces with constant scalar curvature in R^n are spheres (cf. [8]). However all such characterizations use mean curvature, scalar curvature as well as bounds on sectional curvature. Given a hypersurface its geometry is mostly controlled by the restriction on shape operator A and natural invariants associated with A such as $\det A$ (Lipschitz-Killing curvature), the mean curvature (given by $tr.A$) and other invariants given by symmetric functions of eigenvalues of A. The invariants given by the symmetric functions of eigenvalues of A and the mean curvature are used by by Chen in his series of papers to obtain interesting results on submanifolds of Euclidean spaces (cf. [2], [3], [4]). The invariant $\det A$ has been used to define total absolute curvature by Chern and Lashof (cf. [5]) where they used it to obtain a characterization for a compact hypersurface to be homeomorphic to a sphere. We would like to add following brief note on the use of the invariant $\det A$ in the geometry of submanifolds and specially on its role in the fundamental work of Chern-Lashof (cf. [5], [6]):

Recall that in the classical theory of surfaces, the normal mapping of Gauss is used to define the Gauss-Kronecker curvature of the surface and the integral of this curvature for a compact surface is the Euler characteristic of the surface (The Gauss-Bonnet theorem). Then the basic question was to generalize the classical Gauss mapping for the immersed submanifolds in a Euclidean space, which was accomplished without much difficulty. Perhaps the difficult part was to obtain a curvature measure for arbitrary immersed submanifold M^n in a Euclidean space R^{n+k} similar to that of Gauss-Kronecker curvature. The crucial observation of Cher-Lashof theory in meeting this aim was the coincidence that the dimension of the unit normal bundle of the submanifold is same as the dimension of the unit hypersphere in the ambient Euclidean space R^{n+k} centered at the origin. Then the volume form of this unit sphere when pulled back by the generalized Gauss map must be proportional to the volume form of the unit normal bundle (being forms of the same top degree). This proportionality function $G(p, N)$, $p \in M^n$, N is the unit normal vector at p in the unit normal bundle, defined on the nunit normal bundle is then the desired curvature measure called the Lipschitz-Killing curvature similar to the Gauss-Kronecker curvature which on integration gave the total absolute curvature of the submanifold. Chern and Lashof used used height function $h_N : M^n \to R$ with respect to the unit normal vector N to show that the Lipschitz-Killing curvature $G(p, N) = (-1)^n \det(A_N)$, where A_N is the shape operator (Weingarten map) with respect to the unit normal N, (For surfaces in R^3, $n = 2$ and $A_N = A$, $G(p, N) = \det A$ is the classical Gauss-Kronecker curvature). As Gauss-Bonnet theorem relates the geometric invariant (the integral of Gauss-Kronecker curvature) to the topological invariant (Euler characteristic) of the surface, Chern and Lashof (cf. [6]) have shown that for compact immersed submanifolds in a Euclidean space, the geometric invariant the total absolute curvature is related to the topological invariant the total Betti number of the submanifold. They have also proved that if the total absolute curvature of an immersed compact submanifold in a Euclidean space is less than 3 then the submanifold is homeomorphic to a sphere. For an orientable hypersurface M in R^{n+1} as there is only one unit normal vector field N, the shape operator $A_N = A$ and the Lipschitz-Killing curvature $G(p, N) = \det A$, the above mentioned result of Chern and Lashof uses $\det A$ to obtain a characterization for a compact hypersurface in R^{n+1} to be homeomorphic to a sphere.

However this invariant is not used in characterizing the geometry of a sphere in a Euclidean space. In this paper we use this invariant to obtain a characterization of spheres in R^4. Indeed in this paper we prove the following:

Theorem 1. Let M be an orientable compact and connected hypersurface with positive scalar curvature S in the Euclidean space R^4. If the shape operator A and the mean curvature α of M satisfy

$$6\alpha \det A \geq \alpha^2 S + 4 \left\| \nabla \alpha \right\|^2 ,$$

then α is a constant and $M = S^3(\alpha^2)$, where $\nabla \alpha$ is the gradient of α.

Our proof depends on the dimension of the Euclidean space R^4 and therefore it could be an interesting question to extend this result for the hypersurfaces in the Euclidean space R^n, $n > 4$. For the surfaces in R^3 the scalar curvature becomes $2K$, K being the Guassian curvature and the invariant $\det A$ is also K. The condition scalar curvature being positive translates to the surface being positively curved and in this case we prove the following:

Theorem 2. Let M be an orientable compact and connected positively curved surface in the Euclidean space R^3. If the mean curvature α and the Guassian curvature K of M satisfy

$$\alpha^2 K \geq K^2 + \left\| \nabla \alpha \right\|^2 ,$$

then α is a constant and $M = S^2(\alpha^2)$, where $\nabla \alpha$ is the gradient of α.

2. Preliminaries

Let M be an orientable hypersurface of the Euclidean space R^{n+1}. We denote the induced metric on M by g. Let $\overline{\nabla}$ be the Euclidean connection on R^{n+1} and ∇ be the Riemannian connection on M with respect to the induced metric g. Let N be the unit normal vector field to the hypersurface M. Then for any $X \in \mathfrak{X}(M)$, $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M, we have $Xg(N,N) = 0$, which gives $g(\overline{\nabla}_X N, N) = 0$, that is $\overline{\nabla}_X N$ is tangential to M. Thus we can define a map $A : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $A(X) = -\overline{\nabla}_X N$ called the shape operator of M. The shape operator A is symmetric and linear follows from the porperties of the Riemannian connection $\overline{\nabla}$. The Gauss and Weingarten formulas for the hypersurface are (cf. [1])

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX,Y)N, \quad \overline{\nabla}_X N = -AX, \qquad X,Y \in \mathfrak{X}(M). \tag{2.1}$$

We also have the following Gauss and Codazzi equations (cf. [1])

$$R(X,Y;Z,W) = g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W) \tag{2.2}$$

$$(\nabla A)(X,Y) = (\nabla A)(Y,X), \tag{2.3}$$

where R is the curvature tensor field of the hypersurface and $(\nabla A)(X,Y) = \nabla_X AY - A\nabla_X Y$. The mean curvature α of the hypersurface is given by $n\alpha = \sum_i g(Ae_i, e_i)$, where $\{e_1, .., e_n\}$ is a local orthonormal frame on M. If $A = \lambda I$ holds for a constant λ, then the hypersurface is said to be totally umbilical . The square of the length of the shape operator A is given by

$$\|A\|^2 = \sum_{ij} g(Ae_i, e_j)^2 = tr.A^2.$$

The scalar curvature S of the hypersurface is given by

$$S = n^2\alpha^2 - \|A\|^2 . \tag{2.4}$$

The second covariant derivative $(\nabla^2 A)(X,Y,Z)$, $X,Y,Z \in \mathfrak{X}(M)$ of A is defined by

$$(\nabla^2 A)(X,Y,Z) = \nabla_X(\nabla A)(Y,Z) - (\nabla A)(\nabla_X Y, Z) - (\nabla A)(Y, \nabla_X Z)$$

and we have the Ricci identity

$$(\nabla^2 A)(X,Y,Z) - (\nabla^2 A)(Y,X,Z) = R(X,Y)AZ - A(R(X,Y)Z). \tag{2.5}$$

3. Some Lemmas

Let M be an orientable hypersurface of the Euclidean space R^{n+1} and $\nabla\alpha$ be the gradient of the mean curvature function α. Then we have

Lemma 3. Let M be an orientable hypersurface of the Real space form R^{n+1} and $\{e_1, .., e_n\}$ be a local orthonormal frame on the hypersurface M. Then

$$\sum_i (\nabla A)(e_i, e_i) = n\nabla\alpha.$$

The proof is straightforward and follows from the symmetry of A and the equation (2.3).

Lemma 4. Let M be an orientable hypersurface of the Euclidean space R^{n+1}. Then

$$\|\nabla A\|^2 \geq n\,\|\nabla\alpha\|^2\,,$$

where $\|\nabla A\|^2 = \sum_{ij}\|(\nabla A)(e_i,e_j)\|^2$ for a local orthonormal frame $\{e_1,..,e_n\}$ on M.

Proof. Define an operator $B:\mathfrak{X}(M)\to\mathfrak{X}(M)$ by $B=A-\alpha I$. Then we have

$$(\nabla B)(X,Y)=(\nabla A)(X,Y)-(X\alpha)Y,$$

which gives

$$\begin{aligned}\|\nabla B\|^2 &= \|\nabla A\|^2+n\,\|\nabla\alpha\|^2-2\textstyle\sum_{ij}g\left((\nabla A)(e_i,e_j),e_j\right)g(\nabla\alpha,e_i)\\ &= \|\nabla A\|^2+n\,\|\nabla\alpha\|^2-2\textstyle\sum_j g\left(\nabla\alpha,(\nabla A)(e_j,e_j)\right)\\ &= \|\nabla A\|^2-n\,\|\nabla\alpha\|^2\,.\end{aligned}$$

This proves that $\|\nabla A\|^2 \geq n\,\|\nabla\alpha\|^2$.

Lemma 5. Let M be an orientable compact hypersurface of the Euclidean space R^{n+1}. Then

$$\int_M\left(\sum_i g(\nabla_{e_i}(\nabla\alpha),Ae_i)\right)dV=-n\int_M\|\nabla\alpha\|^2\,dV,$$

where $\{e_1,..,e_n\}$ is a local orthonormal frame on M.

Proof. Choosing a point wise covariant constant local orthonormal frame $\{e_1,...,e_n\}$ on M, we compute

$$\begin{aligned}div\,(A(\nabla\alpha)) &= \sum_i e_i g(\nabla\alpha,Ae_i) = \sum_i g(\nabla_{e_i}(\nabla\alpha),Ae_i)+\sum_i g(\nabla\alpha,(\nabla A)(e_i,e_i))\\ &= \sum_i g(\nabla_{e_i}(\nabla\alpha),Ae_i)+n\,\|\nabla\alpha\|^2\,.\end{aligned}$$

Integrating this equation we get the Lemma.

Lemma 6. Let M be an orientable compact hypersurface of the Euclidean space R^{n+1} and $\{e_1,...,e_n\}$ be a local orthonormal frame on M. Then

$$\int_M\left\{\|\nabla A\|^2-n^2\,\|\nabla\alpha\|^2+\sum_{ik}\left[R(e_k,e_i;Ae_k,Ae_i)-R(e_k,e_i;e_k,A^2e_i)\right]\right\}dV=0.$$

Proof. Define a function $f:M\to R$ by $f=\frac{1}{2}\|A\|^2$. Then by a straightforward computation we get the Laplacian Δf of the smooth function f as

$$\Delta f=\|\nabla A\|^2+\sum_{ik}g\left((\nabla^2A)(e_k,e_k,e_i),Ae_i\right). \tag{3.1}$$

Using equations (2.3) and (2.5) in (3.1) we arrive at

$$\begin{aligned}\Delta f &= \|\nabla A\|^2+\sum_{ik}g((\nabla^2A)(e_i,e_k,e_k),Ae_i)\\ &\quad+\sum_{ik}\left[R(e_k,e_i;Ae_k,Ae_i)-R(e_k,e_i;e_k,A^2e_i)\right].\end{aligned} \tag{3.2}$$

We integrate (3.2) and use Lemma 5 together with $\sum_k (\nabla^2 A)(e_i, e_k, e_k) = n\nabla_{e_i}(\nabla\alpha)$ to get the Lemma.

4. Proof of the Theorem 1

Let M be an orientable compact and connected hypersurface of the Euclidean space R^4. Then for a local orthonormal frame $\{e_1, e_2, e_3\}$ on M and equation (2.2) we have

$$\begin{aligned}\sum_{ik} R(e_k, e_i; Ae_k, Ae_i) &= \sum_{ik} [g(Ae_i, Ae_k)g(Ae_k, Ae_i) - g(Ae_k, Ae_k)g(Ae_i, Ae_i)] \\ &= \sum_i g(A^4 e_i, e_i) - \|A\|^4 \\ &= tr.A^4 - \|A\|^4 \qquad (4.1)\end{aligned}$$

and similarly

$$\sum_{ik} R(e_k, e_i; e_k, A^2 e_i) = tr.A^4 - 3\alpha tr.A^3. \qquad (4.2)$$

Thus from the equations (4.1) and (4.2) we arrive at

$$\sum_{ik} \left[R(e_k, e_i; Ae_k, Ae_i) - R(e_k, e_i; e_k, A^2 e_i)\right] = 3\alpha tr.A^3 - \|A\|^4. \qquad (4.3)$$

The Cayley-Hamilton theorem for the linear operator A gives

$$A^3 - 3\alpha A^2 + \frac{S}{2}A - (\det A)I = 0.$$

Multiplying this equation by 3α and taking trace we arrive at

$$3\alpha tr.A^3 = 9\alpha^2 \|A\|^2 - \frac{9}{2}S\alpha^2 + 9\alpha \det A.$$

Using this equation in (4.3) together with equation (2.4) we get

$$\sum_{ik} \left[R(e_k, e_i; Ae_k, Ae_i) - R(e_k, e_i; e_k, A^2 e_i)\right] = \|A\|^2 S - \frac{9}{2}\alpha^2 S + 9\alpha \det A.$$

We use this equation in Lemma 6 to arrive at

$$\int_M \left\{ \begin{array}{c} \left(\|\nabla A\|^2 - 3\|\nabla\alpha\|^2\right) + S\left(\|A\|^2 - 3\alpha^2\right) \\ +\frac{3}{2}\left(6\alpha \det A - \alpha^2 S - 4\|\nabla\alpha\|^2\right) \end{array} \right\} dV = 0.$$

Since the scalar curvature $S > 0$ and the Schwartz inequality implies $\|A\|^2 \geq 3\alpha^2$ (with equality holding if and only if $A = \alpha I$), the last equation together with Lemma 4 and the condition in the statement of the theorem gives $A = \alpha I$. Also as the $\dim M > 2$ from the equation (2.3) it is easy to conclude that α is a constant, that is, M is totally umbilical and consequently that $M = S^3(\alpha^2)$.

5. Proof of Theorem 2

Let M be an orientable compact and connected surface in R^3. Choosing a local orthonormal frame $\{e_1, e_2\}$ on M which diagonalizes A, we get

$$\begin{aligned}\sum_{ik}\left[R(e_k, e_i; Ae_k, Ae_i) - R(e_k, e_i; e_k, A^2e_i)\right] &= \left(\lambda_1^2+\lambda_2^2\right)K - 2\lambda_1\lambda_2 K \\ &= \|A\|^2 K - 2K^2 \qquad (5.1)\\ &= \left(\|A\|^2 - 2\alpha^2\right)K + 2\left(\alpha^2 K - K^2\right),\end{aligned}$$

where $Ae_i = \lambda_i e_i$, and $K = R(e_1, e_2; e_2, e_1) = \det A = \lambda_1\lambda_2$ is the Guassian curvature of M. Thus by Lemma 6 we get

$$\int_M \left\{\begin{array}{c}\left(\|\nabla A\|^2 - 2\|\nabla\alpha\|^2\right) + \left(\|A\|^2 - 2\alpha^2\right)K \\ +2\left(\alpha^2 K - K^2 - \|\nabla\alpha\|^2\right)\end{array}\right\} dV = 0.$$

Using the Schwartz inequality $\|A\|^2 \geq 2\alpha^2$ and Lemma 4 together with the condition in the statement of the theorem in the above integral we get $A = \alpha I$. Finally using equation (2.3) together with $A = \alpha I$ we get that α is a constant, that is, M is totally umbilical and consequently $M = S^2(\alpha^2)$.

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