Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **5** (2011), 80–86. doi:10.2298/AADM110212009B

ASYMPTOTIC EXPANSION FOR THE SUM OF INVERSES OF ARITHMETICAL FUNCTIONS INVOLVING ITERATED LOGARITHMS

Hacène Belbachir, Djamel Berkane

A generalized formula is obtained for the sum of inverses of the prime counting function for a large class of arithmetical functions related to the iterated logarithms.

1. INTRODUCTION AND MAIN RESULT

Let $\pi(x)$ be the number of primes not exceeding x. In 2000, using the asymptotic formula

$$\pi(x) = \frac{x}{\log(x)}\left(\sum_{k=0}^{m-1}\frac{k!}{\log^k(x)} + O\left(\frac{1}{\log^m(x)}\right)\right), \tag{1}$$

L. Panaitopol [**4**] obtained

$$\frac{1}{\pi(x)} = \frac{1}{x}\left(\log(x) - 1 - \frac{k_1}{\log(x)} - \cdots - \frac{k_m}{\log^m(x)} + O\left(\frac{1}{\log^{m+1}(x)}\right)\right),$$

where $m \geq 1$ and $\{k_j\}_j$ is the sequence of integers given by the recurrence relation

$$k_n + 1!\, k_{n-1} + 2!\, k_{n-2} + \cdots + (n-1)!\, k_1 = n \cdot n!\ .$$

2000 Mathematics Subject Classification. 11N37, 11N05.
Keywords and Phrases. Asymptotic expansion, arithmetical function, prime number theorem.

Two years later, A. Ivić [**3**] proved that

$$\sum_{2\le n\le x}\frac{1}{\pi(n)}=\frac{1}{2}\log^2(x)-\log(x)-\log\log(x)+C$$
$$+\frac{k_2}{\log(x)}+\cdots+\frac{k_m}{(m-1)\log^{m-1}(x)}+O\left(\frac{1}{\log^m(x)}\right),$$

where C is an absolute constant not depending on m.

In 2009, the first author and F. Bencherif [**1**] derived an asymptotic formula for the sum of reciprocals of a large class of arithmetic functions having the following expansion

$$f(n)=\frac{n}{\log(n)}\left(a_0+\frac{a_1}{\log(n)}+\cdots+\frac{a_{m-1}}{\log^{m-1}(n)}+O\left(\frac{1}{\log^m(n)}\right)\right),\ \text{with } a_0\neq 0,$$

they obtained

$$\sum_{2\le n\le x}{}'\frac{1}{f(n)}=\frac{b_0}{2}\log^2(x)+b_1\log(x)+b_2\log\log(x)+C_0$$
$$-\frac{b_3}{\log(x)}-\cdots-\frac{b_{m+1}}{(m-1)\log^{m-1}(x)}+O\left(\frac{1}{\log^m(x)}\right),$$

where $\sum_{2\le n\le x}{}'\frac{1}{f(n)}$ is a sum restricted to integers n for which $f(n)\neq 0$ and $b_j=A_j(a_0,a_1,\dots,a_j)$ for $0\le j\le m+1$, with

$$A_0(t_0)=\frac{1}{t_0},\qquad A_1(t_0,t_1)=-\frac{t_1}{t_0^2},$$

$$A_n(t_0,t_1,\cdots,t_n)=\frac{(-1)^n}{t_0^{n+1}}\cdot\begin{vmatrix} t_1 & t_2 & \cdots & \cdots & t_n\\ t_0 & t_1 & \cdots & \cdots & t_{n-1}\\ 0 & t_0 & t_1 & \cdots & t_{n-2}\\ 0 & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & 0 & t_0 & t_1\end{vmatrix},\quad (n\ge 1).$$

More recently, the authors in [**2**] studied the arithmetical function $nK(n)$, where

$$K(x):=\max\{k\in\mathbb{N}\ /\ p_1p_2\cdots p_k\le x\},$$

and p_k is the k^{th} prime number. Using the asymptotic expansion

$$K(x)=\frac{\log(x)}{\log\log(x)}\left(\sum_{j=0}^{m}\frac{j!}{[\log\log(x)]^j}+O\left(\frac{1}{[\log\log(x)]^{m+1}}\right)\right),\tag{2}$$

they get a similar result to the one in A. Ivić [**3**], with three levels of logarithmic iterations x, $\log x$, $\log\log x$,

$$\sum_{2\le n\le x}\frac{1}{nK(n)}=\frac{1}{2}\log^2\log(x)-\log\log(x)-\log\log\log(x)+C_1$$
$$+\frac{k_2}{\log\log(x)}+\cdots+\frac{k_m}{(m-1)\log^{m-1}\log(x)}+O\left(\frac{1}{\log^m\log(x)}\right),$$

where C_1 is an absolute constant not depending on m.

Let $s\ge 0$ be an integer. We define the function

$$\pounds_s(x):=\prod_{i=0}^{s}\log_i(x),\quad \text{with } \log_i(x)=\underbrace{\log\log\ldots\log}_{i\text{ times}}(x) \text{ and } \log_0(x)=x.$$

For $s=2$, $\pounds_2(x)=x\log(x)\log\log(x)$.

Let f_s be the arithmetical function admitting, for all $m\ge 1$, the following asymptotic formula

$$f_s(n)=\frac{\pounds_s(n)}{\log_{s+1}(n)}\left\{\sum_{i=0}^{m-1}\frac{a_i}{\log_{s+1}^i(n)}+O\left(\frac{1}{\log_{s+1}^m(n)}\right)\right\},\ a_0\ne 0. \tag{3}$$

For $s=0$ and $a_i=i!$, we obtain (1), which corresponds to $\pi(n)$. For $s=1$ with $a_i=i!$, we find (2), which corresponds to $nK(n)$.

Considering the above background, here is our main result:

Theorem 1. *For all integers $m\ge 1$ and $s\ge 0$, we have*

$$\sum_{n\le x}{}'\frac{1}{f_s(n)}=\frac{\delta_0}{2}\log_{s+1}^2(x)+\delta_1\log_{s+1}(x)+\delta_2\log_{s+2}(x)+C_s$$
$$-\frac{\delta_3}{\log_{s+1}(x)}-\cdots-\frac{\delta_{m+1}}{(m-1)\log_{s+1}^{m-1}(x)}+O\left(\frac{1}{\log_{s+1}^m(x)}\right),$$

where $\sum_{n\le x}{}'\frac{1}{f_s(n)}$ *is a sum restricted to integers $e(s)<n\le x$ for which $f_s(n)\ne 0$, C_s is an absolute constant not depending on m, $\{\delta_i\}_i$ is the sequence given by the recurrence relation*

$$a_0\delta_n+a_1\delta_{n-1}+\cdots+a_n\delta_0=0,\ a_0\delta_0=1,$$

and $e(s):=\underbrace{\exp\exp\ldots\exp}_{s\text{ times}}(0)$.

For $a_i=i!$ and $s=0$ and $s=1$, respectively we find the results of A. Ivić [**3**] and H. Belbachir and D. Berkane [**2**].

2. LEMMAS AND PROOF OF THE MAIN RESULT

Let $\{\delta_i\}_i$ be the sequence of real numbers defined by expanding the following expression of the rational function Δ, for $y > 0$ we consider

$$\Delta(y) := \left(\sum_{i=0}^{m} \frac{a_i}{y^{i+1}}\right)\left(\sum_{i=0}^{m+1} \frac{\delta_i}{y^{i-1}}\right),\ m \geq 1,$$

such that $a_0\delta_0 = 1$, and terms with $\frac{1}{y^i}$, $1 \leq i \leq m$ vanish.

Then, when $y \to \infty$, we obtain

$$\Delta = 1 + O\left(\frac{1}{y^{m+1}}\right). \tag{4}$$

Lemma 1. *The coefficient δ_n, $n \geq 1$, is given by the relation*

$$\delta_n = \frac{1}{a_0^{n+1}} \begin{vmatrix} 0 & a_1 & \dots & a_{n-1} & a_n \\ 0 & a_0 & \dots & a_{n-2} & a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & a_0 & a_1 \\ 1 & 0 & \dots & 0 & a_0 \end{vmatrix}.$$

Proof. From the definition of $\Delta(y)$, we notice that the vector $\delta = (\delta_0, ..., \delta_n)$, is the unique solution to the following Cramer's system

$$\begin{cases} a_0\delta_n + a_1\delta_{n-1} + \cdots + a_n\delta_0 = & 0 \\ a_0\delta_{n-1} + \cdots + a_{n-1}\delta_0 = & 0 \\ & \vdots \\ a_0\delta_1 + a_1\delta_0 = & 0 \\ a_0\delta_0 = & 1. \end{cases}$$

□

Lemma 2. *For n sufficiently large, we have*

$$f_s(n) = \frac{\pounds_s(n)}{\delta_0 \log_{s+1}(n) + \delta_1 + \varepsilon(n)},$$

where $\lim_{n\to\infty} \varepsilon(n) = 0$.

Proof. From (3), we have

$$f_s(n) = \pounds_s(n)\left(\sum_{j=0}^{m} \frac{a_j}{\log_{s+1}^{j+1}(n)}\right) + O\left(\frac{\pounds_s(n)}{\log_{s+1}^{m+2}(n)}\right), \tag{5}$$

and from (4) it follows

$$\sum_{j=0}^{m} \frac{a_j}{y^{j+1}} = \frac{1 + O\left(\frac{1}{y^{m+1}}\right)}{\delta_0 y + \sum_{j=1}^{m+1} \frac{\delta_j}{y^{j-1}}} = \frac{1}{\delta_0 y + \sum_{j=1}^{m+1} \frac{\delta_j}{y^{j-1}}} + O\left(\frac{1}{y^{m+2}}\right). \tag{6}$$

The substitution of $y = \log_{s+1}(n)$ in (6) and in relation (5) gives

$$f_s(n) = \frac{\pounds_s(n)}{\delta_0 \log_{s+1}(n) + \delta_1 + \frac{\delta_2}{\log_{s+1}(n)} + \frac{\delta_3}{\log^2_{s+1}(n)} + \cdots + \frac{\delta_{m+1}}{\log^m_{s+1}(n)}} + O\Big(\frac{\pounds_s(n)}{\log^{m+2}_{s+1}(n)}\Big). \tag{7}$$

Thus we can write

$$f_s(n) = \frac{\pounds_s(n)}{\delta_0 \log_{s+1}(n) + \delta_1 + \varepsilon(n)},$$

with $\varepsilon(n) = O\Big(\frac{1}{\log_{s+1}(n)}\Big)$ from which it follows that $\lim_{n\to\infty} \varepsilon(n) = 0$.

The case $s = 0$ and $a_i = i!$, gives the approximation given by L. Panaitopol [**4**],

$$\pi\,(n) = \frac{n}{\log(n) - 1 - \varepsilon\,(n)}.$$

□

Proof of the main result. Simplifying formula (7), we can write for all $m \geq 1$,

$$f_s(n) = \frac{\pounds_s(n)}{\delta_0 \log_{s+1}(n) + \delta_1 + \frac{\delta_2}{\log_{s+1}(n)} + \frac{\delta_3}{\log^2_{s+1}(n)} + \cdots + \frac{\delta_{m+1}\big(1+\varepsilon_m(n)\big)}{\log^m_{s+1}(n)}},$$

with

$$\varepsilon_m(n) \ll_m \frac{1}{\log_{s+1}(n)}.$$

Then, for all $m \geq 1$ and all $n > e(s)$, we obtain

$$\frac{1}{f_s(n)} = \frac{1}{\pounds_s(n)}\Big(\delta_0 \log_{s+1}(n) + \delta_1 + \frac{\delta_2}{\log_{s+1}(n)} + \frac{\delta_3}{\log^2_{s+1}(n)} + \cdots + \frac{\delta_{m+1}\big(1+\varepsilon_m(n)\big)}{\log^m_{s+1}(n)}\Big),$$

and by summation, we obtain

$$\sum_{n\leq x}{}' \frac{1}{f_s(n)} = A_1 + A_2 + A_3 + \sum_{r=2}^{m} B_r + \sum_{e(s)<n\leq x} \frac{\delta_{m+1}\varepsilon_m(n)}{\pounds_s(n)\log^m_{s+1}(n)}, \tag{8}$$

with

$$A_1 = \sum_{e(s)<n\leq x} \frac{\delta_0 \log_{s+1}(n)}{\pounds_s(n)}, \qquad A_2 = \sum_{e(s)<n\leq x} \frac{\delta_1}{\pounds_s(n)},$$

$$A_3 = \sum_{e(s)<n\leq x} \frac{\delta_2}{\pounds_s(n)\log_{s+1}(n)},\ B_r = \sum_{e(s)<n\leq x} \frac{\delta_{r+1}}{\pounds_s(n)\log^r_{s+1}(n)},\ 2 \leq r \leq m.$$

Let us evaluate these sums. First we can notice that the functions involved in the previous sums are all positive and decreasing for a given constant $\omega \geq e(s)$. Let's compose for A_1,

$$\sum_{\lfloor\omega\rfloor<n\leq x} \frac{\log_{s+1}(n)}{\pounds_s(n)} = \int_{\lceil\omega\rceil}^{x} \frac{\log_{s+1}(t)}{\pounds_s(t)}\mathrm{d}t + O\bigg(\frac{\log_{s+1}(x)}{\pounds_s(x)}\bigg).$$

Thus there is a constant α_1 which includes the sum $\displaystyle\sum_{n=2}^{\lfloor\omega\rfloor} \frac{\log_{s+1}(n)}{L_s(n)}$ such that

$$A_1 = \frac{\delta_0}{2}\log^2_{s+1}(x) + \alpha_1 + O\bigg(\frac{\log_{s+1}(x)}{\pounds_s(x)}\bigg).$$

Using similar argument, we also obtain

$$\begin{aligned}
A_2 &= \delta_1 \log_{s+1}(x) + \alpha_2 + O\bigg(\frac{1}{\pounds_s(x)}\bigg),\\
A_3 &= \delta_2 \log_{s+2}(x) + \alpha_3 + O\bigg(\frac{1}{\pounds_s(x)\log_{s+1}(x)}\bigg),\\
B_r &= \frac{-\delta_{r+1}}{(r-1)\log^{r-1}_{s+1}(x)} + \beta_r + O\bigg(\frac{1}{\pounds_s(x)\log^r_{s+1}(x)}\bigg).
\end{aligned}$$

As $\varepsilon_m(n)$ is bounded and the series

$$\sum_{n>e(s)} \frac{1}{\pounds_s(n)\log^m_{s+1}(n)},$$

is convergent for all $m \geq 2$ (Bertrand's series), with the sum noted S_m, we deduce that

$$\sum_{e(s)<n\leq x} \frac{\delta_{m+1}\varepsilon_m(n)}{\pounds_s(n)\log^m_{s+1}(n)} = S_m + O\bigg(\frac{1}{\log^m_{s+1}(x)}\bigg).$$

Putting together the above expression in (8) we infer that

$$\begin{aligned}
{\sum_{n\leq x}}' \frac{1}{f_s(n)} = \frac{\delta_0}{2}&\log^2_{s+1}(x) + \delta_1\log_{s+1}(x) + \delta_2\log_{s+2}(x)\\
&+ \alpha_1 + \alpha_2 + \alpha_3 + \sum_{r=2}^{m}\beta_r + S_m\\
&- \frac{\delta_3}{\log_{s+1}(x)} - \cdots - \frac{\delta_{m+1}}{(m-1)\log^{m-1}_{s+1}(x)} + O\bigg(\frac{1}{\log^m_{s+1}(x)}\bigg).
\end{aligned}$$

Setting $C_s = \alpha_1 + \alpha_2 + \alpha_3 + \displaystyle\sum_{r=2}^{m}\beta_r + S_m$ we find the formula mentioned in the main Theorem. This constant is independent of the value of m because the difference between two developments of $\displaystyle{\sum_{n\leq x}}' \frac{1}{f_s(n)}$ is a quantity which is absorbed by the roundness when $x \to +\infty$. □

Acknowledgments. The authors wish to warmly thank their referee for valuable advice and comments which helped to improve the quality of this paper.

REFERENCES

1. H. Belbachir, F. Bencherif: *Développement asymptotique de la somme des inverses d'une fonction arithmétique.* Ann. Math. Blaise Pascal, **16** (2009), 93–99.
2. H. Belbachir, D. Berkane: *Asymptotic expansion of a sum involving the largest product of primes* (submitted).
3. A. Ivić: *On a sum involving the prime counting function.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., **13** (2002), 85–88.
4. L. Panaitopol: *A formula for $\pi(x)$ applied to a result of Koninck-Ivić.* Nieuw Arch. Wiskd. (5), **1** (1) (2000), 55–56.

USTHB, Faculty of Mathematics,
Po. Box 32, El Alia 16111,
Algiers, Algeria
E-mail: hacenebelbachir@gmail.com, hbelbachir@usthb.dz

(Received October 15, 2010)
(Revised February 6, 2011)

Department of Mathematics,
University of Blida
Po. Box 270,
Route de Soumaa,
Blida, Algeria
E-mail: djameberkan@gmail.com