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A monotone projection algorithm for fixed points of nonlinear operators

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Abstract
In this paper, a monotone projection algorithm is investigated for equilibrium and fixed point problems. Strong convergence theorems for common solutions of the two problems are established in the framework of reflexive Banach spaces.

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Keywords: asymptotically quasi-ϕ-nonexpansive mapping; generalized asymptotically quasi-ϕ-nonexpansive mapping; bifunction; equilibrium problem; fixed point

1 Introduction and preliminaries

Let E be a real Banach space with the dual E^*. Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Let $B_E = \{x \in E : \|x\| = 1\}$ be the unit ball of E. Recall that E is said to be smooth iff $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in B_E$. It is also said to be uniformly smooth iff the above limit is attained uniformly for $x, y \in B_E$. E is said to be strictly convex iff $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex iff $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. It is well known that E is uniformly smooth if and only if E^* is uniformly convex. In what follows, we use $\rightharpoonup$ and $\to$ to stand for weak and strong convergence, respectively. Recall that E enjoys the Kadec-Klee property iff for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property. Let E be a smooth Banach space. Let us consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

Recently, Alber [1] introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection P_C in Hilbert spaces. Recall that the generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem $\phi(\bar{x}, x) = \min_{y\in C} \phi(y, x)$. Existence and uniqueness of the operator Π_C follows from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x,y) = 0$ if and only if $x = y$. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of function ϕ that $(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|y\| + \|x\|)^2$, $\forall x,y \in E$.

Let $\mathbb{R}$ be the set of real numbers. Let F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Let $\varphi : C \to \mathbb{R}$ be a real-valued function and $A : C \to E^*$ be a mapping. The so-called generalized mixed equilibrium problem is to find $p \in C$ such that

$$F(p,y) + \langle Ap, y - p\rangle + \varphi(y) - \varphi(p) \geq 0, \quad \forall y \in C. \tag{1.1}$$

We use $GMEP(F,A,\varphi)$ to denote the solution set of the equilibrium problem. That is,

$$GMEP(F,A,\varphi) := \bigl\{p \in C : F(p,y) + \langle Ap, y - p\rangle + \varphi(y) - \varphi(z) \geq 0, \forall y \in C\bigr\}.$$

Next, we give some special cases:

If $A = 0$, then problem (1.1) is equivalent to finding $p \in C$ such that

$$F(p,y) + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \tag{1.2}$$

which is called the mixed equilibrium problem.

If $F = 0$, then problem (1.1) is equivalent to finding $p \in C$ such that

$$\langle Ap, y - p\rangle + \varphi(y) - \varphi(z) \geq 0, \quad \forall y \in C, \tag{1.3}$$

which is called the mixed variational inequality of Browder type.

If $\varphi = 0$, then problem (1.1) is equivalent to finding $p \in C$ such that

$$F(p,y) + \langle Ap, y - p\rangle \geq 0, \quad \forall y \in C, \tag{1.4}$$

which is called the generalized equilibrium problem.

If $A = 0$ and $\varphi = 0$, then problem (1.1) is equivalent to finding $p \in C$ such that

$$F(p,y) \geq 0, \quad \forall y \in C, \tag{1.5}$$

which is called the equilibrium problem.

For solving the above problem, let us assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$, $\forall x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$, $\forall x,y \in C$;

(A3)

$$\limsup_{t \downarrow 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y), \quad \forall x,y,z \in C;$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

Iterative algorithms have emerged as an effective and powerful tool for studying a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization; see [2–27] and the references therein. The computation of solutions of nonlinear operator equations (inequalities) is important in the study of many real world problems. Recently, the study of the convergence of various iterative algorithms for solving various nonlinear mathematical models forms the major part of numerical mathematics.

Let C be a nonempty subset of E, and let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to stand for the fixed point set of T. Recall that T is said to be asymptotically regular on C iff for any bounded subset K of C, $\limsup_{n\to\infty}\{\|T^{n+1}x - T^n x\| : x \in K\} = 0$. Recall that T is said to be closed iff for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. Recall that a point p in C is said to be an asymptotic fixed point of T iff C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. T is said to be relatively nonexpansive iff $\widetilde{F}(T) = F(T) \neq \emptyset$ and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, \forall p \in F(T).$$

T is said to be relatively asymptotically nonexpansive iff $\widetilde{F}(T) = F(T) \neq \emptyset$ and

$$\phi\big(p, T^n x\big) \le (1 + \mu_n)\phi(p, x), \quad \forall x \in C, \forall p \in F(T), \forall n \ge 1,$$

where $\{\mu_n\} \subset [0, \infty)$ is a sequence such that $\mu_n \to 0$ as $n \to \infty$.

Recall that T is said to be quasi-ϕ-nonexpansive iff $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, \forall p \in F(T).$$

Recall that T is said to be asymptotically quasi-ϕ-nonexpansive iff there exists a sequence $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$F(T) \neq \emptyset, \qquad \phi\big(p, T^n x\big) \le (1 + \mu_n)\phi(p, x), \quad \forall x \in C, \forall p \in F(T), \forall n \ge 1.$$

Remark 1.1 The class of relatively asymptotically nonexpansive mappings, which is an extension of the class of relatively nonexpansive mappings, was first introduced in [28].

Remark 1.2 The class of asymptotically quasi-ϕ-nonexpansive mappings, which is an extension of the class of quasi-ϕ-nonexpansive mappings, was considered in [29–31]. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more general than the class of relatively nonexpansive mappings and the class of relatively asymptotically nonexpansive mappings. Quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive mappings do not require the restriction $F(T) = \widetilde{F}(T)$.

Recall that T is said to be generalized asymptotically quasi-ϕ-nonexpansive iff $F(T) \neq \emptyset$, and there exist two nonnegative sequences $\{\mu_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\{\xi_n\} \subset [0, \infty)$ with $\xi_n \to 0$ as $n \to \infty$ such that

$$\phi\big(p, T^n x\big) \le (1 + \mu_n)\phi(p, x) + \xi_n, \quad \forall x \in C, \forall p \in F(T), \forall n \ge 1.$$

Remark 1.3 The class of generalized asymptotically quasi-ϕ-nonexpansive mappings [32] is a generalization of the class of generalized asymptotically quasi-nonexpansive mappings in the framework of Banach spaces which was studied by Agarwal *et al.* [33].

In this paper, we consider a projection algorithm for a common solution of a family of generalized asymptotically quasi-ϕ-nonexpansive mappings and generalized mixed equilibrium problems. A strong convergence theorem is established in a Banach space. In order to prove our main results, we need the following lemmas.

Lemma 1.4 [21] Let E be a uniformly convex Banach space, and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and

$$\left\| \sum_{i=1}^{\infty} (\alpha_i x_i) \right\|^2 \leq \sum_{i=1}^{\infty} \left(\alpha_i \|x_i\|^2\right) - \alpha_i \alpha_j g\left(\|x_i - x_j\|\right), \quad \forall i, j \in \{1, 2, \ldots, N\}$$

for all $x_1, x_2, \ldots, \in B_r = \{x \in E : \|x\| \leq r\}$ and $\alpha_1, \alpha_2, \ldots, \in [0, 1]$ such that $\sum_{i=1}^{\infty} \alpha_i = 1$.

Lemma 1.5 [1] Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and $x \in E$. Then

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 1.6 [1] Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 1.7 [32] Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a generalized asymptotically quasi-ϕ-nonexpansive mapping. Then $F(T)$ is closed and convex.

Lemma 1.8 [34] Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E. Let $A : C \to E^*$ be a continuous and monotone mapping, let $\varphi : C \to \mathbb{R}$ be convex and lower semi-continuous, and let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that

$$F(z, y) + \langle Az, y - z \rangle + \varphi(y) - \varphi(z) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $T_r : E \to C$ by

$$T_r x = \left\{ z \in C : F(z, y) + \langle Az, y - z \rangle + \varphi(y) - \varphi(z) + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}.$$

Then the following conclusions hold:

(1) T_r is a single-valued firmly nonexpansive-type mapping, *i.e.*, for all $x, y \in E$, $\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle$;

(2) $F(T_r) = GMEP(F, A, \varphi)$ is closed and convex;
(3) T_r is quasi-ϕ-nonexpansive;
(4) $\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x)$, $\forall q \in F(T_r)$.

2 Main results

Theorem 2.1 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and let C be a nonempty closed and convex subset of E. Let Δ be an index set and N be an integer. Let $A_j : C \to E^*$ be a continuous and monotone mapping and $\varphi_j : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let F_j be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *for every $j \in \Delta$. Let T_0 be an identity mapping, and let $T_i : C \to C$ be a generalized asymptotically quasi-ϕ-nonexpansive mapping for every $1 \leq i \leq N$. Assume that T_i is closed asymptotically regular on C and $\Psi := \bigcap_{i=1}^N F(T_i) \cap \bigcap_{j\in\Delta} GMEP(F_j, A_j, \varphi_j)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily}, \\ C_{1,j} = C, \\ C_1 = \bigcap_{j\in\Delta} C_{1,j}, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\sum_{i=0}^N \alpha_{n,i} J T_i^n x_n), \\ u_{n,j} \in C \text{ such that } F_j(u_{n,j}, y) + \langle A_j u_{n,j}, y - u_{n,j}\rangle + \varphi_j(y) - \varphi_j(u_{n,j}) \\ \qquad + \frac{1}{r_{n,j}} \langle y - u_{n,j}, J u_{n,j} - J y_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1,j} = \{z \in C_n : \phi(z, u_{n,j}) \leq \phi(z, x_n) + \sum_{i=1}^N \mu_{n,i} M_n + N\xi_n\}, \\ C_{n+1} = \bigcap_{J\in\Delta} C_{n+1,J}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\{\alpha_{n,i}\}$ is a real number sequence in $(0,1)$ for every $i \leq 1$, $\{r_{n,j}\}$ is a real number sequence in $[r, \infty)$, where r is some positive real number, and $M_n = \sup\{\phi(z, x_n) : z \in \Psi\}$. Assume that $\sum_{i=0}^N \alpha_{n,i} = 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ for every $1 \leq i \leq N$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_\Psi x_0$, where Π_Ψ is the generalized projection from E onto Ψ.

Proof The proof is split into five steps.

Step 1. Show that the common solution set Ψ is convex and closed.

This step is clear in view of Lemma 1.7 and Lemma 1.8.

Step 2. Show that the set C_n is convex and closed.

To show Step 2, it suffices to show, for any fixed but arbitrary $i \in \Delta$, that $C_{n,i}$ is convex and closed. This can be proved by induction. It is clear that $C_{1,j} = C$ is convex and closed. Assume that $C_{m,j}$ is closed and convex for some $m \geq 1$. We next prove that $C_{m+1,j}$ is convex and closed. It is clear that $C_{m+1,j}$ is closed. We only prove they are convex. Indeed, $\forall x, y \in C_{m+1,j}$, we find that $x, y \in C_{m,j}$, and

$$\phi(x, u_{m,j}) \leq \phi(x, x_m) + \sum_{i=1}^N \mu_{n,i} M_n + N\xi_n,$$

and

$$\phi(y, u_{m,j}) \le \phi(y, x_m) + \sum_{i=1}^{N} \mu_{n,i} M_n + N\xi_n.$$

Notice that the above two inequalities are equivalent to the following inequalities, respectively:

$$2\langle x, Jx_m - Ju_{m,j}\rangle \le \|x_m\|^2 - \|u_{m,j}\|^2 + \sum_{i=1}^{N} \mu_{n,i} M_n + N\xi_n$$

and

$$2\langle y, Jx_m - Ju_{m,j}\rangle \le \|x_m\|^2 - \|u_{m,j}\|^2 + \sum_{i=1}^{N} \mu_{n,i} M_n + N\xi_n.$$

These imply that

$$2\bigl\langle ax + (1-a)y, Jx_m - Ju_{m,j}\bigr\rangle \le \|x_m\|^2 - \|u_{m,j}\|^2 + \sum_{i=1}^{N} \mu_{n,i} M_n + N\xi_n, \quad \forall a \in (0,1).$$

Since $C_{m,j}$ is convex, we see that $ax + (1-a)y \in C_{m,j}$. Notice that the above inequality is equivalent to

$$\phi\bigl(ax + (1-a)y, u_{m,j}\bigr) \le \phi\bigl(ax + (1-a)y, x_m\bigr) + \sum_{i=1}^{N} \mu_{n,i} M_n + N\xi_n.$$

This proves that $C_{m+1,j}$ is convex. This proves that C_n is closed and convex. This completes Step 2.

Step 3. Show that $\Psi \subset C_n$.

It suffices to claim that $\Psi \subset C_{n,j}$ for every $j \in \Delta$. Note that $\Psi \subset C_{1,j} = C$. Suppose that $\Psi \subset C_{m,j}$ for some m and for every $j \in \Delta$. Then, for $\forall z \in \Psi \subset C_{m,j}$, we have

$$\begin{aligned}
\phi(z, u_{m,j}) &= \phi(z, T_{r_{m,j}} y_m) \\
&\le \phi(z, y_m) \\
&= \phi\left(z, J^{-1}\left(\alpha_{m,0} Jx_m + \sum_{i=1}^{N} \alpha_{m,i} JT_i^m x_m\right)\right) \\
&= \|z\|^2 - 2\left\langle z, \alpha_{m,0} Jx_m + \sum_{i=1}^{N} \alpha_{m,i} JT_i^m x_m\right\rangle + \left\|\alpha_{m,0} Jx_m + \sum_{i=1}^{N} \alpha_{m,i} JT_i^m x_m\right\|^2 \\
&\le \|z\|^2 - 2\alpha_{m,0}\langle z, Jx_m\rangle - 2\sum_{i=1}^{N} \alpha_{m,i}\bigl\langle z, JT_i^m x_m\bigr\rangle \\
&\quad + \alpha_{m,0}\|x_m\|^2 + \sum_{i=1}^{N} \alpha_{m,i}\bigl\|T_i^m x_m\bigr\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_{m,0}\phi(z,x_m) + \sum_{i=1}^{N} \alpha_{m,i}\phi\bigl(z, T_i^m x_m\bigr) \\
&\leq \alpha_{m,0}\phi(z,x_m) + \sum_{i=1}^{N} \alpha_{m,i}\phi(z,x_m) + \sum_{i=1}^{N} \alpha_{m,i}\mu_{m,i}\phi(z,x_m) + \sum_{i=1}^{N} \alpha_{m,i}\xi_m \\
&\leq \phi(z,x_m) + \sum_{i=1}^{N} \mu_{m,i}\phi(z,x_m) + \sum_{i=1}^{N} \alpha_{m,i}\xi_m \\
&\leq \phi(z,x_m) + \sum_{i=1}^{N} \mu_{m,i}M_m + \sum_{i=1}^{N} \alpha_{m,i}\xi_m \\
&\leq \phi(z,x_m) + \sum_{i=1}^{N} \mu_{m,i}M_m + N\xi_m,
\end{aligned}
\tag{2.1}
$$

which proves that $z \in C_{m+1,j}$. This completes Step 3.

Step 4. Show that $x_n \to p$, where $p \in \Psi$.

In view of Lemma 1.5, we find that $\phi(x_n,x_0) \leq \phi(w,x_0) - \phi(w,x_n) \leq \phi(w,x_0)$ for $\forall w \in \Psi \subset C_n$. This shows that the sequence $\phi(x_n,x_0)$ is bounded. It follows that $\{x_n\}$ is also bounded. Since the framework of the space is reflexive, we may, without loss of generality, assume that $x_n \rightharpoonup p$, where $p \in C_n$. Note that $\phi(x_n,x_0) \leq \phi(p,x_0)$. It follows that

$$
\phi(p,x_0) \leq \liminf_{n\to\infty} \phi(x_n,x_0) \leq \limsup_{n\to\infty} \phi(x_n,x_0) \leq \phi(p,x_0).
$$

This gives that $\lim_{n\to\infty} \phi(x_n,x_0) = \phi(p,x_0)$. Hence, we have $\lim_{n\to\infty} \|x_n\| = \|p\|$. Since the space E enjoys the Kadec-Klee property, we find that $x_n \to p$ as $n \to \infty$.

Now, we are in a position to show that $p \in \bigcap_{j\in\Delta} GMEP(F_j, A_j, \varphi_j)$. By the construction of C_n, we have that $C_{n+1} \subset C_n$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_n$. It follows that

$$
\begin{aligned}
\phi(x_{n+1},x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_0) \\
&\leq \phi(x_{n+1},x_0) - \phi(\Pi_{C_n} x_0, x_0) \\
&= \phi(x_{n+1},x_0) - \phi(x_n,x_0).
\end{aligned}
$$

Letting $n \to \infty$, we obtain that $\phi(x_{n+1},x_n) \to 0$. In view of $x_{n+1} \in C_{n+1}$, we see that

$$
\phi(x_{n+1},u_{n,j}) \leq \phi(x_{n+1},x_n) + \sum_{i=1}^{N} \mu_{n,i}M_n + N\xi_n.
$$

We, therefore, obtain that $\lim_{n\to\infty} \phi(x_{n+1},u_{n,j}) = 0$. It follows that $\lim_{n\to\infty} \|u_{n,j}\| = \|p\|$. It follows that $\lim_{n\to\infty} \|Ju_{n,j}\| = \|Jp\|$. This implies that $\{Ju_{n,j}\}$ is bounded. Note that E is reflexive and E^* is also reflexive. We may assume that $Ju_{n,j} \rightharpoonup u^{*,j} \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists $u^j \in E$ such that $Ju^j = u^{*,j}$. It follows that $\phi(x_{n+1},u_n) = \|x_{n+1}\|^2 - 2\langle x_{n+1}, Ju_n\rangle + \|Ju_n\|^2$. Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields that

$$\begin{aligned}0 &\geq \|p\|^2 - 2\langle p, u^{*,j}\rangle + \|u^{*,j}\|^2 \\ &= \|p\|^2 - 2\langle p, Ju^j\rangle + \|Ju^j\|^2 \\ &= \|p\|^2 - 2\langle p, Ju^j\rangle + \|u^j\|^2 \\ &= \phi(p, u^j).\end{aligned}$$

That is, $p = u^j$, which in turn implies that $Jp = u^{*,j}$. It follows that $Ju_{n,j} \rightharpoonup Jp \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain that $Ju_{n,j} - Jp \to 0$ as $n \to \infty$. Since $J^{-1} : E^* \to E$ is demicontinuous, it follows that $u_{n,j} \rightharpoonup p$. Since E enjoys the Kadec-Klee property, we obtain that $u_{n,j} \to p$ as $n \to \infty$. Note that $\|x_n - u_{n,j}\| \leq \|x_n - p\| + \|p - u_{n,j}\|$. This gives that

$$\lim_{n\to\infty} \|x_n - u_{n,j}\| = 0. \tag{2.2}$$

Since J is uniformly norm-to-norm continuous on any bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_{n,j}\| = 0. \tag{2.3}$$

Notice that

$$\begin{aligned}\phi(z, x_n) - \phi(z, u_{n,j}) &= \|x_n\|^2 - \|u_{n,j}\|^2 - 2\langle z, Jx_n - Ju_{n,j}\rangle \\ &\leq \|x_n - u_{n,j}\|\big(\|x_n\| + \|u_{n,j}\|\big) + 2\|z\|\|Jx_n - Ju_{n,j}\|.\end{aligned}$$

It follows from (2.2) and (2.3) that

$$\lim_{n\to\infty} \phi(z, x_n) - \phi(z, u_{n,j}) = 0. \tag{2.4}$$

From (2.1), we find that $\phi(z, y_n) \leq \phi(z, x_n) + \sum_{i=1}^{N} \mu_{n,j} M_n + N\xi_n$, where $z \in \Psi$. In view of $u_{n,j} = S_{r_{n,j}} y_n$, we find from Lemma 1.8 that

$$\begin{aligned}\phi(u_{n,j}, y_n) &= \phi(S_{r_{n,j}} y_n, y_n) \\ &\leq \phi(z, y_n) - \phi(z, S_{r_{n,j}} y_n) \\ &\leq \phi(z, x_n) - \phi(z, S_{r_{n,j}} y_n) + \sum_{i=1}^{N} \mu_{n,j} M_n + N\xi_n \\ &= \phi(z, x_n) - \phi(z, u_{n,j}) + \sum_{i=1}^{N} \mu_{n,j} M_n + N\xi_n.\end{aligned}$$

From (2.4), we obtain that

$$\lim_{n\to\infty} \phi(u_{n,j}, y_n) = 0.$$

This implies that $\|u_{n,j}\| - \|y_n\| \to 0$ as $n \to \infty$. Since $u_{n,j} \to p$ as $n \to \infty$, we arrive at $\lim_{n\to\infty} \|y_n\| = \|p\|$. It follows that $\lim_{n\to\infty} \|Jy_n\| = \|Jp\|$. Since E^* is also reflexive, we may assume that $Jy_n \rightharpoonup y^* \in E^*$. In view of $J(E) = E^*$, we see that there exists $y \in E$ such that $Jy = y^*$. It follows that

$$\phi(u_{n,j}, y_n) = \|u_{n,j}\|^2 - 2\langle u_{n,j}, Jy_n\rangle + \|Jy_n\|^2.$$

Taking $\liminf_{n\to\infty}$ on the both sides of the equality above yields that $0 \ge \phi(p, y)$. That is, $p = y$, which in turn implies that $y^* = Jp$. It follows that $Jy_n \rightharpoonup Jp \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain that $Jy_n - Jp \to 0$ as $n \to \infty$. Note that $J^{-1} : E^* \to E$ is demicontinuous. It follows that $y_n \rightharpoonup p$. Since E enjoys the Kadec-Klee property, we obtain that $y_n \to p$ as $n \to \infty$. Since $\|u_{n,j} - y_n\| \le \|u_{n,j} - p\| + \|p - y_n\|$, we find that $\lim_{n\to\infty} \|u_{n,i} - y_n\| = 0$. Since J is uniformly norm-to-norm continuous on any bounded sets, we have $\lim_{n\to\infty} \|Ju_{n,j} - Jy_n\| = 0$. From the assumption $r_{n,i} \ge r$, we see that $\lim_{n\to\infty} \frac{\|Ju_{n,j}-Jy_n\|}{r_{n,j}} = 0$. Notice that

$$f_j(u_{n,j}, y) + \frac{1}{r_{n,j}}\langle y - u_{n,j}, Ju_{n,j} - Jy_n\rangle \ge 0, \quad \forall y \in C,$$

where $f_j(u_{n,j}, y) = F_j(u_{n,j}, y) + \langle A_j u_{n,j}, y - u_{n,j}\rangle + \varphi_j(y) - \varphi_j(u_{n,j})$. From (A2), we find that

$$\|y - u_{n,j}\| \frac{\|Ju_{n,j} - Jy_n\|}{r_{n,j}} \ge \frac{1}{r_{n,j}}\langle y - u_{n,j}, Ju_{n,j} - Jy_n\rangle \ge f_j(y, u_{n,j}), \quad \forall y \in C.$$

Taking the limit as $n \to \infty$, we find that $f_j(y, p) \le 0$, $\forall y \in C$. For $0 < t_j < 1$ and $y \in C$, define $y_{t_j} = t_j y + (1 - t_j)p$. It follows that $y_{t,j} \in C$, which yields that $f_j(y_{t,j}, p) \le 0$. It follows from conditions (A1) and (A4) that $0 = f_j(y_{t,j}, y_{t,j}) \le t_j f_j(y_{t,j}, y) + (1 - t_j) f_j(y_{t,j}, p) \le t_j f_j(y_{t,j}, y)$. This yields that $f_j(y_{t,j}, y) \ge 0$. Letting $t_j \downarrow 0$, we find from condition (A3) that $f_j(p, y) \ge 0$, $\forall y \in C$. This implies that $p \in EP(f_j) = GMEP(F_j, A_j, \varphi_j)$ for every $j \in \Delta$.

Next, we state $p \in \bigcap_{i=1}^N F(T_i)$. Since E is uniformly smooth, we know that E^* is uniformly convex. It follows from Lemma 1.4 that

$$\begin{aligned}
\phi(z, u_{n,j}) &= \phi(z, S_{r_{n,j}} y_n) \\
&\le \phi(z, y_n) \\
&= \phi\left(z, J^{-1}\left(\alpha_{n,0} Jx_n + \sum_{i=1}^N \alpha_{n,i} JT_i^n x_n\right)\right) \\
&= \|z\|^2 - 2\left\langle z, \alpha_{n,0} Jx_n + \sum_{i=1}^N \alpha_{n,i} JT_i^n x_n\right\rangle + \left\|\alpha_{n,0} Jx_n + \sum_{i=1}^N \alpha_{n,i} JT_i^n x_n\right\|^2 \\
&\le \|z\|^2 - 2\alpha_{n,0}\langle z, Jx_n\rangle - 2\sum_{i=1}^N \alpha_{n,i}\langle z, JT_i^n x_n\rangle \\
&\quad + \alpha_{n,0}\|x_n\|^2 + \sum_{i=1}^N \alpha_{n,i}\|T_i^n x_n\|^2 - \alpha_{n,0}(1 - \alpha_{n,i}) g\big(\|Jx_n - JT_i^n x_n\|\big) \\
&= \alpha_{n,0}\phi(z, x_m) + \sum_{i=1}^N \alpha_{n,i}\phi\big(z, T_i^n x_n\big) - \alpha_{n,0}(1 - \alpha_{n,i}) g\big(\|Jx_n - JT_i^n x_n\|\big) \\
&\le \alpha_{n,0}\phi(z, x_m) + \sum_{i=1}^N \alpha_{n,i}\phi(z, x_m) + \sum_{i=1}^N \alpha_{n,i}\mu_{n,i}\phi(z, x_n) + \sum_{i=1}^N \alpha_{n,i}\xi_n
\end{aligned}$$

$$
\begin{aligned}
&\quad - \alpha_{n,0}(1-\alpha_{n,i})g\left(\left\|Jx_n - JT_i^n x_n\right\|\right) \\
&\leq \phi(z,x_n) + \sum_{i=1}^{N} \mu_{n,i}\phi(z,x_m) + \sum_{i=1}^{N} \alpha_{n,i}\xi_n \\
&\quad - \alpha_{n,0}(1-\alpha_{n,i})g\left(\left\|Jx_n - JT_i^n x_n\right\|\right) \\
&\leq \phi(z,x_n) + \sum_{i=1}^{N} \mu_{n,i}M_n + N\xi_n - \alpha_{n,0}(1-\alpha_{n,i})g\left(\left\|Jx_n - JT_i^n x_n\right\|\right).
\end{aligned}
$$

This yields that

$$
\alpha_{n,0}(1-\alpha_{n,i})g\left(\left\|Jx_n - JT_i^n x_n\right\|\right) \leq \phi(z,x_n) - \phi(z,u_{n,j}) + \sum_{i=1}^{N} \mu_{n,i}M_n + N\xi_n.
$$

In view of $\liminf_{n\to\infty} \alpha_{n,0}(1-\alpha_{n,i}) > 0$, we see from (2.4) that $\lim_{n\to\infty} g(\|Jx_n - JT_i^n x_n\|) = 0$ It follows from the property of g that

$$
\lim_{n\to\infty} \left\|Jx_n - JT_i^n x_n\right\| = 0. \tag{2.5}
$$

Since $x_n \to p$ as $n \to \infty$ and $J : E \to E^*$ is demicontinuous, we obtain that $Jx_n \rightharpoonup Jp \in E^*$. Note that $|\|Jx_n\| - \|Jp\|| = |\|x_n\| - \|p\|| \leq \|x_n - p\|$. This implies that $\|Jx_n\| \to \|Jp\|$ as $n \to \infty$. Since E^* enjoys the Kadec-Klee property, we see that

$$
\lim_{n\to\infty} \|Jx_n - Jp\| = 0. \tag{2.6}
$$

On the other hand, we have $\|JT_i^n x_n - Jp\| \leq \|JT_i^n x_n - Jx_n\| + \|Jx_n - Jp\|$. Combining (2.5) with (2.6), one obtains that $\lim_{n\to\infty} \|JT_i^n x_n - Jp\| = 0$. Since $J^{-1} : E^* \to E$ is demicontinuous, one sees that $T_i^n x_n \rightharpoonup p$. Notice that $|\|T_i^n x_n\| - \|p\|| \leq \|JT_i^n x_n - Jp\|$. This yields that $\lim_{n\to\infty} \|T_i^n x_n\| = \|p\|$. Since the space E enjoys the Kadec-Klee property, we obtain that $\lim_{n\to\infty} \|T_i^n x_n - p\| = 0$. Note that $\|T^{n+1}x_n - p\| \leq \|T^{n+1}x_n - T^n x_n\| + \|T^n x_n - p\|$. Since T is asymptotically regular, we find that $\lim_{n\to\infty} \|T_i^{n+1}x_n - p\| = 0$. That is, $T_i T_i^n x_n - p \to 0$ as $n \to \infty$. It follows from the closedness of T_i that $T_i p = p$ for every $i \in \{1, 2, \ldots, N\}$. This completes Step 4.

Step 5. Show that $p = \Pi_\Psi x_0$.

Since $x_n = \Pi_{C_n} x_0$, we see that

$$
\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall z \in C_n.
$$

Since $\Psi \subset C_n$, we find that

$$
\langle x_n - w, Jx_0 - Jx_n\rangle \geq 0, \quad \forall w \in \Psi.
$$

Letting $n \to \infty$, we arrive at

$$
\langle p - w, Jx_0 - Jp\rangle \geq 0, \quad \forall w \in \Psi.
$$

From Lemma 1.6, we can immediately obtain that $p = \Pi_\Psi x_0$. This completes the proof. □

Remark 2.2 Theorem 2.1 mainly improves the corresponding results in Kim [20], Yang *et al.* [21], Hao [23], Qin *et al.* [31], Qin *et al.* [35].

Remark 2.3 The framework of the space in Theorem 2.1 can be applicable to L^p, $p \geq 1$.

If $N = 2$ and $\Delta = \{1\}$, then Theorem 2.1 is reduced to the following.

Corollary 2.4 *Let E be a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property, and let C be a nonempty closed and convex subset of E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $T_i : C \to C$ be a generalized asymptotically quasi-ϕ-nonexpansive mapping for every $i \in \{1, 2\}$. Assume that each T_i is closed asymptotically regular on C and $F(T_1) \cap F(T_2) \cap EP(F)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E, \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}(\alpha_{n,0} J x_n + \alpha_{n,1} J T_1^n x_n + \alpha_{n,2} J T_2^n x_n), \\ u_n \in C \text{ such that } F(u_n, y) + \frac{1}{r_n} \langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + (\mu_{n,1} + \mu_{n,2}) M_n + 2\xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where $\{\alpha_{n,0}\}$, $\{\alpha_{n,1}\}$, and $\{\alpha_{n,2}\}$ are real number sequences in $(0,1)$, $\{r_n\}$ is a real number sequence in $[r, \infty)$, where r is some positive real number, and $M_n = \sup\{\phi(z, x_n) : z \in F(T_1) \cap F(T_2) \cap EF(F)\}$. Assume that $\sum_{i=0}^{2} \alpha_{n,i} = 1$ and $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T_1)\cap F(T_2)\cap EP(F)} x_0$, where $\Pi_{F(T_1)\cap F(T_2)\bigcap EF(F)}$ is the generalized projection from E onto $F(T_1) \cap F(T_2) \cap EP(F)$.

Remark 2.5 Corollary 2.4 mainly improves the corresponding results in Qin *et al.* [31]. To be more clear, the mapping is extended from quasi-ϕ-nonexpansive mappings to generalized asymptotically quasi-ϕ-nonexpansive mappings and the framework of spaces is extended from a uniformly smooth and uniformly convex Banach space to a uniformly smooth and strictly convex Banach space.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this manuscript. Both authors read and approved the final manuscript.

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