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Functional type Caristi-Kirk theorem on two metric spaces and applications

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Abstract

In this paper, we give some generalizations of the functional type Caristi-Kirk theorem (see Functional Type Caristi-Kirk Theorems, 2005) for two mappings on metric spaces. We investigate the existence of some fixed points for two simultaneous projections to find the optimal solutions of the proximity two functions via Caristi-Kirk fixed point theorem.

MSC: Primary 47H10; secondary 54H25

Keywords: complete metric space; Caristi-Kirk fixed point theorem; locally bounded function; ordered metric space; maximal element; simultaneous projection

1 Introduction

Recall that a real-valued function ϕ defined on a metric space X is said to be lower (upper) semi-continuous if for any sequence $(x_n)_n$ of X which converges to $x \in X$, we have $\phi(x) \leq \liminf_n \phi(x_n)$ ($\phi(x) \geq \limsup_n \phi(x_n)$).

In 1976, Caristi (see [2]) obtained the following fixed point theorem on complete metric spaces, known as the Caristi fixed point theorem.

Theorem 1.1 *Let (X,d) be a complete metric space, $T : X \to X$ be a mapping and $\psi : X \to \mathbb{R}^+$ be a lower semi-continuous function such that, for all $x \in X$,*

$$d(x, Tx) \leq \psi(x) - \psi(Tx). \tag{1}$$

Then T has a fixed point in X.

Let M be a nonempty set partially ordered by $\leq$. We will say that $x \in M$ is a maximal element of M if and only if ($x \leq y, y \in M \Rightarrow x = y$).

Theorem 1.2 (I. Ekeland [3]) *Let (X,d) be a complete metric space and $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous function. Define a relation $\leq$ by for all $x, y \in X$,*

$$x \leq y \quad \Leftrightarrow \quad d(x,y) \leq \phi(x) - \phi(y), \quad (x,y) \in X^2.$$

Then $(X, \leq)$ is partially ordered and it has a maximal element.

It is noted that Theorems 1.1 and 1.2 are equivalent.

In 1994, Bae, Cho, and Yeom (see [4]) proved some functional versions of the Caristi-Kirk fixed point theorem; each of these including Theorem 1.1 as a particular case.

Let $c : \mathbb{R}^+ \to \mathbb{R}^+$ be some function. Denote, for $\alpha \in \mathbb{R}^+$,

$$\liminf_{t\to\alpha^+} c(t) = \sup_{\varepsilon>0} \inf c\bigl([\alpha, \alpha + \varepsilon]\bigr), \qquad \limsup_{t\to\alpha^+} c(t) = \inf_{\varepsilon>0} \sup c\bigl([\alpha, \alpha + \varepsilon]\bigr).$$

Clearly $\liminf_{t\to\alpha^+} c(t) \le c(\alpha) \le \limsup_{t\to\alpha^+} c(t)$. And we say that c is right lower (upper) semi-continuous at α if $\liminf_{t\to\alpha^+} c(t) = c(\alpha)$ ($\limsup_{t\to\alpha^+} c(t) = c(\alpha)$).

We obtain a sequential characterization of these local properties:

Proposition 1.3 *c is right lower (upper) semi-continuous at α if and only if for all sequence $(t_n)_n$ such that $t_n \to \alpha$ and $t_n \ge \alpha$ for all n, we have:*

$$\phi(\alpha) \le \liminf_n \phi(t_n) \quad \Bigl(\phi(\alpha) \ge \liminf_n \phi(t_n)\Bigr).$$

Proposition 1.4 *If c is right lower (upper) semi-continuous at α then it is right locally bounded below (above) at α: $\exists\lambda = \lambda(\alpha) > 0$, such that $\inf(c([\alpha, \alpha + \lambda])) > -\infty$ ($\sup(c([\alpha, \alpha + \lambda])) < \infty$).*

Theorem 1.5 (see [4]) *Let $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous function and $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a upper semi-continuous function from the right such that, for all $x \in X$,*

$$d(x, Tx) \le \max\bigl\{c\bigl(\phi(x)\bigr), c\bigl(\phi(Tx)\bigr)\bigr\}\bigl[\phi(x) - \phi(Tx)\bigr].$$

Then T has a fixed point in X.

If $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, let us consider the functional Caristi-Kirk type contraction

$$d(x, Tx) \le H\bigl(c\bigl(\phi(x)\bigr), c\bigl(\phi(Tx)\bigr)\bigr)\bigl[\phi(x) - \phi(Tx)\bigr]. \tag{2}$$

Theorem 1.6 *Let $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous function. If $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a right locally bounded from above and $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a locally bounded function such that, for all $x \in X$,*

$$d(x, Tx) \le H\bigl(c\bigl(\phi(x)\bigr), c\bigl(\phi(Tx)\bigr)\bigr)\bigl[\phi(x) - \phi(Tx)\bigr].$$

Then T has at least one fixed point in X.

For $H(s, t) = s$, we obtain the following.

Theorem 1.7 (see [5]) *Let $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous function and $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a right locally bounded from above such that, for all $x \in X$,*

$$d(x, Tx) \le c\bigl(\phi(x)\bigr)\bigl[\phi(x) - \phi(Tx)\bigr].$$

Then T has at least one fixed point in X.

The following definitions (see [6]) will be needed.

Let H be a Hilbert space, C_i be a nonempty closed convex subset of H where $i \in I = \{1,\dots,m\}$,

$$\Delta_m = \{u = (u_1,\dots,u_m) \in \mathbb{R}^m; u_i \geq 0, \forall i \text{ and } u_1 + \cdots + u_m = 1\}$$

and $P_{C_i} : H \to C_i$, $1 \leq i \leq m$, the metric projection onto C_i.

Definition 1.8 (A Cegielski [6])

1. The operator $T = \sum_{i\in I} w_i P_{C_i}$, where $(w_1,\dots,w_m) \in \Delta_m$ and $I = \{1,\dots,m\}$, is called a simultaneous projection.
2. The function $f : H \to \mathbb{R}^+$ defined by

$$f(x) = \frac{1}{2}\sum_{i\in I} w_i \|P_{C_i}x - x\|^2, \quad x \in H, \tag{3}$$

   called the proximity function.
3. The set defined by

$$\operatorname*{Argmin}_{x\in C} f(x) = \{z \in C; f(z) \leq f(x) \text{ for all } x \in C\},$$

   where $C \subset H$ and $f : C \to \mathbb{R}$, is called a subset of minimizers of f.

The set of all fixed points of self mapping T of a metric space X will be denoted by $\operatorname{Fix}(T)$.

Recently, Farskid Khojasteh and Erdal Karapinar (see [7]) proved the following result.

Theorem 1.9 *Let $T = \sum_{i\in I} w_i P_{C_i}$ be a simultaneous projection, where $w \in \Delta_m$ and a proximity function $f : H \to \mathbb{R}$ defined by equation* (3).

Then we have

$$\operatorname{Fix}(T) = \operatorname*{Argmin}_{x\in H} f(x).$$

Moreover, if $\|x - Tx\| \geq 1$, for all $x \in K$, where

$$K = \bigcap_{n=1}^{\infty} \{x \in H; T^{n+1}x \neq T^n x\},$$

then $\operatorname{Fix}(T) \neq \emptyset$.

2 Main results

We prove a functional version of Caristi-Kirk theorem for two pairs of mappings on metric spaces.

Theorem 2.1 *Let (X,d) be a complete metric space, $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous function and $T, S : X \to X$ two mappings such that, for all $x \in X$,*

$$\begin{cases} d(x,Sx) \leq H(c(\phi(x)), c(\phi(Tx)))(\phi(x) - \phi(Tx)), \\ d(x,Tx) \leq H(c(\phi(x)), c(\phi(Sx)))(\phi(x) - \phi(Sx)), \end{cases} \tag{4}$$

where $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a right locally bounded from above and H a locally bounded function from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{R}^+$. Then there exists an element $x^* \in X$ such that $Tx^* = x^* = Sx^*$.

Proof *First step*. Let $\alpha = \inf \phi(X)$; as c is locally bounded from above, there exists $\lambda = \lambda(\alpha) > 0$ such that $\mu = \sup c([\alpha, \alpha + \lambda]) < \infty$. It follows that there exists $\nu = \nu(\mu) > 0$ such that $H(t, s) \leq \nu$ for all $s, t \in [0, \mu]$.

For some $x_0 \in X$ such that $\alpha \leq \phi(x_0) \leq \alpha + \lambda$, we define the set X_0 by

$$X_0 = \{x \in X; \phi(x) \leq \phi(x_0)\};$$

X_0 is a nonempty closed subset of X. By (4), we have

$$\phi(Tx) \leq \phi(x) \leq \phi(x_0) \quad \text{and} \quad \phi(Sx) \leq \phi(x) \leq \phi(x_0)$$

for all $x \in X_0$; and consequently, $T(X_0) \subset X_0$ and $S(X_0) \subset X_0$. And since $\phi(x), \phi(Tx), \phi(Sx) \in [\alpha, \alpha + \lambda]$, for all $x \in X_0$, we obtain

$$c(\phi(x)), c(\phi(Tx)), c(\phi(Sx)) \leq \mu;$$

and then

$$\max(H(c(\phi(x)), c(\phi(Tx)), H(c(\phi(x)), c(\phi(Sx))))) \leq \nu.$$

Second step. We define a partial order $\leq$ on X_0 as follows: for $x, y \in X_0$

$$x \leq y \quad \Leftrightarrow \quad d(x, y) \leq \nu(\phi(x) - \phi(y)).$$

Since ϕ is lower semi-continuous function on the complete metric space (X_0, d), we see by the Ekeland theorem (see [3]) that $(X_0, \leq)$ has a maximal element x^* such that

$$\begin{cases} d(x^*, Sx^*) \leq \nu(\phi(x^*) - \phi(Tx^*)), \\ d(x^*, Tx^*) \leq \nu(\phi(x^*) - \phi(Sx^*)). \end{cases}$$

If $\phi(Sx^*) \leq \phi(Tx^*)$, we obtain $d(x^*, Sx^*) \leq \nu(\phi(x^*) - \phi(Sx^*))$; then $x^* \leq Sx^*$, which implies $Sx^* = x^*$ and $Tx^* = x^*$.

The same conclusion holds in the case $\phi(Tx^*) \leq \phi(Sx^*)$. □

Corollary 2.2 *Let (X, d) be a complete metric space, $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous function and $T, S : X \to X$ two mappings such that, for all $x \in X$,*

$$\begin{cases} d(x, Sx) \leq c(\phi(x))[\phi(x) - \phi(Tx)], \\ d(x, Tx) \leq c(\phi(x))[\phi(x) - \phi(Sx)], \end{cases}$$

where $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a right locally bounded from above. Then there exists an element $x^* \in X$ such that $Tx^* = x^* = Sx^*$.

Corollary 2.3 *Let (X,d) be a complete metric space, $\phi : X \to \mathbb{R}^+$ a lower semi-continuous function and $T, S : X \to X$ two mappings such that, for all $x \in X$,*

$$\begin{cases} d(x,Sx) \leq \phi(x) - \phi(Tx), \\ d(x,Tx) \leq \phi(x) - \phi(Sx). \end{cases}$$

Then there exists an element $x^* \in X$ such that $Tx^* = x^* = Sx^*$.

Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be locally bounded above in the sense that g is bounded above on each $[0,a]$, $(a > 0)$.

Corollary 2.4 *Let (X,d) be a complete metric space, $\phi : X \to \mathbb{R}^+$ be a lower semi-continuous function and $T, S : X \to X$ two mappings such that, for all $x \in X$,*

$$\begin{cases} d(x,Sx) \leq \min\{\phi(x), g(d(x,Tx))(\phi(x) - \phi(Tx))\}, \\ d(x,Tx) \leq \min\{\phi(x), g(d(x,Sx))(\phi(x) - \phi(Sx))\}. \end{cases} \tag{5}$$

Then there exists an element $x^* \in X$ such that $Tx^* = x^* = Sx^*$.

Proof We define a function c on $\mathbb{R}^+$ by $\forall t \in \mathbb{R}^+$, $c(t) = \sup g([0,t])$. c is increasing and then it is right locally bounded above. By (5), we have, for all $x \in X$,

$$\begin{cases} g(d(x,Tx)) \leq c(d(x,Tx)) \leq c(\phi(x)), \\ g(d(x,Sx)) \leq c(d(x,Sx)) \leq c(\phi(x)), \end{cases}$$

which implies

$$\begin{cases} d(x,Sx) \leq c(\phi(x))[\phi(x) - \phi(Tx)], \\ d(x,Tx) \leq c(\phi(x))[\phi(x) - \phi(Sx)], \end{cases}$$

for all $x \in X$. By Corollary 2.2, T and S have a common fixed point. □

Theorem 2.5 *Let (X,d) be a complete metric space, $\phi, \psi : X \to \mathbb{R}^+$ be a lower semi-continuous functions and $T, S : X \to X$ two continuous mappings such that, for all $x \in X$,*

$$\begin{cases} d(x,Sx) \leq H(c((\phi + \psi)(x)), c((\phi + \psi)(Tx)))(\psi(x) - \phi(Tx)), \\ d(x,Tx) \leq H(c((\phi + \psi)(x)), c((\phi + \psi)(Sx)))(\phi(x) - \psi(Sx)), \end{cases} \tag{6}$$

where $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a right locally bounded from above and $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ be a locally bounded function.

Assume that there exists $x_0 \in X$ such that $\psi(Tx_0) \leq \psi(Sx_0)$ and $\phi(Sx_0) \leq \phi(Tx_0)$. Then there exists an element $x^* \in X$ such that $Tx^* = x^* = Sx^*$.

Proof The set $X_0 = \{x \in X; \psi(Tx) \leq \psi(Sx) \text{ and } \phi(Sx) \leq \phi(Tx)\}$ is nonempty ($x_0 \in X_0$) and closed (hence complete), because $\phi \circ T$, $\phi \circ S$, $\psi \circ T$, and $\psi \circ S$ are lower semi-continuous.

First case. Let $\alpha = \inf(\phi + \psi)(X_0)$; since the function c is locally bounded from above, there exists $\lambda = \lambda(\alpha) > 0$ such that $\mu = \sup c([\alpha, \alpha + \lambda]) < \infty$. Also there exists $\nu = \nu(\mu) > 0$ with $H(t,s) \leq \nu$, whenever $(t,s) \in [0,\mu]^2$.

Let $x_1 \in X_0$ such that $\alpha \leq (\phi + \psi)(x_1) \leq \alpha + \lambda$. And let

$$X_1 = \{x \in X_0; (\phi + \psi)(x) \leq (\phi + \psi)(x_1)\}.$$

X_1 is nonempty ($x_1 \in X_1$) and closed since $\phi + \psi$ is lower semi-continuous.

By (6), we obtain

$$\begin{aligned} x \in X_1 \quad &\Rightarrow \quad (\phi + \psi)(Tx) \leq \psi(x) + \psi(Sx) \leq \psi(x) + \phi(x) \leq (\psi + \psi)(x_1), \\ x \in X_1 \quad &\Rightarrow \quad (\phi + \psi)(Sx) \leq \phi(Tx) + \phi(x) \leq \psi(x) + \phi(x) \leq (\psi + \psi)(x_1). \end{aligned}$$

Hence, for all $x \in X_1$, $Tx, Sx \in X_1$. For all $x \in X_1$, we have

$$(\phi + \psi)(x), (\phi + \psi)(Tx), (\phi + \psi)(Sx) \in [\alpha, \alpha + \lambda],$$

then $\max\{c((\phi + \psi)(x)), c((\phi + \psi)(Tx)), c((\phi + \psi)(Sx))\} \leq \mu$; and, consequently,

$$\begin{cases} H(c((\phi + \psi)(x)), c((\phi + \psi)(Tx))) \leq \nu, \\ H(c((\phi + \psi)(x)), c((\phi + \psi)(Sx))) \leq \nu. \end{cases}$$

Second case. We introduce the partial order $\leq$ on X_1 by

$$x \leq y \quad \Leftrightarrow \quad d(x,y) \leq \frac{\nu}{2}\big((\psi + \phi)(x) - (\psi + \phi)(y)\big).$$

Since $\psi + \phi$ are lower semi-continuous functions, $(X_1, \leq)$ has a maximal element x^*, by the Ekeland theorem. If $d(x^*, Sx^*) \leq d(x^*, Tx^*)$, we obtain

$$d(x^*, Sx^*) \leq \frac{\nu}{2}\big((\psi + \phi)(x^*) - (\phi(Sx^*) + \psi(Sx^*))\big).$$

It follows that $x^* \leq Sx^*$ and then $Sx^* = x^*$. And since

$$\phi(x^*) = \phi(Sx^*) \leq \phi(Tx^*) \leq \psi(x^*) = \psi(Sx^*) \leq \phi(x^*),$$

we conclude $\phi(x^*) = \psi(Sx^*)$, and $d(x^*, Tx^*) = 0$ *i.e.* $Tx^* = x^*$.

If $d(x^*, Tx^*) \leq d(x^*, Sx^*)$, we obtain $d(x^*, Tx^*) = 0$ and $d(x^*, Sx^*) = 0$ by the same arguments. □

Example 2.6 Consider the space $X = [0, +\infty[$ with the usual metric d and define T, S, ψ, and ϕ by

$$Tx = \begin{cases} 1, & x \in [0,1], \\ x, & x \in]1, +\infty[, \end{cases}$$

$$Sx = \begin{cases} 2 - x, & x \in [0,1], \\ 1, & x \in]1, +\infty[, \end{cases}$$

$$\psi(x) = \begin{cases} 1, & x \in [0,1[, \\ \frac{1}{2}, & x = 1, \\ x, & x \in]1, +\infty[, \end{cases}$$

$$\phi(x) = \begin{cases} x, & x \in [0,1[, \\ \frac{1}{2}, & x \in [1, +\infty[. \end{cases}$$

Let $H(t,s) = \max\{t,s\}$, $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$, and $c(y) = 1$, for each $y \in \mathbb{R}^+$.

For all $x \in X$, we have

$$\begin{cases} d(x, Sx) \leq H(c((\phi + \psi)(x)), c((\phi + \psi)(Tx)))[\psi(x) - \phi(Tx)], \\ d(x, Tx) \leq H(c((\phi + \psi)(x)), c((\phi + \psi)(Sx)))[\phi(x) - \psi(Sx)]. \end{cases} \tag{7}$$

For $x_0 = \frac{1}{2}$, we have $\psi(Tx_0) = \frac{1}{2} < \psi(Sx_0)$ and $\phi(Sx_0) = \frac{1}{2} = \phi(Tx_0)$. Note that $x^* = 1$ is a common fixed point of T and S.

Theorem 2.7 *Let d and δ be two metrics on a nonempty set X. Assume that (X,d) is complete. Let $(T_n)_n$ be a sequence of lower semi-continuous self mappings on X such that, for all $x \in X$ and for all $n,m \in \mathbb{N}^*$, we have*

$$\begin{cases} \max\{\delta(x, T_n x), d(x, T_n T_m x)\} \leq H(c(\phi(x)), c(\phi(T_m x)))(\phi(x) - \phi(T_m x)), \\ \delta(x, T_m T_n x) \leq H(c(\phi(x)), c(\phi(T_n x)))(\phi(x) - \phi(T_n x)), \end{cases} \tag{8}$$

where $c : \mathbb{R}^+ \to \mathbb{R}^+$ be a right locally bounded from above and H a locally bounded function from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{R}^+$. Then there exists an element $x^* \in X$ such that, for all $n \in \mathbb{N}^*$, $T_n x^* = x^*$.

Proof As in the proof of Theorem 2.1, there exists a complete subset X_o of X such that, for all $n,m \in \mathbb{N}^*$, $T_n X_0 \subset X_0$, and for all $x \in X_0$,

$$\begin{cases} H(c(\phi(x)), c(\phi(T_m x))) \leq \nu, \\ H(c(\phi(x)), c(\phi(T_n x))) \leq \nu, \end{cases}$$

where $\nu \in \mathbb{R}^+$. By (8), we have

$$d(x, T_n T_m x) \leq \nu\bigl(\phi(x) - \phi(T_m x)\bigr) \leq \phi(x) - \phi(T_n T_m x),$$

for each $x \in X$ and $n, m\mathbb{N}^*$. since ϕ is lower semi-continuous and (X,d) is complete, the Caristi fixed point theorem implies that there exists $x_{n,m} \in X$ such that $T_n T_m x_{n,m} = x_{n,m}$.

We have

$$\begin{cases} 0 \leq \phi(x_{n,m}) - \phi(T_m x_{n,m}), \\ 0 \leq \phi(T_m x_{n,m}) - \phi(T_n T_m x_{n,m}) = \phi(T_m x_{n,m}) - \phi(x_{n,m}). \end{cases}$$

Then $\phi(x_{n,m}) = \phi(T_m x_{n,m})$. By (8), we obtain

$$\delta(x_{n,m}, T_n x_{n,m}) \leq \phi(x_{n,m}) - \phi(T_m x_{n,m}) = 0,$$

which leads to $T_n x_{n,m} = x_{n,m}$. By the second relation of (8), we have

$$\delta(x_{n,m}, T_m x_{n,m}) = \delta(x_{n,m}, T_m T_n x_{n,m}) \leq \phi(x_{n,m}) - \phi(T_n x_{n,m}) = 0,$$

which leads to $T_m x_{n,m} = x_{n,m}$.

Hence, there exists $x_{n,m} \in X$ such that $T_n(x_{n,m}) = x_{n,m} = T_m(x_{n,m})$.

Let $m_o \in \mathbb{N}^*$. For each $n, m \in \mathbb{N}^*$,

$$\delta(x_{n,m_o}, T_m T_n x_{n,m_o}) \leq \phi(x_{n,m_o}) - \phi(T_n x_{n,m_o}) = 0.$$

Consequently, for $n = m_o$ and for all $m \in \mathbb{N}^*$, we obtain $T_m x_{m_o,m_o} = x_{m_o,m_o}$. □

Theorem 2.8 *Let (X,d) and (Y,δ) be two complete metric spaces. Let $T : X \to Y$, $S : Y \to X$ be two mappings and $\psi : X \to \mathbb{R}^+$, $\phi : Y \to \mathbb{R}^+$ two lower semi-continuous functions such that, for all $(x,y) \in X \times Y$,*

$$\begin{cases} d(x, STx) \leq H(c(\psi(x) + \phi(y)), c(\psi(Sy) + \phi(Tx)))[\psi(x) - \phi(Tx)], \\ \delta(y, TSy) \leq H(c(\psi(x) + \phi(y)), c(\psi(Sy) + \phi(Tx)))[\phi(y) - \psi(Sy)], \end{cases} \tag{9}$$

where $c : \mathbb{R}^+ \to \mathbb{R}^+$ is a right locally bounded from above and $H : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a locally bounded function. Then there exists a couple $(x^*, y^*) \in X \times Y$ such that $STx^* = x^*$ and $TSy^* = y^*$. Also, then $Tx^* = y^*$ and $Sy^* = x^*$.

Proof First case. Let $\alpha = \inf(\psi(X) + \phi(Y))$. The function c is locally bounded from above, there exists $\lambda = \lambda(\alpha) > 0$ in such a way that $\beta = \sup([\alpha, \alpha + \lambda]) < \infty$ and there exists $\nu = \nu(\beta) > 0$ with $H(t,s) \leq \nu$ for each $s, t \in [0, \beta]$.

By definition of α, there exists $(x_0, y_0) \in X \times Y$ such that

$$\alpha \leq \psi(x_0) + \phi(y_0) \leq \alpha + \lambda.$$

The set $A = \{(x,y) \in X \times Y; \psi(x) + \phi(y) \leq \psi(x_0) + \phi(y_0)\}$ is nonempty and closed.

By (9), we obtain

$$(x,y) \in A \quad\Rightarrow\quad \phi(Tx) + \psi(Sy) \leq \psi(x) + \phi(y) \leq \psi(x_0) + \phi(y_0) \quad\Rightarrow\quad (Sy, Tx) \in A.$$

For all $(x,y) \in A$, we have

$$\psi(x) + \phi(y), \psi(Sy) + \phi(Tx) \in [\alpha, \alpha + \lambda].$$

Then $c(\psi(x) + \phi(y)), c(\psi(Sy) + \phi(Tx)) \leq \beta$, and hence

$$\begin{cases} H(c(\psi(x) + \phi(y)), c(\psi(Sy) + \phi(Tx))) \leq \nu, \\ H(c(\psi(x) + \phi(y)), c(\psi(Sy) + \phi(Tx))) \leq \nu. \end{cases}$$

Second case. Let $(x,y) \in A$; we have

$$\begin{cases} d(x, STx) \leq \nu(\psi(x) - \phi(Tx)), \\ \delta(y, TSy) \leq \nu(\phi(y) - \psi(Sy)). \end{cases}$$

Since $(Sy, Tx) \in A$, we have

$$\begin{cases} d(Sy, STSy) \leq \nu(\psi(Sy) - \phi(TSy)), \\ \delta(Tx, TSTx) \leq \nu(\phi(Tx) - \psi(STx)), \end{cases}$$

and then

$$\begin{cases} d(x, STx) \leq \nu(\psi(x) - \psi(STx)), \\ \delta(y, TSy) \leq \nu(\phi(y) - \phi(TSy)). \end{cases} \tag{10}$$

Define the partial order $\preceq$ in A as follows: for $(x,y), (x',y') \in A$

$$(x,y) \preceq (x',y') \quad \Leftrightarrow \quad d(x,x') \leq \nu(\psi(x) - \psi(x')) \quad \text{and} \quad \delta(y,y') \leq \nu(\phi(y) - \phi(y')).$$

Let $(x_\alpha, y_\alpha)_\alpha$ be some chain of A;

$$\begin{aligned} \alpha \leq \beta \quad &\Leftrightarrow \quad (x_\alpha, y_\alpha) \preceq (x_\beta, y_\beta) \\ &\Leftrightarrow \quad d(x_\alpha, x_\beta) \leq \nu(\psi(x_\alpha) - \psi(x_\beta)) \quad \text{and} \quad \delta(y_\alpha, y_\beta) \leq \nu(\phi(y_\alpha) - \phi(y_\beta)). \end{aligned}$$

$(\psi(x_\alpha)_\alpha)$ and $(\phi(y_\alpha)_\alpha)$ are increasing bounded and thus convergent sequences.

Let $\gamma = \lim_\alpha \psi(x_\alpha)$ and $\eta = \lim_\alpha \phi(y_\alpha)$.

For $\varepsilon > 0$, there exists α_o such that, for all $\beta \geq \alpha \geq \alpha_o$,

$$\begin{cases} \psi(x_\alpha) - \psi(x_\beta) \leq (\varepsilon + \gamma) - \gamma \leq \varepsilon, \\ \phi(y_\alpha) - \phi(y_\beta) \leq (\varepsilon + \eta) - \eta \leq \varepsilon, \end{cases}$$

which implies that $((x_\alpha, y_\alpha))_\alpha$ is a Cauchy sequence in the complete space (A, d_∞) where d_∞ is defined by $d_\infty((x,y),(x',y')) = \max\{d(x,x'), \delta(y,y')\}$. It follows that there exists $(x^*, y^*) \in A$ such that $\lim_\alpha x_\alpha = x^*$ and $\lim_\alpha y_\alpha = y^*$.

We obtain

$$\begin{cases} d(x_\alpha, x^*) \leq \nu(\psi(x_\alpha) - \psi(x^*)), \\ \delta(y_\alpha, y^*) \leq \nu(\phi(y_\alpha) - \phi(y^*)). \end{cases}$$

Hence, $(x_\alpha, y_\alpha) \preceq (x^*, y^*)$. And by the Ekeland theorem, $(A, \preceq)$ has a maximal element $(\bar{x}, \bar{y})$. By (10), we have

$$\begin{cases} d(\overline{x}, ST\overline{x}) \leq \nu[\psi(\overline{x}) - \psi(ST\overline{x})], \\ \delta(\overline{y}, TS\overline{y}) \leq \nu[\phi(\overline{y}) - \phi(TS\overline{y})]. \end{cases}$$

And since (A, d_∞) is complete and $(STx, TSy) \in A$, for all $(x, y) \in A$, there exists $(\bar{x}, \bar{y}) \in A$ such that $(ST\overline{x}, TS\overline{y}) = (\overline{x}, \overline{y})$. □

For $h(t,s) = 1$, for all $(t,s) \in \mathbb{R}^+ \times \mathbb{R}^+$, we have the following.

Theorem 2.9 *Let (X,d) and (Y,δ) be two metric spaces such that (X,d) is complete. Let $T : X \to Y$, $S : Y \to X$ be two mappings and $\psi : X \to \mathbb{R}^+$, $\phi : Y \to \mathbb{R}^+$ two lower semi-continuous functions such that, for all $(x,y) \in X \times Y$,*

$$\begin{cases} d(x, Sy) \le \psi(x) - \phi(TSy), \\ \delta(y, Tx) \le \phi(y) - \psi(STx). \end{cases}$$

Then there exists an unique couple $(x^*, y^*) \in X \times Y$ such $STx^* = x^*$, $TSy^* = y^*$; and then $Tx^* = y^*$ and $Sy^* = x^*$.

Proof Let $x \in X$, $y = Tx$, and $y' = TSTx$; we have

$$\begin{cases} d(x, STx) \le \psi(x) - \phi(TSTx), \\ \delta(y', Tx) = \delta(TSTx, Tx) \le \phi(TSTx) - \psi(STx). \end{cases}$$

It follows that $\phi(TSTx) \ge \psi(STx)$ and $d(x, STx) \le \psi(x) - \psi(STx)$, for all $x \in X$. By the Caristi theorem, there exists $x^* \in X$ such that $STx^* = x^*$.

Let $y^* = Tx^*$; for $x = STx^*$ and $y = y^*$, we have

$$\begin{cases} d(STx^*, Sy^*) \le \psi(STx^*) - \phi(TSy^*), \\ \delta(y^*, TSTx^*) \le \phi(y^*) - \psi(STSTx^*), \end{cases}$$

$$\begin{cases} 0 \le d(x^*, Sy^*) \le \psi(x^*) - \phi(TSy^*) = \psi(x^*) - \phi(y^*), \\ \delta(y^*, Tx^*) \le \phi(y^*) - \psi(x^*). \end{cases}$$

Then $\phi(y^*) = \psi(x^*)$ and $x^* = Sy^*$. Hence, $TSy^* = y^*$.

For uniqueness, let $(x,y) \in X \times Y$ such that $STx = x$ end $TSy = y$. We have

$$\begin{cases} d(x, Sy^*) \le \psi(x) - \phi(TSy^*), \\ \delta(y, Tx^*) \le \phi(y) - \psi(STx^*), \end{cases}$$

$$\begin{cases} d(x, x^*) \le \psi(x) - \phi(y^*), \\ \delta(y, y^*) \le \phi(y) - \psi(x^*). \end{cases}$$

Similarly

$$\begin{cases} d(x^*, x) \le \psi(x^*) - \phi(y), \\ \delta(y^*, y) \le \phi(y^*) - \psi(x). \end{cases}$$

So, $\psi(x^*) = \phi(y)$ and $\phi(y^*) - \psi(x)$. Then $x = x^*$ and $y = y^*$. □

Theorem 2.10 *Let (X,d) and (Y,δ) be metric spaces. Assume that (X,d) is complete. Let $T:X\to Y$, $S:Y\to X$ be two mappings and $\psi:X\to\mathbb{R}^+$, $\phi:Y\to\mathbb{R}^+$ two lower semi-continuous functions such that, for all $(x,y)\in X\times Y$,*

$$\begin{cases} d(x,Sy)\leq\psi(x)-\phi(Tx),\\ \delta(y,Tx)\leq\phi(y)-\psi(Sy).\end{cases}$$

Then there exists an unique $(x^*,y^*)\in X\times Y$ such that $STx^*=x^*$ and $TSy^*=y^*$. And then $Tx^*=y^*$ and $Sy^*=x^*$.

Proof For $y=Tx$, $x\in X$, we have

$$\begin{cases} d(x,STx)\leq\psi(x)-\phi(Tx),\\ 0\leq\phi(Tx)-\psi(STx).\end{cases}$$

So, for all $x\in X$, $d(x,STx)\leq\psi(x)-\psi(STx)$. By the Caristi theorem, there exists $x^*\in X$ such that $STx^*=x^*$.

Let $y^*=Tx^*$. For $y=y^*$ and $x=x^*$, we have

$$\begin{cases} d(x^*,Sy^*)\leq\psi(x^*)-\phi(Tx^*)=\psi(x^*)-\phi(y^*),\\ \delta(y^*,Tx^*)\leq\phi(y^*)-\psi(Sy^*)=\phi(y^*)-\psi(x^*),\end{cases}$$

which leads to $\phi(y^*)=\psi(x^*)$ and $x^*=Sy^*$. Hence, $TSy^*=y^*$.

For uniqueness, $(x,y)\in X\times Y$ such that $STx=x$ and $TSy=y$; we have

$$\begin{cases} d(x^*,Sy)\leq\psi(x^*)-\phi(Tx^*),\\ \delta(y^*,Tx)\leq\phi(y^*)-\psi(Sy^*),\end{cases}$$

$$\begin{cases} d(x^*,Sy)\leq\psi(x^*)-\phi(y^*),\\ \delta(y^*,Tx)\leq\phi(y^*)-\psi(x^*).\end{cases}$$

So $\psi(x^*)=\phi(y^*)$. We obtain $x^*=Sy$ and $y^*=Tx$. Thereby, $y=TSy=Tx^*=y^*$ and $x=STx=Sy^*=x^*$. □

Example 2.11 Let $X=[0,+\infty[$ and $Y=[0,\frac{1}{2}]\cup\{1\}$; we use the usual metric d and the metric δ given by

$$\delta(x,y)=\begin{cases} x+y & \text{if } x\neq y,\\ 0 & \text{if } x=y.\end{cases}$$

We define $T:X\longrightarrow Y$ and $S:Y\longrightarrow X$ by

$$Tx=\begin{cases} 1 & \text{if } x\in[0,1[,\\ \frac{1}{1+x} & \text{if } x\in[1,+\infty[,\end{cases}$$

and $Sy=1$, for all $y\in Y$.

Let ψ and ϕ be defined, respectively, on X and Y by

$$\psi(x) = \begin{cases} 1 & \text{if } x \in [0,1[, \\ 0 & \text{if } x = 1, \\ x+1 & \text{if } x \in]1,+\infty[, \end{cases}$$

$$\phi(y) = \begin{cases} y+1 & \text{if } y \in [0,\frac{1}{2}[, \\ 0 & \text{if } y = \frac{1}{2}, \\ \frac{1}{2} & \text{if } y = 1. \end{cases}$$

The functions c and H are defined by $c(t) = 6t$ and $H(t,s) = t$, for all $s,t \in [0,+\infty[$.

We have $STx = 1$ and $TSy = \frac{1}{2}$, for all $(x,y) \in X \times Y$.

We discuss the following cases:

Case 1: $x \in [0,1[$ and $y \in [0,\frac{1}{2}[$.

We obtain $d(x,STx) = 1-x$, $\delta(y,TSy) = y+\frac{1}{2}$, $\psi(x)+\phi(y) = y+2$, $\psi(x)-\phi(Tx) = \frac{1}{2}$ and $\phi(y)-\psi(Sy) = y+1$. So

$$\begin{cases} d(x,STx) \leq H(c(\psi(x)+\phi(y)), c(\psi(Sy)+\phi(Tx)))(\psi(x)-\phi(Tx)), \\ \delta(y,TSy) \leq H(c(\psi(x)+\phi(y)), c(\psi(Sy)+\phi(Tx)))(\phi(y)-\psi(Sy)). \end{cases}$$

Case 2: $x \in [0,1[$ and $y = \frac{1}{2}$.

We obtain $d(x,STx) = 1-x$, $\delta(\frac{1}{2},TS\frac{1}{2}) = 0$, $\psi(x)+\phi(\frac{1}{2}) = 1$, $\psi(x)-\phi(Tx) = \frac{1}{2}$, and $\phi(\frac{1}{2})-\psi(S\frac{1}{2}) = 0$. So

$$\begin{cases} d(x,STx) \leq H(c(\psi(x)+\phi(\frac{1}{2})), c(\psi(S\frac{1}{2})+\phi(Tx)))(\psi(x)-\phi(Tx)), \\ \delta(\frac{1}{2},TS\frac{1}{2}) \leq H(c(\psi(x)+\phi(\frac{1}{2})), c(\psi(Sy)+\phi(Tx)))(\phi(\frac{1}{2})-\psi(S\frac{1}{2})). \end{cases}$$

Case 3: $x \in [0,1[$ and $y = 1$.

We obtain $d(x,STx) = 1-x$, $\delta(1,TS1) = \frac{3}{2}$, $\psi(x)+\phi(1) = \frac{3}{2}$, $\psi(x)-\phi(Tx) = \frac{1}{2}$, and $\phi(1)-\psi(S1) = \frac{1}{2}$. So

$$\begin{cases} d(x,STx) \leq H(c(\psi(x)+\phi(1)), c(\psi(S1)+\phi(Tx)))(\psi(x)-\phi(Tx)), \\ \delta(1,TS1) \leq H(c(\psi(x)+\phi(1)), c(\psi(S1)+\phi(Tx)))(\phi(1)-\psi(S1)). \end{cases}$$

Case 4: $x = 1$ and $y \in [0,\frac{1}{2}[$.

We obtain $d(1,ST1) = 0$, $\delta(y,TSy) = y+\frac{1}{2}$, $\psi(1)+\phi(y) = y+1$, $\psi(1)-\phi(T1) = 0$ and $\phi(y)-\psi(Sy) = y+1$. So

$$\begin{cases} d(1,ST1) \leq H(c(\psi(1)+\phi(y)), c(\psi(Sy)+\phi(T1)))(\psi(1)-\phi(T1)), \\ \delta(y,TSy) \leq H(c(\psi(1)+\phi(y)), c(\psi(Sy)+\phi(T1)))(\phi(y)-\psi(Sy)). \end{cases}$$

Case 5: $x = 1$ and $y = \frac{1}{2}$.

We obtain $d(1,ST1)=0$, $\delta(\frac{1}{2},TS\frac{1}{2})=0$, $\psi(1)+\phi(\frac{1}{2})=0$, $\psi(1)-\phi(T1)=0$, and $\phi(\frac{1}{2})-\psi(S\frac{1}{2})=0$. So

$$\begin{cases} d(1,ST1)\leq H(c(\psi(1)+\phi(\frac{1}{2})),c(\psi(S\frac{1}{2})+\phi(T1)))(\psi(1)-\phi(T1)),\\ \delta(\frac{1}{2},TS\frac{1}{2})\leq H(c(\psi(1)+\phi(\frac{1}{2})),c(\psi(S\frac{1}{2})+\phi(T1)))(\phi(\frac{1}{2})-\psi(S\frac{1}{2})).\end{cases}$$

Case 6: $x\in]1,+\infty[$ and $y\in[0,\frac{1}{2}[$.

We obtain $d(x,STx)=x-1$, $\delta(y,TSy)=y+\frac{1}{2}$, $\psi(x)+\phi(y)=x+y+2$, $\psi(x)-\phi(Tx)=x-\frac{1}{x+1}$, and $\phi(y)-\psi(Sy)=y+1$. So

$$\begin{cases} d(x,STx)\leq H(c(\psi(x)+\phi(1)),c(\psi(S1)+\phi(Tx)))(\psi(x)-\phi(Tx)),\\ \delta(y,TSy)\leq H(c(\psi(x)+\phi(y)),c(\psi(Sy)+\phi(Tx)))(\phi(y)-\psi(Sy)).\end{cases}$$

Case 7: $x\in]1,+\infty[$ and $y=\frac{1}{2}$.

We obtain $d(x,STx)=x-1$, $\delta(\frac{1}{2},TS\frac{1}{2})=0$, $\psi(x)+\phi(\frac{1}{2})=x+1$, $\psi(x)-\phi(Tx)=x-\frac{1}{x+1}$, and $\phi(\frac{1}{2})-\psi(S\frac{1}{2})=0$. So

$$\begin{cases} d(x,STx)\leq H(c(\psi(x)+\phi(\frac{1}{2})),c(\psi(S\frac{1}{2})+\phi(Tx)))(\psi(x)-\phi(Tx)),\\ \delta(\frac{1}{2},TS\frac{1}{2})\leq H(c(\psi(x)+\phi(\frac{1}{2})),c(\psi(S\frac{1}{2})+\phi(Tx)))(\phi(\frac{1}{2})-\psi(S\frac{1}{2})).\end{cases}$$

Case 8: $x\in]1,+\infty[$ and $y=1$.

We obtain $d(x,STx)=x-1$, $\delta(1,TS1)=\frac{3}{2}$, $\psi(x)+\phi(1)=x+\frac{3}{2}$, $\psi(x)-\phi(Tx)=x-\frac{1}{x+1}$, and $\phi(1)-\psi(S1)=\frac{1}{2}$. So

$$\begin{cases} d(x,STx)\leq H(c(\psi(x)+\phi(1)),c(\psi(S1)+\phi(Tx)))(\psi(x)-\phi(Tx)),\\ \delta(1,TS1)\leq H(c(\psi(x)+\phi(1)),c(\psi(S1)+\phi(Tx)))(\phi(1)-\psi(S1)).\end{cases}$$

Case 9: $x=y=1$.

We have $d(1,ST1)=0$, $\delta(1,TS1)=\frac{3}{2}$, $\psi(1)+\phi(1)=\frac{1}{2}$, $\psi(1)-\phi(T1)=0$, and $\phi(1)-\psi(S1)=\frac{1}{2}$. So

$$\begin{cases} d(1,ST1)\leq H(c(\psi(1)+\phi(1)),c(\psi(S1)+\phi(T1)))(\psi(1)-\phi(T1)),\\ \delta(1,TS1)\leq H(c(\psi(1)+\phi(1)),c(\psi(S1)+\phi(T1)))(\phi(1)-\psi(S1)).\end{cases}$$

Note that $T1=\frac{1}{2}$ and $S\frac{1}{2}=1$.

Example 2.12 Let $X=[0,1]$ and $Y=[0,1]\cup\{2,3,\dots,p\}$, where $p\in\mathbb{N}\setminus\{0,1\}$; we consider the following metrics:

$$d(x,x')=|x-x'|\quad\text{for all } x,x'\in X$$

and

$$\delta(y,y')=\begin{cases}|y-y'| & \text{if } y,y'\in[0,1]; y\neq y',\\ y+y' & \text{if } y \text{ or } y'\notin[0,1] \text{ and } y\neq y',\\ 0 & \text{if } y=y'.\end{cases}$$

We define $T: X \to Y$ and $S: Y \to X$ by $Tx = 1$ and $Sy = 1$. We define ψ and ϕ on X and Y resp. by

$$\psi(x) = \begin{cases} 1 & \text{if } x \in [0,1[, \\ 0 & \text{if } x = 1, \end{cases}$$

and

$$\phi(y) = \begin{cases} e^y & \text{if } y \in [0,1[\cup \{2,3,\ldots\}, \\ 0 & \text{if } y = 1. \end{cases}$$

We have $STx = 1$ and $TSy = 1$, for all $(x,y) \in X \times Y$.

Case 1: $x, y \in [0,1[$. We have

$$\begin{cases} d(x, Sy) = 1 - x \leq \psi(x) - \phi(TSy) = \psi(x) - \phi(1) = 1, \\ \delta(y, Tx) = 1 - y \leq \phi(y) - \psi(STx) = \phi(y) - \psi(1) = e^y. \end{cases}$$

Case 2: $x \in [0,1[$ and $y \in \{2,\ldots,p\}$

$$\begin{cases} d(x, Sy) = 1 - x \leq \psi(x) - \phi(TSy) = \psi(x) - \phi(1) = 1, \\ \delta(y, Tx) = y + 1 \leq \phi(y) - \psi(STx) = \phi(y) - \psi(1) = e^y. \end{cases}$$

Case 3: $x = 1$ and $y \in [0,1[$.

$$\begin{cases} d(1, Sy) = 0 = \psi(1) - \phi(TSy) = \psi(1) - \phi(1), \\ \delta(y, T1) = 1 - y \leq \phi(y) - \psi(ST1) = \phi(y) - \psi(1) = e^y. \end{cases}$$

Case 4: $x = 1$ and $y \in \{2,\ldots,p\}$.

$$\begin{cases} d(1, Sy) = 0 = \psi(1) - \phi(TSy) = \psi(1) - \phi(1), \\ \delta(y, T1) = y + 1 \leq \phi(y) - \psi(ST1) = \phi(y) - \psi(1) = e^y. \end{cases}$$

Case 5: $x \in [0,1[$ and $y = 1$.

$$\begin{cases} d(x, S1) = 1 - x \leq \psi(x) - \phi(TS1) = \psi(x) - \phi(1) = 1, \\ \delta(1, Tx) = 0 = \phi(1) - \psi(STx) = \phi(1) - \psi(1). \end{cases}$$

Case 6: $x = y = 1$.

$$\begin{cases} d(1, S1) = 0 = \psi(1) - \phi(TS1) = \psi(1) - \phi(1), \\ \delta(1, T1) = 0 = \phi(1) - \psi(ST1) = \phi(1) - \psi(1). \end{cases}$$

Note that $T1 = 1$ and $S1 = 1$.

3 Application

Theorem 3.1 *Let (X,d) and (Y,δ) be two metric spaces such that (X,d) is complete. Let $\psi: X \to \mathbb{R}^+$, $\phi: Y \to \mathbb{R}^+$ be two lower semi-continuous functions. Assume that, for $(u,v) \in X \times Y$ such that $\psi(u) > \inf_{x\in X} \psi(x)$ and $\phi(v) > \inf_{y\in Y} \phi(y)$, there exists $(u',v') \in X \times Y$, $(u',v') \neq (u,v)$ such that*

$$\phi(v') + d(u,u') \leq \psi(u) \quad \textit{and} \quad \psi(u') + \delta(v,v') \leq \phi(v), \tag{11}$$

then there exists $(u_o, v_o) \in X \times Y$ such that

$$\psi(u_o) = \inf_{x\in X} \psi(x) \quad \textit{or} \quad \phi(v_o) = \inf_{y\in Y} \phi(y).$$

Proof Assume $\psi(u) > \inf_{x\in X} \psi(x)$ and $\phi(v) > \inf_{y\in Y} \phi(y)$ for all $(u,v) \in X \times Y$.

For each $(u,v) \in X \times Y$. There exists $(u',v') \in X \times Y$ such that

$$(u,v) \neq (u',v'), \qquad \phi(v') + d(u,u') \leq \psi(u) \quad \text{and} \quad \psi(u') + \delta(v,v') \leq \phi(v).$$

Define the set

$$E(u,v) = \{(u',v') \in X \times Y, \text{such that } (u',v') \neq (u,v) \text{ and (11) is satisfied}\}.$$

For all $(u,v) \in X \times Y$, we have $E(u,v) \neq \emptyset$ and $(u,v) \notin E(u,v)$.

We define the mappings $T: X \to Y$ and $S: Y \to Y$ $Tu = v'$ and $Sv = u'$ where $(u',v') \in E(u,v)$. For all $(u,v) \in X \times Y$, we have

$$\begin{cases} d(u,Sv) \leq \psi(u) - \phi(Tu), \\ \delta(v,Tu) \leq \phi(v) - \psi(Sv). \end{cases}$$

By Theorem 2.9, there exists $(u^*,v^*) \in X \times Y$ such that $Tu^* = v^*$ and $Sv^* = u^*$. Hence, $(Sv^*, Tu^*) = (u^*,v^*) \in E(u^*,v^*)$ which is absurd. □

4 Caristi's fixed point theorem for two pairs of mappings in Hilbert space

In this section, we prove the existence of fixed points for two simultaneous projections to find the optimal solutions for some proximity functions via the Caristi fixed point theorem.

Let H be a Hilbert space, $I = \{1,\dots,m\}$ and $J = \{1,\dots,p\}$; for each $(i,j) \in I \times J$, we consider two nonempty closed convex subsets C_i and D_j of H and we define the metric projections $P_{C_i}: H \to C_i$ and $P_{D_j}: H \to D_j$.

For $k \in \mathbb{N}^*$, we define Δ_k by

$$\Delta_k = \{u = (u_1,\dots,u_k) \in \mathbb{R}^k, u_i \geq 0 \text{ and } u_1 + \cdots + u_k = 1\}.$$

For each $u = (u_1,\dots,u_m) \in \Delta_m$ and $w = (w_1,\dots,w_p) \in \Delta_p$, we define the proximity functions $f: H \to \mathbb{R}^+$ and $g: H \to \mathbb{R}^+$ by

$$f(x) = \frac{1}{2}\sum_{i\in I} u_i \|P_{C_i}x - x\|^2 \quad \text{and} \quad g(x) = \frac{1}{2}\sum_{j\in J} w_j \|P_{D_j}x - x\|^2. \tag{12}$$

The set of all minimizers of f and g is defined by

$$\operatorname*{Argmin}_{x\in H}\{f(x),g(x)\} = \{z \in H, f(z) \le f(x) \text{ and } g(z) \le g(x), \forall x \in H\}.$$

Theorem 4.1 *Let $T = \sum_{i\in I} u_i P_{C_i}$ and $S = \sum_{j\in J} w_j P_{D_j}$ be simultaneous projections, where I and J are defined as above, and define $f : H \to \mathbb{R}$ and $g : H \to \mathbb{R}$ by* (12).

Then we have

$$\operatorname{Fix}(T) \cap \operatorname{Fix}(S) = \operatorname*{Argmin}_{x\in H}\{f(x),g(x)\}.$$

Moreover, if

1. $\|x - Tx\| \ge 1$, *for all $x \in K$, where*

$$K = \{x \in H; T^{n+1}x \neq T^n x \text{ for all } n \in \mathbb{N}^*\},$$

2. *there exists $x_0 \in H$ such that, for all $n \in \mathbb{N}$, $g(T^{n+1}x_0) \le g(ST^n x_0)$,*

then $\operatorname{Fix}(T) \cap \operatorname{Fix}(S) \neq \emptyset$.

Proof f, g are convex and differentiable functions. Moreover, for all $x \in H$,

$$\begin{cases} Df(x) = \sum_{i\in I} u_i(x - P_{C_i}(x)) = x - Tx, \\ Dg(x) = \sum_{i\in J} w_j(x - P_{D_i}(x)) = x - Sx. \end{cases}$$

Therefore, the sufficient and necessary optimality yields

$$\begin{aligned} z \in \operatorname*{Argmin}_{x\in H}\{f(x),g(x)\} \quad &\Leftrightarrow \quad Df(z) = z - Tz = 0 \quad \text{and} \quad Dg(z) = z - Sz = 0 \\ &\Leftrightarrow \quad z \in \operatorname{Fix}(T) \cap \operatorname{Fix}(S). \end{aligned}$$

Let

$$X_0 = \{x \in X; g(T^{n+1}x) \le g(ST^n x), \forall n \in \mathbb{N}\}.$$

The set X_0 is nonempty ($x_0 \in X_0$) and closed (complete).

First step. $\overline{K} \cap X_0 = \emptyset$, there exists $z \in X_0$ such that $z \notin \overline{K}$, so there exists $p \in \mathbb{N}^*$ such that $T^{p+1}z = T^p z$.

Since $\|P_{D_j}(ST^p z) - ST^p z\| \le \|P_{D_j}(T^p z) - ST^p z\|$, we have

$$\begin{aligned} g(ST^p z) &= \frac{1}{2}\sum_{j\in J} w_j \|P_{D_j}(ST^p z) - ST^p z\|^2 \\ &\le \frac{1}{2}\sum_{j\in J} w_j \|P_{D_j}(T^p z) - ST^p z\|^2 \\ &\le \frac{1}{2}\sum_{j\in J} w_j \|P_{D_j}(T^p z) - T^p z\|^2 + \frac{1}{2}\sum_{j\in J} w_j \|T^p z - ST^p z\|^2 \end{aligned}$$

$$
\begin{aligned}
&- \sum_{j\in J} w_j \langle P_{D_j}(T^p z) - T^p z, ST^p z - T^p z\rangle \\
&\leq \frac{1}{2}\sum_{j\in J} w_j \|P_{D_j}(T^p z) - T^p z\|^2 + \frac{1}{2}\|T^p z - ST^p z\|^2 - \|ST^p z - T^p z\|^2 \\
&\leq g(T^p z) - \frac{1}{2}\|T^p z - ST^p z\|^2.
\end{aligned}
$$

We obtain

$$\frac{1}{2}\|T^p z - ST^p z\|^2 \leq g(T^p z) - g(ST^p z) \leq g(T^p z) - g(T^{p+1} z) = 0.$$

Thus,

$$T(T^p z) = T^p z = ST^p z.$$

Second step. $\overline{K} \cap X_0 \neq \emptyset$. Prove that $T(\overline{K} \cap X_0) \subset \overline{K} \cap X_0$.

Let $x \in K$; for all $n \in \mathbb{N}^*$, we have

$$T^{n+1}(Tx) = T^{n+2}x \neq T^{n+1}x = T^n(Tx),$$

which gives $T(K) \subset K$; and since T is continuous, we obtain $T(\overline{K}) \subset \overline{T(K)} \subset \overline{K}$.

For any $x \in \overline{K} \cap X_0$, there exists a sequence $(z_n)_{n\geq 0}$ of K such that $\lim_n z_n = x$. Let $n \in \mathbb{N}$. Since $\|z_n - Tz_n\| \geq 1$, we have

$$
\begin{aligned}
f(Tz_n) &= \frac{1}{2}\sum_{i\in I} u_i \|P_{C_i}(Tz_n) - Tz_n\|^2 \\
&\leq \frac{1}{2}\sum_{i\in I} u_i \|P_{C_i}(z_n) - Tz_n\|^2 \\
&\leq \frac{1}{2}\sum_{i\in I} u_i \|P_{C_i}(z_n) - z_n\|^2 + \frac{1}{2}\sum_{i\in I} u_i \|z_n - Tz_n\|^2 \\
&\quad - \sum_{i\in I} u_j \langle P_{C_i}(Tz_n) - z_n, Tz_n - z_n\rangle \\
&\leq \frac{1}{2}\sum_{i\in I} u_i \|P_{C_i}(z_n) - z_n\|^2 + \frac{1}{2}\|z_n - Tz_n\|^2 - \|Tz_n - z_n\|^2 \\
&\leq f(z_n) - \frac{1}{2}\|z_n - Tz_n\|^2 \\
&\leq f(z_n) - \frac{1}{2}\|z_n - Tz_n\|.
\end{aligned}
$$

This leads, for all $n \in \mathbb{N}$, to

$$\frac{1}{2}\|z_n - Tz_n\| \leq f(z_n) - f(Tz_n).$$

We make n to $+\infty$, which gives

$$\frac{1}{2}\|x - Tx\| \leq f(x) - f(Tx). \tag{13}$$

Since $\overline{K} \cap X_0$ is complete, so by the first inequality of equation (13), there exists $x^* \in \overline{K} \cap X_0$ such that $Tx^* = x^*$. And since

$$\frac{1}{2}\|x^* - Sx^*\|^2 \leq g(x^*) - g(Sx^*) \leq g(x^*) - g(Tx^*) = 0,$$

we conclude $\mathrm{Fix}(T) \cap \mathrm{Fix}(S) \neq \emptyset$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally in this paper.

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