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Asymptotical stability of a nonlinear non-differentiable dynamic system in microbial continuous cultures

Jiajia Lv*, Liping Pang and Enmin Feng

*Correspondence:
ljiajia2849@163.com
School of Mathematical Sciences,
Dalian University of Technology,
Dalian, 116024, China

Abstract
In this paper, we consider a nonlinear non-differentiable dynamic system in microbial continuous cultures involving all possible metabolic pathways of the inhibition mechanisms of 3-hydroxypropionaldehyde onto the cell growth and the transport systems of glycerol and 1,3-PD across the cell membrane. First, the existence of the equilibrium point of the system proved. And by numerical calculation, the equilibrium point of the system is obtained. Subsequently, we derive the local bounded properties of the Jacobian, tensor and Hessian matrices of the system. Finally, the local asymptotical stability of the system at equilibrium point is proved.

Keywords: nonlinear dynamic system; equilibrium point; tensor; asymptotical stability

1 Introduction

The bioconversion of glycerol to 1,3-propanediol (1,3-PD) by Klebsiella pneumoniae is particularly attractive to industry because of renewable feedbacks and potential uses of 1,3-PD [1]. Especially continuous culture is of interest because of its high productivity, stable product quality and high automation.

In recent years, many efforts have been made to understand and express, in mathematical terms, the above mentioned bioconversion. The original model was proposed by Zeng *et al.* [2, 3], in which the concentrations of biomass, glycerol and products (1,3-PD, acetate and ethanol) in reactor were considered. In addition, ignoring the influence of acetate and ethanol on the fermentation process, Xiu *et al.* [4] discussed the multiplicity of a three-dimensional dynamical system. Based on the model, Xiu *et al.* [5] proposed a five-dimensional one later. On the basis of models, Sun [6] developed and re-constructed an eight-dimensional dynamic system considering intracellular substances: 3-hydroxy propionaldehyde (3-HPA), 1,3-PD and glycerol, which is more reasonable to describe the consumption of substrates and the formation of products.

With the development of fermentation models, there has been a lot of theoretical and numerical research about these models considering the extracellular substances, such as parameter identification works [7–11]. Since the continuous fermentation proceeds under a steady state (that is, a little change of the initial conditions would not cause a great change of the solutions), the existence of an equilibrium solution and the stability analysis are necessary to evaluate a model describing the continuous fermentation. The stability of equilibrium solutions suggests that proper operating conditions could be chosen to obtain the expected productions' concentrations. However, the stability results are seldom discussed for the complexity of the mathematical models. Li *et al.* [12] mainly discussed the stability of the three-dimensional system by introducing its linearization. Ye *et al.* [13] examined the existence of equilibrium points and proposed an efficient method to calculate the equilibrium points of the five-dimensional model in continuous fermentations. Wang *et al.* [14] discussed the stability of an eight-dimensional system taking the changes of concentrations of intracellular substances into consideration. Chen and Gao [15, 16] studied positive linear systems with the common linear copositive Lyapunov functions and presented four algorithms to compute common infinity-norm Lyapunov functions. Choi *et al.* [17, 18] characterized the h-stability for nonlinear difference systems. Wang *et al.* [19] introduced the h-stability for differential systems with different initial time and formulated the stability criteria. However, especially for the eight-dimensional system being in consideration of both the inhibition mechanisms of 3-HPA onto the cell growth and the transport systems of glycerol and 1,3-PD across the cell membrane, the properties of the system and the stability analysis are scarcely discussed.

In this paper, we consider a nonlinear non-differentiable dynamic system in microbial continuous culture, develop Ye's [13] method to derive the existence of equilibrium points, and present its local asymptotically stability for the eight-dimensional nonlinear dynamical system presented in [8]. Different from the system of [14], the eight-dimensional dynamic system considers all possible metabolic pathways of the inhibition mechanisms of 3-HPA onto the cell growth and the transport systems of glycerol and 1,3-PD across the cell membrane, and it is a non-differentiable system; especially, we construct differentiable domains to derive the local bounded properties of the Jacobian, tensor and Hessian matrices of the system and the local asymptotically stability on these differentiable domains under some conditions. Numerical experiments are carried out using the parameter values of [8] to evaluate the validity of the theoretical work.

The rest of this paper is organized as follows. Section 2 describes the nonlinear non-differentiable dynamic system and present the existence of the equilibrium point of the system. By numerical calculation, the equilibrium point of the system is obtained. The local asymptotically stability of Jacobian and Hessian matrices of the system on differential domains are derived in Section 3. We present the local asymptotical stability under some conditions of the system at equilibrium point. Final conclusions follow in Section 4.

2 Nonlinear non-differentiable dynamical system and existence of the equilibrium point

In consideration of the fact that both the inhibition mechanisms of 3-HPA onto the cell growth and the transport systems of glycerol and 1,3-PD across the cell membrane are still unclear, we consider the nonlinear non-differentiable system of [8]

$$\begin{cases} \dot{x}(t) = F(x(t), u), & t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{1}$$

Here $x(t) = (x_1(t), x_2(t), \dots, x_8(t))^T \in \mathbb{R}^8$. $x_1(t)$ is biomass concentrations whose unit is $g \cdot L^{-1}$; $x_2(t), \dots, x_8(t)$ are concentrations of extracellular glycerol, extracellular 1,3-PD, acetate, ethanol, intracellular glycerol, intracellular 3-HPA, and intracellular 1,3-PD at time t, respectively, whose units are mmol $\cdot L^{-1}$; $x_0 = (x_{01}, x_{02}, \dots, x_{08})^T \in \mathbb{R}^8$; t is time, $T > 0$ is the terminal time, $[0, T]$ denotes the interval of reaction time, and the unit is the hour. $u = (u_1, \dots, u_{21})^T \in \mathbb{R}_+^{21}$ is the control variable. To simply notation, set $\mathbb{I}_n = \{1, 2, \dots, n\}$, $n \in \mathbb{Z}^+$; and we denote $x(t)$ as x, if there is no confusion. x_0 is the initial state. $F(x, u) = (f_1(x, u), f_2(x, u), \dots, f_8(x, u))^T$ is the reaction rate, and its components are

$$f_1(x,u) = \bigl(\mu(x) - 0.27\bigr)x_1, \tag{2}$$

$$f_2(x,u) = 0.27(435 - x_2) - q_2(x)x_1, \tag{3}$$

$$f_3(x,u) = q_3(x)x_1 - 0.27x_3, \tag{4}$$

$$f_4(x,u) = q_4(x)x_1 - 0.27x_4, \tag{5}$$

$$f_5(x,u) = q_5(x)x_1 - 0.27x_5, \tag{6}$$

$$f_6(x,u) = \frac{1}{u_7}\left[\frac{u_8 x_2}{x_2 + u_9} + u_{10}(x_2 - x_6)\delta(x_2 - x_6) - \left(2.2 + \frac{\mu(x)}{0.0082} + 28.58\frac{x_2}{x_2 + 11.43}\right)\right] - \mu(x)x_6, \tag{7}$$

$$f_7(x,u) = \frac{u_{11}x_6}{0.53[1 + \frac{x_7}{u_{13}}\delta(x_7 - u_{14})] + x_6} - \frac{u_{15}x_7}{0.14 + x_7 + \frac{x_7^2}{u_{17}}\delta(x_7 - u_{18})} - \mu(x)x_7, \tag{8}$$

$$f_8(x,u) = \frac{u_{15}x_7}{0.14 + x_7 + \frac{x_7^2}{u_{17}}\delta(x_7 - u_{18})} - \frac{u_{19}x_8}{x_8 + u_{20}} - u_{21}(x_8 - x_3)\delta(x_8 - x_3) - \mu(x)x_8. \tag{9}$$

Here $\delta(x)$ is indicator function, $\mu(x)$ is the specific cellular growth rate, due to the influence of 3-HPA, it can be expressed as

$$\mu(x) := \frac{0.67x_2}{x_2 + 0.28}\left(1 - \frac{x_7}{x_{U7}}\right)\prod_{i=2}^{5}\left(1 - \frac{x_i}{x_{Ui}}\right). \tag{10}$$

Here $x_{Ui} = \max_{t \in [0,T]}\{x_i(t)\}$, $i \in \mathbb{I}_8$. $q_2(x)$ is the specific consumption rate of extracellular glycerol, $q_3(x), q_4(x)$ and $q_5(x)$ are the specific formation rates of 1,3-PD, acetate and ethanol, which are defined by

$$q_2(x) := \frac{u_1 x_2}{x_2 + u_2} + u_3(x_2 - x_6)\delta(x_2 - x_6), \tag{11}$$

$$q_3(x) := \frac{u_4 x_8}{x_8 + u_5} + u_6(x_8 - x_3)\delta(x_8 - x_3), \tag{12}$$

$$q_4(x) := -0.97 + 33.07\mu(x) + 5.74\frac{x_2}{x_2 + 85.71}, \tag{13}$$

$$q_5(x) := 5.26 + 11.66\mu(x). \tag{14}$$

According to the actual fermentation process and Ref. [8], some assumptions are presented.

Assumption 2.1 For the control variable $u = (u_1, \dots, u_{21})^T \in \mathbb{R}^{21}$, there exist $0 < u_{Li} < u_{Ui}$, such that $u_{Li} \le u \le u_{Ui}$, $i \in \mathbb{I}_{21}$. That is,

$$u_1 \in [30, 70], \qquad u_2 \in [1, 5], \qquad u_3 \in [100, 5{,}000],$$

$$
\begin{aligned}
&u_4 \in [10,100], \quad u_5 \in [1,10], \\
&u_6 \in [20,30], \quad u_7 \in [5,10], \quad u_8 \in [40,70], \\
&u_9 \in [0.5,3], \quad u_{10} \in [100,5{,}000], \\
&u_{11} \in [1,50], \quad u_{12} \in [100,300], \quad u_{13} \in [100,300], \\
&u_{14} \in [0.1,5], \quad u_{15} \in [1,50], \\
&u_{16} \in [0.01,2], \quad u_{17} \in [0.01,2], \quad u_{18} \in [0.1,5], \\
&u_{19} \in [0.5,20], \quad u_{20} \in [1,30], \\
&u_{21} \in [1,100].
\end{aligned}
$$

Assumption 2.2 For the lower bound and upper bound of the state variable $x(t)$ are

$$
\begin{aligned}
&x_L = [0.01,0,0,0,0,0,0,0]^T, \\
&x_U = [15,2{,}039,939.5,1{,}026,360.9,2{,}039,12.6,939.5]^T.
\end{aligned}
$$

So the admissible set of the state vector is

$$
\mathcal{W}_a \triangleq [x_L, x_U] = \prod_{i=1}^{8} [x_{Li}, x_{Ui}].
$$

Assumption 2.3 Let $f_7(x,u) = 0$ and $f_8(x,u) = 0$, we can find two expressions of x_7 and x_8. Take note that $x_7 = F_7(x_2,x_3,x_4,x_5,x_6,u) > 0$ and $x_8 = F_8(x_2,x_3,x_4,x_5,x_7,u) > 0$.

Assumption 2.4 For any $x \in \mathcal{W}_a$ and $u \in \prod_{i=1}^{21}[u_{Li}, u_{Ui}]$, define

$$
y_1(x_2,u) := \frac{0.27}{q_2(x_2)}(435 - x_2), \tag{15}
$$

$$
y_3(x_2,u) := \frac{q_3(x_2)}{q_2(x_2)}(435 - x_2), \tag{16}
$$

$$
y_4(x_2,u) := \frac{q_4(x_2)}{q_2(x_2)}(435 - x_2), \tag{17}
$$

$$
y_5(x_2,u) := \frac{q_5(x_2)}{q_2(x_2)}(435 - x_2), \tag{18}
$$

$$
\begin{aligned}
y_6(x_2,u) := {}& \frac{1}{u_7\mu(x_2) + u_{10}\delta(x_2 - x_6)} \biggl[u_8 \frac{x_2}{x_2 + u_9} + u_{10}x_2\delta(x_2 - x_6) \\
&- \biggl(2.2 + \frac{\mu(x_2)}{0.0082} + 28.58 \frac{x_2}{x_2 + 11.43} \biggr) \biggr],
\end{aligned} \tag{19}
$$

$$
y_7(x_2,u) := F_7\bigl(x_2, y_3(x_2), y_4(x_2), y_5(x_2), y_6(x_2), u\bigr), \tag{20}
$$

$$
y_8(x_2,u) := F_8\bigl(x_2, y_3(x_2), y_4(x_2), y_5(x_2), y_7(x_2), u\bigr). \tag{21}
$$

Here

$$
\mu(x_2) = \frac{0.67x_2}{x_2 + 0.28} \biggl(1 - \frac{x_2}{x_{U2}} \biggr) \biggl(1 - \frac{x_7}{x_{U7}} \biggr) \prod_{i=3}^{5} \biggl(1 - \frac{q_i(x_2)}{\mu(x_2)x_{Ui}} (435 - x_2) \biggr),
$$

$$q_2(x_2) = u_1 \frac{x_2}{x_2 + u_2} + u_3(x_2 - x_6)\delta(x_2 - x_6),$$
$$q_3(x_2) = u_4 \frac{x_8}{x_8 + u_5} + u_6\left(x_8 - \frac{q_3(x_2)}{\mu(x_2)}(435 - x_2)\right)\delta\left(x_8 - \frac{q_3(x_2)}{\mu(x_2)}(435 - x_2)\right),$$
$$q_4(x_2) = -0.97 + 33.07\mu(x_2) + 5.74\frac{x_2}{x_2 + 85.71},$$
$$q_5(x_2) = 5.26 + 11.66\mu(x_2).$$

For $x_2 = 435$, there exists $\tilde{u} \in \prod_{i=1}^{21}[u_{Li}, u_{Ui}]$, such that $y_i(435, \tilde{u}) = 0$, $i \in \mathbb{I}_8 \backslash \{2\}$.

On the basis of the above four assumptions, define the admissible set of the control vector to be

$$\begin{aligned} \mathcal{U}_a := \Bigg\{ u \in \prod_{i=1}^{21}[u_{Li}, u_{Ui}] \Bigg| & \text{there exists } x_2 \in [x_{L2}, x_{U2}], \text{such that} \\ & \frac{0.67x_2}{x_2 + 0.28}\left(1 - \frac{x_2}{2{,}039}\right)\left(1 - \frac{y_3(x_2,u)}{939.5}\right)\left(1 - \frac{y_4(x_2,u)}{1{,}026}\right) \\ & \times \left(1 - \frac{y_5(x_2,u)}{360.9}\right)\left(1 - \frac{y_7(x_2,u)}{12.6}\right) - 0.27 = 0, \\ & \text{and } x_{Lj} \le y_j \le x_{Uj}, j \in \mathbb{I}_8 \backslash \{2\} \Bigg\}. \end{aligned}$$

Therefore, we can obtain the properties of $F(x,u)$ on the basis of Ref. [8].

Property 2.1 Suppose that Assumptions 2.1-2.4 hold and $F(x,u) \in \mathbb{R}^8$ is defined by equations (2)-(14), then $F(x,u)$ is Lipschitz continuous for any $(x,u) \in \mathcal{W}_a \times \mathcal{U}_a$, and satisfies the linear growth condition.

Property 2.2 According to Property 2.1, we can see that, for any $u \in \mathcal{U}_a$, there exists one unique solution satisfying system (1) and denote it as $x(t) = x(t,u)$.

On the basis of the above four assumptions and the two properties, the next theorem is concerned with the existence of equilibrium point of system (1).

Theorem 2.1 *Suppose that Assumptions* 2.1-2.4 *hold and* $F(x,u)$ *is defined by equations* (2)-(14), *then, for any* $u \in \mathcal{U}_a$, *there exists at least one equilibrium point* $\bar{x} \in \mathcal{W}_a$ *of system* (1).

Proof For any $x_2 \in [x_{L2}, x_{U2}] = [0, 2{,}039]$ and $u \in \mathcal{U}_a$, define

$$\begin{aligned} H(x_2,u) := {} & \frac{0.67x_2}{x_2 + 0.28}\left(1 - \frac{x_2}{2{,}039}\right)\left(1 - \frac{y_3(x_2,u)}{939.5}\right)\left(1 - \frac{y_4(x_2,u)}{1{,}026}\right) \\ & \times \left(1 - \frac{y_5(x_2,u)}{360.9}\right)\left(1 - \frac{y_7(x_2,u)}{12.6}\right) - 0.27. \end{aligned} \tag{22}$$

Taking $x_2 = 0$ of equation (22), we have $H(0,u) = -0.27 < 0$. According to Assumption 2.2, we have $435 \in [0, 2{,}039]$. Thus taking $x_2 = 435$ of equation (22), from Assumption 2.4, we can see that there exist $\tilde{u} \in \prod_{i=1}^{21}[u_{Li}, u_{Ui}]$, such that $y_i(435, \tilde{u}) = 0$, $i \in \mathbb{I}_8\backslash\{2\}$. Then we have

$$H(435, u) = \frac{0.67 \times 435}{435 + 0.28}\left(1 - \frac{435}{2{,}039}\right) - 0.27 \doteq 0.256723 > 0.$$

From Property 2.1, we know $H(x_2, u)$ is continuous on $[0, 2{,}039]$, so it is continuous on $[0, 435]$. According to the intermediate value theorem and the definition of $\mathcal{U}_a$, there exists at least $u^* \in \mathcal{U}_a$ and one point $\bar{x}_2 \in (0, 435)$, such that $H(\bar{x}_2, u^*) = 0$. And $x_{L2} = 0 < 435 < x_{U2}$, so $\bar{x}_2 \in (x_{L2}, x_{U2})$. Substituting $\bar{x}_2$ into equations (15)-(21), we can see that $y_1(\bar{x}_2, u^*) := \bar{x}_1$, $y_3(\bar{x}_2, u^*) := \bar{x}_3$, $y_4(\bar{x}_2, u^*) := \bar{x}_4$, $y_5(\bar{x}_2, u^*) := \bar{x}_5$, $y_6(\bar{x}_2, u^*) := \bar{x}_6$, $y_7(\bar{x}_2, u^*) := \bar{x}_7$, and $y_8(\bar{x}_2, u^*) := \bar{x}_8$ are positive. Therefore we obtain an equilibrium point $\bar{x} := (\bar{x}_1, \ldots, \bar{x}_8)^T \in \mathcal{W}_a$ of system (1). □

Since it is difficult to find the accurate solution of nonlinear equations, we define a so-called approximate equilibrium solution in $\mathcal{W}_a$, dependent on some steady accuracy ε. Let

$$\mathcal{U}_{\text{app}} := \{u \in \mathcal{U}_a | \|F(x, u)\|_2 \leq \zeta, x \in \mathcal{W}_a\},$$
$$\mathcal{W}_{\text{app}} := \{x \in \mathcal{W}_a | \|F(x, u)\|_2 \leq \eta, u \in \mathcal{U}_a\}.$$

Here ζ and η are very little positive real numbers. It is easy to see that $\mathcal{W}_{\text{app}} \neq \Phi$ and there exist $\bar{x} \in \mathcal{W}_a$ and $u^* \in \mathcal{U}_a$ such that $\|F(\bar{x}, u^*)\|_2 \leq \varepsilon$. According to Ref. [8], the initial vector and u^* are chosen

$$x^{(0)} = (2.592063, 3.005114, 312.234314, 78.247200, 80.898071, 2.920278,$$
$$0.973960, 0.480131)^T,$$
$$u^* = \begin{pmatrix} 64.2859, & 2.28551, & 100.008, & 99.991, & 1, & 26.1141, & 9.67773, \\ 40.1389, & 72.97655, & 336.545, & 4.08171, & 281.569, & 171.374, & 4.21562, \\ 3.6506, & 1.9986, & 0.0143828, & 3.81709, & 19.9962, & 2.65421, & 96.5665 \end{pmatrix}.$$

Taking $\varepsilon = 10^{-10}$, we calculate the approximate equilibrium point of system (1) using Newton's method. The approximate equilibrium point is

$$\bar{x} = (2.601233, 2.977254, 312.176110, 78.534049, 81.006248,$$
$$2.892513, 0.966451, 0.479418)^T.$$

Because of the existence of the indicator function $\delta(x)$, system (1) is not differentiable at $\{x \in \mathcal{W}_a | x_2 = x_6\}$ or $\{x \in \mathcal{W}_a | x_3 = x_8\}$ or $\{x \in \mathcal{W}_a | x_7 = u_{14}\}$ or $\{x \in \mathcal{W}_a | x_7 = x_{18}\}$, which makes it difficult to discuss the Jacobian matrix and Hessian matrix of $F(x)$ for $x \in \mathcal{W}_a$ and the stability of the approximate equilibrium point $\bar{x}$ of system (1). Next we will show the local asymptotical stability of the equilibrium point $\bar{x}$ of system (1) in a specified domain of $\mathcal{W}_a$.

3 Asymptotical stability of nonlinear dynamical system

In this section, we will derive the stability of the nonlinear dynamic system (1). For any $x \in \mathcal{W}_a$ and $u \in \mathcal{U}_a$, simplify $x(t, u^*)$ as $x(t)$ and $F(x, u^*)$ as $F(x)$. That is,

$$F(x) := F(x, u^*) = (f_1(x, u^*), \ldots, f_8(x, u^*))^T = (f_1(x), \ldots, f_8(x))^T.$$

Since $F(x)$ is not differentiable at some sub-domains of $\mathcal{W}_a$, first we will construct a domain of $\mathcal{W}_a$ to derive the Jacobian matrix and Hessian matrix of $F(x)$ and their bounded properties. Second, we will get the local asymptotical stability of the equilibrium point of system (1). For any $x \in \mathcal{W}_a$, define

$$\bar{\delta} := \max\{\|x - \bar{x}\| | x \in \mathcal{W}_a, \delta_i(x) = \delta_i(\bar{x}), i \in \mathbb{I}_4\}, \tag{23}$$

and a ball

$$\bar{\mathbb{B}}(\bar{x}, \bar{\delta}) := \{x \in \mathbb{C}([0, T], \mathcal{W}_a) | \|x - \bar{x}\| \le \bar{\delta}\} \subset \mathbb{C}([0, T], \mathbb{R}^8),$$

it is obvious that $\bar{\mathbb{B}}(\bar{x}, \bar{\delta}) \subseteq \mathbb{C}([0, T], \mathcal{W}_a)$. Since $x(t)$ is sufficiently smooth on $\bar{\mathbb{B}}(\bar{x}, \bar{\delta})$, $\bar{\mathbb{B}}(\bar{x}, \bar{\delta})$ is a non-empty compact set. In the next theorem, we will see that $F(x, u)$ is twice continuously differentiable on $(x, u) \in \bar{\mathbb{B}}(\bar{x}, \bar{\delta}) \times \mathcal{U}_a$.

Theorem 3.1 *Suppose that Assumptions* 2.1-2.4 *hold and* $F(x, u) \in \mathbb{R}^8$ *is defined by equations* (2)-(14), *then* $F(x, u)$ *is twice continuously differentiable on* $(x, u) \in \bar{\mathbb{B}}(\bar{x}, \bar{\delta}) \times \mathcal{U}_a$.

Proof From equations (2)-(14) and the definition of $\bar{\mathbb{B}}(\bar{x}, \bar{\delta})$, we can see that there is a two-order partial derivative of $F(x, u)$ on $(x, u) \in \bar{\mathbb{B}}(\bar{x}, \bar{\delta}) \times \mathcal{U}_a$, and all two-order partial derivatives are continuous on $(x, u) \in \bar{\mathbb{B}}(\bar{x}, \bar{\delta}) \times \mathcal{U}_a$. Thus the proof is completed. □

Next we will discuss the local asymptotical stability in $\bar{\mathbb{B}}(\bar{x}, \bar{\delta})$.

3.1 Local bounded properties of $F'(x)$, $F''(x)$ and $f_k''(x)$

Denote $F'(x)$ be the Jacobian matrix of $F(x)$, $f_k''(x), k \in \mathbb{I}_8$ be the Hessian matrix of $f_k(x), k \in \mathbb{I}_8$, $F''(x)$ be the Hessian matrix of $F(x)$. From Theorem 3.1, we can see that $f_k(x)$, $\frac{\partial f_k}{\partial x_i}$, $\frac{\partial^2 f_k}{\partial x_i \partial x_j} \in \mathbb{C}^2(\bar{\mathbb{B}}(\bar{x}, \bar{\delta}))$ for any $i, j, k \in \mathbb{I}_8$, so the Jacobian matrix $F'(x)$, the tensor $F''(x)$ and the Hessian matrix $f_k''(x)$ exist. To get the local asymptotical stability of the equilibrium point $\bar{x}$ of system (1), in this section, we will derive the bounded properties of $F'(x)$, $F''(x)$ and $f_k''(x), k \in \mathbb{I}_8$. For this purpose, the definitions of tensor and norm are shown first.

Definition 3.1 Let $n \in \mathbb{Z}^+$, $G(x) : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently differentiable, where $G(x) = (g_1(x), \ldots, g_n(x))^T$, $x = (x_1, \ldots, x_n)^T$. Define the Jacobian matrix $G'(x)$ to be

$$G'(x) := \left[\frac{\partial g_i(x)}{\partial x_j}\right]_{ij} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}_{n \times n}, \quad \forall i, j \in \mathbb{I}_n.$$

Hessian matrix $H_kG(x)$ is defined to be

$$\begin{aligned}
H_kG(x) &:= \left[\frac{\partial^2 g_k(x)}{\partial x_i\,\partial x_j}\right]_{ij} \\
&= \begin{bmatrix} \frac{\partial^2 g_k}{\partial x_1^2} & \cdots & \frac{\partial^2 g_k}{\partial x_1\,\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 g_k}{\partial x_n\,\partial x_1} & \cdots & \frac{\partial^2 g_k}{\partial x_n^2} \end{bmatrix}_{n\times n}, \quad \forall i,j,k \in \mathbb{I}_n,
\end{aligned}$$

and the tensor $G''(x)$ to be

$$\begin{aligned}
G''(x) &:= \bigl(G'(x)\bigr)' = \begin{bmatrix} H_1G(x) \\ \vdots \\ H_nG(x) \end{bmatrix} \\
&= \begin{bmatrix} [\frac{\partial^2 g_1}{\partial x_i\,\partial x_j}]_{ij} \\ \vdots \\ [\frac{\partial^2 g_n}{\partial x_i\,\partial x_j}]_{ij} \end{bmatrix}_{n^2\times n}, \quad \forall i,j \in \mathbb{I}_n.
\end{aligned}$$

Definition 3.2 Let $n \in \mathbb{Z}^+$, vector $x = (x_1,\dots,x_n)^T \in \mathbb{R}^n$, matrix $A = [a_{ij}]_{n\times n} \in \mathbb{R}^{n\times n}$, vector function $y(t) : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$, matrix function $B(t) : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^{n\times n}$, their norms are defined by

$$\begin{aligned}
\|x\| &:= \sum_{i=1}^{n} |x_i|, \\
\|A\| &:= \max_{j\in\mathbb{I}_n} \sum_{i=1}^{n} |a_{ij}|, \\
\bigl\|y(t)\bigr\| &:= \max_{s\in\Omega} \bigl\|y(s)\bigr\|, \\
\bigl\|B(t)\bigr\| &:= \max_{s\in\Omega} \bigl\|B(s)\bigr\|.
\end{aligned}$$

Remark 3.1 From Definitions 3.1 and 3.2, we can see that the norms of $G'(x)$, $G''(x)$ and $H_kG(x)$ are

$$\begin{aligned}
\bigl\|G'(x)\bigr\| &= \max_{x\in\Omega} \max_{j\in\mathbb{I}_n} \sum_{i=1}^{n} \left|\frac{\partial g_i(x)}{\partial x_j}\right|, \\
\bigl\|H_kG(x)\bigr\| &= \max_{j\in\mathbb{I}_n} \sum_{i=1}^{n} \left|\frac{\partial^2 g_k(x)}{\partial x_i\,\partial x_j}\right|, \\
\bigl\|G''(x)\bigr\| &= \sum_{k=1}^{n} \bigl\|H_kG(x)\bigr\| = \sum_{k=1}^{n} \left\|\left[\frac{\partial^2 g_k(x)}{\partial x_i\,\partial x_j}\right]_{ij}\right\|.
\end{aligned}$$

Theorem 3.2 *Suppose that the conditions of Theorem* 3.1 *hold, then* $F'(x)$, $F''(x)$ *and* $f_k''(x), k \in \mathbb{I}_8$, *are bounded on* $x \in \bar{\mathbb{B}}(\bar{x}, \bar{\delta})$, *where* $\bar{\delta}$ *is defined by equation* (23).

Proof For any $x \in \mathbb{B}(\bar{x}, \bar{\delta})$, the Jacobian matrix $F'(x)$ of $F(x)$ is

$$F'(x) = \begin{bmatrix} \nabla f_1(x)^T \\ \vdots \\ \nabla f_8(x)^T \end{bmatrix}$$

$$= \begin{bmatrix} \mu(x) - 0.27 & x_1 \frac{\partial \mu}{\partial x_2} & x_1 \frac{\partial \mu}{\partial x_3} & x_1 \frac{\partial \mu}{\partial x_4} & x_1 \frac{\partial \mu}{\partial x_5} & 0 & x_1 \frac{\partial \mu}{\partial x_7} & 0 \\ -q_2(x) & -0.27 - x_1 \frac{\partial q_2}{\partial x_2} & 0 & 0 & 0 & -x_1 \frac{\partial q_2}{\partial x_6} & 0 & 0 \\ q_3(x) & 0 & -0.27 & 0 & 0 & 0 & 0 & x_1 \frac{\partial q_3}{\partial x_8} \\ q_4(x) & x_1 \frac{\partial q_4}{\partial x_2} & x_1 \frac{\partial q_4}{\partial x_3} & x_1 \frac{\partial q_4}{\partial x_4} - 0.27 & x_1 \frac{\partial q_4}{\partial x_5} & 0 & x_1 \frac{\partial q_4}{\partial x_7} & 0 \\ q_5(x) & x_1 \frac{\partial q_5}{\partial x_2} & x_1 \frac{\partial q_5}{\partial x_3} & x_1 \frac{\partial q_5}{\partial x_4} & x_1 \frac{\partial q_5}{\partial x_5} - 0.27 & 0 & x_1 \frac{\partial q_5}{\partial x_7} & 0 \\ 0 & \frac{\partial f_6}{\partial x_2} & \frac{\partial f_6}{\partial x_3} & \frac{\partial f_6}{\partial x_4} & \frac{\partial f_6}{\partial x_5} & \frac{\partial f_6}{\partial x_6} & \frac{\partial f_6}{\partial x_7} & 0 \\ 0 & -x_7 \frac{\partial \mu}{\partial x_2} & -x_7 \frac{\partial \mu}{\partial x_3} & -x_7 \frac{\partial \mu}{\partial x_4} & -x_7 \frac{\partial \mu}{\partial x_5} & \frac{\partial f_7}{\partial x_6} & \frac{\partial f_7}{\partial x_7} & 0 \\ 0 & -x_8 \frac{\partial \mu}{\partial x_2} & \frac{\partial f_8}{\partial x_3} & -x_8 \frac{\partial \mu}{\partial x_4} & -x_8 \frac{\partial \mu}{\partial x_5} & 0 & \frac{\partial f_8}{\partial x_7} & \frac{\partial f_8}{\partial x_8} \end{bmatrix}. \tag{24}$$

Here

$$\frac{\partial \mu}{\partial x_2} = \left(1 - \frac{x_7}{12.6}\right)\left[\left(1 - \frac{x_2}{2{,}039}\right)\frac{0.1876}{x_2 + 0.28} - \frac{0.28}{2{,}039(x_2 + 0.28)^2}\right]\prod_{i=3}^{5}\left(1 - \frac{x_i}{x_{Ui}}\right),$$

$$\frac{\partial \mu}{\partial x_3} = -\frac{0.67x_2}{939.5(x_2 + 0.28)}\left(1 - \frac{x_7}{12.6}\right)\left(1 - \frac{x_2}{2{,}039}\right)\left(1 - \frac{x_4}{1{,}026}\right)\left(1 - \frac{x_5}{360.9}\right),$$

$$\frac{\partial \mu}{\partial x_4} = -\frac{0.67x_2}{1{,}026(x_2 + 0.28)}\left(1 - \frac{x_7}{12.6}\right)\left(1 - \frac{x_2}{2{,}039}\right)\left(1 - \frac{x_3}{939.5}\right)\left(1 - \frac{x_5}{360.9}\right),$$

$$\frac{\partial \mu}{\partial x_5} = -\frac{0.67x_2}{360.9(x_2 + 0.28)}\left(1 - \frac{x_7}{12.6}\right)\prod_{i=2}^{4}\left(1 - \frac{x_i}{x_{Ui}}\right),$$

$$\frac{\partial \mu}{\partial x_7} = -\frac{0.67x_2}{12.6(x_2 + 0.28)}\prod_{i=2}^{5}\left(1 - \frac{x_i}{x_{Ui}}\right),$$

$$\frac{\partial q_2}{\partial x_2} = \frac{u_1 u_2}{(x_2 + u_2)^2} + u_3 \delta(x_2 - x_6),$$

$$\frac{\partial q_2}{\partial x_6} = -u_3 \delta(x_2 - x_6), \qquad \frac{\partial q_3}{\partial x_3} = -u_6 \delta(x_8 - x_3),$$

$$\frac{\partial q_3}{\partial x_8} = \frac{u_4 u_5}{(x_8 + u_5)^2} + u_6 \delta(x_8 - x_3),$$

$$\frac{\partial q_4}{\partial x_2} = 33.07\frac{\partial \mu}{\partial x_2} + 5.74\frac{85.71}{(x_2 + 85.71)^2},$$

$$\frac{\partial q_4}{\partial x_i} = 33.07\frac{\partial \mu}{\partial x_i} \quad (i = 3, 4, 5, 7),$$

$$\frac{\partial q_5}{\partial x_i} = 11.66\frac{\partial \mu}{\partial x_i} \quad (i = 2, 3, 4, 5, 7),$$

$$\frac{\partial f_6}{\partial x_2} = \frac{1}{u_7}\left[\frac{u_8 u_9}{(x_2 + u_9)^2} + u_{10}\delta(x_2 - x_6) - \frac{1}{0.0082}\frac{\partial \mu}{\partial x_2} - 28.58\frac{11.43}{(x_2 + 11.43)^2}\right] - x_6\frac{\partial \mu}{\partial x_2},$$

$$\frac{\partial f_6}{\partial x_i} = -\left(\frac{1}{0.0082u_7} + x_6\right)\frac{\partial \mu}{\partial x_i} \quad (i = 3, 4, 5, 7),$$

$$
\begin{aligned}
\frac{\partial f_6}{\partial x_6} &= -\frac{u_{10}}{u_7}\delta(x_2 - x_6) - \mu(x), \\
\frac{\partial f_7}{\partial x_6} &= \frac{0.53u_{11}(1 + \frac{x_7}{u_{13}}\delta(x_7 - u_{14}))}{(0.53(1 + \frac{x_7}{u_{13}}\delta(x_7 - u_{14})) + x_6)^2}, \\
\frac{\partial f_7}{\partial x_7} &= \frac{-0.53u_{11}x_6\delta(x_7 - u_{14})}{u_{13}(0.53(1 + \frac{x_7}{u_{13}}\delta(x_7 - u_{14})) + x_6)^2} \\
&\quad - \frac{u_{15}(0.14 + x_7 + \frac{x_7^2}{u_{17}}\delta(x_7 - u_{18}) - x_7(1 + \frac{2}{u_{17}}\delta(x_7 - u_{18})))}{(0.14 + x_7 + k_{13}\frac{x_7^2}{u_{16}} + \frac{x_7^2}{u_{17}}\delta(x_7 - u_{18}))^2} \\
&\quad - \mu(x) - x_7\frac{\partial \mu}{\partial x_7}, \\
\frac{\partial f_8}{\partial x_3} &= u_{21}\delta(x_8 - x_3) - x_8\frac{\partial \mu}{\partial x_3}, \\
\frac{\partial f_8}{\partial x_7} &= \frac{u_{15}(0.14 + x_7 + \frac{x_7^2}{u_{17}}\delta(x_7 - u_{18}) - \frac{2x_7}{u_{17}}\delta(x_7 - u_{18}))}{(0.14 + x_7 + \frac{x_7^2}{u_{17}}\delta(x_7 - u_{18}))^2} - x_8\frac{\partial \mu}{\partial x_7}, \\
\frac{\partial f_8}{\partial x_8} &= -\frac{u_{19}u_{20}}{(x_8 + u_{20})^2} - u_{21}\delta(x_8 - x_3) - \mu(x).
\end{aligned}
$$

And the tensor $F''(x)$ of $F(x)$ is

$$
F''(x) = \begin{bmatrix} H_1F(x) \\ \vdots \\ H_8F(x) \end{bmatrix} = \begin{bmatrix} Hf_1(x) \\ \vdots \\ Hf_8(x) \end{bmatrix} = \begin{bmatrix} [\frac{\partial^2 f_1}{\partial x_i\,\partial x_j}]_{ij} \\ \vdots \\ [\frac{\partial^2 f_8}{\partial x_i\,\partial x_j}]_{ij} \end{bmatrix}_{8^2\times 8}, \tag{25}
$$

which can be viewed as a matrix of dimension $8^2 \times 8$.

The Hessian matrix of $f_k(x), k \in \mathbb{I}_8$ is

$$
f_k''(x) = Hf_k(x) = \left[\frac{\partial^2 f_k(x)}{\partial x_i\,\partial x_j}\right]_{8\times 8}, \quad k \in \mathbb{I}_8.
$$

Since $f_k(x), \frac{\partial f_k}{\partial x_i}, \frac{\partial^2 f_k}{\partial x_i\,\partial x_j} \in \mathbb{C}^2(\bar{\mathbb{B}}(\bar{x},\bar{\delta}))$ for any $i,j,k \in \mathbb{I}_8$, according to Definition 3.2 and $\bar{\mathbb{B}}(\bar{x},\bar{\delta}) \subset \mathbb{C}([0,T],\mathbb{R}^8)$ is a non-empty compact set, $\|f_k(x)\|, |\frac{\partial f_k}{\partial x_i}|$ and $|\frac{\partial^2 f_k}{\partial x_i\,\partial x_j}|$ are bounded on $\bar{\mathbb{B}}(\bar{x},\bar{\delta})$, whose bounds are defined by $\widetilde{M}(>0)$. Then, for any $x \in \bar{\mathbb{B}}(\bar{x},\bar{\delta})$,

$$
\begin{aligned}
\|F'(x)\| &= \max_{x\in\Omega}\max_{j\in\mathbb{I}_8}\sum_{i=1}^{8}\left|\frac{\partial f_i}{\partial x_j}\right| \le 8\widetilde{M}, \\
\|f_k''(x)\| &= \left\|\left[\frac{\partial^2 f_k}{\partial x_i\,\partial x_j}\right]_{ij}\right\| = \max_{j\in\mathbb{I}_8}\sum_{i=1}^{8}\left|\frac{\partial^2 f_k}{\partial x_i\,\partial x_j}\right| \le 8\widetilde{M}, \\
\|F''(x)\| &= \sum_{k=1}^{8}\|f_k''(x)\| \le 64\widetilde{M}.
\end{aligned}
$$

Therefore, $F'(x)$, $F''(x)$ and $f_k''(x), k \in \mathbb{I}_8$ are bounded for any $x \in \bar{\mathbb{B}}(\bar{x},\bar{\delta})$. □

3.2 Local asymptotical stability of the equilibrium point

From Theorem 2.1, we obtain the equilibrium point $\bar{x}$ of system (1) corresponding one $u^* \in \mathbb{U}_a$. According to Property 2.1, expanding $F(x)$ defined by equations (2)-(9) with a Taylor formula around $x = \bar{x}$ and considering $F(\bar{x}) = (f_1(\bar{x}), \ldots, f_8(\bar{x}))^T = 0$, we can get

$$f_k(x) = \nabla f_k(\bar{x})^T (x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T f_k''(\xi)(x - \bar{x}), \quad k \in \mathbb{I}_8,$$

where $\exists \theta_k \in (0,1), k \in \mathbb{I}_8$, such that $\xi = x + \theta_k(\bar{x} - x) \in \bar{\mathbb{B}}(\bar{x}, \bar{\delta})$. $\nabla f_k(x)$ is the gradient of $f_k(x)$ at $\bar{x}$. $f_k''(\xi)$ is the Hessian matrix at ξ. $F'(\bar{x}) = (\nabla f_1(\bar{x}), \ldots, \nabla f_8(\bar{x}))^T$ is the Jacobian matrix of $F(x)$ at $\bar{x}$ as defined by equation (24). Then system (1) can be approximately transformed into

$$\begin{cases} \dot{x}_k(t) = \nabla f_k(\bar{x})(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^T f_k''(\xi)(x - \bar{x}), & k \in \mathbb{I}_8, t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{26}$$

Obviously $\bar{x}$ is the equilibrium point of system (26), and is also that of the following linear system:

$$\begin{cases} \dot{x}_k(t) = \nabla f_k(\bar{x})(x - \bar{x}), & k \in \mathbb{I}_8, t \in [0, T], \\ x(0) = x_0. \end{cases} \tag{27}$$

To prove the local asymptotical stability of system (1), two lemmas are put forward.

Lemma 3.1 ([20]) *Suppose that λ_i is an eigenvalue of Jacobian matrix $F'(\bar{x}) = (\nabla f_1(\bar{x}), \ldots, \nabla f_8(\bar{x}))^T$, and each λ_i satisfies the condition* $\mathrm{Re}(\lambda_i) < 0$, *then the equilibrium point $x = \bar{x}$ of system* (27) *is asymptotically stable.*

Lemma 3.2 ([21]) *Suppose that $\bar{x}$ is an equilibrium point of a linear time-varying system* (27) *and asymptotically stable. If $F'(\bar{x})$ is bounded, and the nonlinear perturbation term $g_k(\bar{x}, t) = \frac{1}{2}(x - \bar{x})^T f_k''(\xi)(x - \bar{x})$, $\xi = x + \theta_k(\bar{x} - x) \in \bar{\mathbb{B}}(\bar{x}, \bar{\delta})$, $\theta_k \in (0,1), k \in \mathbb{I}_8$, satisfies*

$$\lim_{\|x - \bar{x}\| \to 0} \sup_{t \ge 0} \frac{\|g_k(x - \bar{x}, t)\|}{\|x - \bar{x}\|} = 0,$$

then the equilibrium point $x = \bar{x}$ of the nonlinear system (26) *is local asymptotically stable.*

From Lemmas 3.1 and 3.2, it can be seen that we can use a linear approximate system to get the local asymptotically stable property of $\bar{x}$.

Theorem 3.3 *Suppose that the conditions of Theorem* 2.1 *hold, $\bar{x}$ is an equilibrium point of system* (1) *corresponding $u^* \in \mathcal{U}_a$; for any $x \in \bar{\mathbb{B}}(\bar{x}, \bar{\delta})$, then the equilibrium point $\bar{x}$ of system* (1) *is local asymptotically stable on $\bar{\mathbb{B}}(\bar{x}, \bar{\delta})$.*

Proof From Theorem 3.1, we obtain the equilibrium point $\bar{x}$ of system (1) corresponding $u^* \in \mathbb{U}_a$. First, we will prove that the equilibrium point $\bar{x}$ of the linear system (27) is asymptotically stable. We calculate eigenvalues of Jacobian matrix $F'(\bar{x})$ to be

$$\lambda_1 = -3.0748, \qquad \lambda_2 = -0.0568, \qquad \lambda_3 = -0.0174, \qquad \lambda_4 = -0.0069,$$

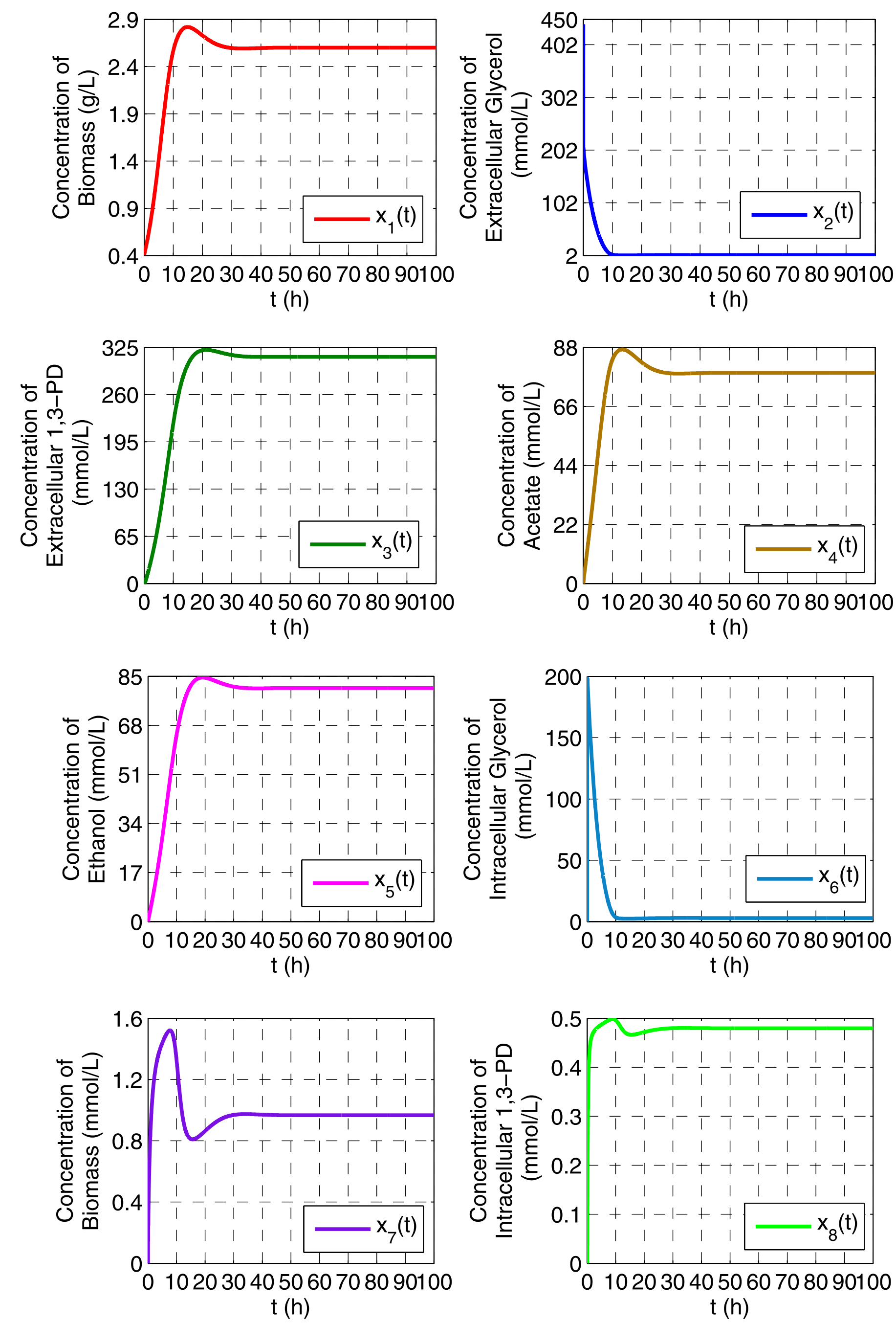

Figure 1 Change of the state variable $x(t)$ with $t \in [0, 100]$.

$$\lambda_5 = -0.0017 + 0.0017i, \qquad \lambda_6 = -0.0017 - 0.0017i,$$

$$\lambda_7 = -0.0027, \qquad \lambda_8 = -0.0027.$$

For any $i \in \mathbb{I}_8$, we have $\mathrm{Re}(\lambda_i) < 0$. Thus according to Lemma 3.1, we can see that the equilibrium point $\bar{x}$ of the linear system (27) is asymptotically stable.

Next we will derive that the equilibrium point $\bar{x}$ of system (26) is local asymptotically stable. From Theorem 3.2, we have $f_k''(\xi)$ is bounded in $\bar{\mathbb{B}}(\bar{x},\bar{\delta}) \subset \mathbb{C}([0,T],\mathbb{R}^8)$. According to Definition 3.2, we consider the following limit:

$$\lim_{\|x-\bar{x}\|\to 0} \sup_{t\geq 0} \frac{1}{2} \frac{\|(x(t)-\bar{x})^T f_k''(\xi)(x(t)-\bar{x})\|}{\|x(t)-\bar{x}\|} = \lim_{\|x-\bar{x}\|\to 0} \sup_{t\geq 0} \frac{1}{2} \|f_k''\| \cdot \|x(t)-\bar{x}\| = 0.$$

Moreover, $F'(\bar{x})$ is bounded in $\bar{\mathbb{B}}(\bar{x},\bar{\delta})$ according to Theorem 3.2, and the equilibrium point $\bar{x}$ of the linear system (27) is asymptotically stable. Therefore, from Lemma 3.2, the equilibrium point $\bar{x}$ of system (26) is local asymptotically stable. That is, the equilibrium point $\bar{x}$ of system (1) is local asymptotically stable. □

Remark 3.2 Euler method is performed with the step $\frac{1}{3{,}600}$ hour. The terminal time is taken $T = 100$ hour. Figure 1 shows the change of every sub-vector of the state vector $x(t)$ with $t \in [0,100]$. From this figure, we can conclude that every curve is stable.

4 Conclusions

In this work, we have considered the stability of a nonlinear non-differentiable dynamic system in a microbial continuous culture. The existence of the equilibrium point has been derived. Because of its non-differentiable property, we have constructed a differentiable domain, in which the stability of the system has been derived. The local bounded properties of the Jacobian, tensor and Hessian matrices of the system have been proved. A numerical experiment has been performed to evaluate its effectiveness.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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