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On the spectrum of the pencil of high order differential operators with almost periodic coefficients

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Abstract

In this paper, the spectrum and the resolvent of the operator L_λ which is generated by the differential expression $\ell_\lambda(y) = y^{(m)} + \sum_{\gamma=1}^{m}(\sum_{k=0}^{\gamma} \lambda^k p_{\gamma k}(x))y^{(m-\gamma)}$ has been investigated in the space $L_2(\mathbb{R})$. Here the coefficients $p_{\gamma k}(x) = \sum_{n=1}^{\infty} p_{\gamma kn}e^{i\alpha_n x}$, $k = 0,1,\ldots,\gamma-1$; $p_{\gamma\gamma}(x) = p_{\gamma\gamma}$, $\gamma = 1,2,\ldots,m$, are constants, $p_{mm} \neq 0$ and $p_{\gamma k}^{(\nu)}(x)$, $\nu = 0,1,2,\ldots,m-\gamma$, are Bohr almost-periodic functions whose Fourier series are absolutely convergent. The sequence of Fourier exponents of coefficients (these are positive) has a unique limit point at $+\infty$. It has been shown that if the polynomial $\phi(z) = z^m + p_{11}z^{m-1} + p_{22}z^{m-2} + \cdots + p_{m-1,m-1}z + p_{mm}$ has the simple roots $\omega_1,\omega_2,\ldots,\omega_m$ (or one multiple root ω_0), then the spectrum of operator L_λ is pure continuous and consists of lines $\mathrm{Re}(\lambda\omega_k) = 0$, $k = 1,2,\ldots,m$ (or of line $\mathrm{Re}(\lambda\omega_0) = 0$). Moreover, a countable set of spectral singularities on the continuous spectrum can exist which coincides with numbers of the form $\lambda = 0$, $\lambda_{sjn} = i\alpha_n(\omega_j - \omega_s)^{-1}$, $n \in \mathbb{N}$, $s,j = 1,2,\ldots,m$, $j \neq s$. If $\phi(z) = (z-\omega_0)^m$, then the spectral singularity does not exist. The resolvent L_λ^{-1} is an integral operator in $L_2(\mathbb{R})$ with the kernel of Karleman type for any $\lambda \in \rho(L_\lambda)$.

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Keywords: almost-periodic; spectrum; resolvent; eigenvalue; spectral singularity

1 Introduction

In this study, we investigate the spectrum and the resolvent of the maximal differential operator L_λ which is generated by the linear differential expression

$$\ell_\lambda(y) = y^{(m)} + \sum_{\gamma=1}^{m} p_\gamma(x,\lambda)y^{(m-\gamma)}$$

in the space $L_2(\mathbb{R})$, where λ is a complex parameter,

$$\begin{aligned} &p_\gamma(x,\lambda) = \sum_{k=0}^{\gamma} \lambda^k p_{\gamma k}(x), \qquad p_{\gamma\gamma}(x) = p_{\gamma\gamma}, \\ &p_{\gamma k}(x) = \sum_{n=1}^{\infty} p_{\gamma kn}e^{i\alpha_n x}, \quad \gamma = 1,2,\ldots,m,\ k = 0,1,\ldots,\gamma-1, \end{aligned} \tag{1}$$

with $p_{\gamma\gamma}, p_{\gamma kn} \in \mathbb{C}$, $p_{mm} \neq 0$ and the condition

$$\sum_{\gamma=1}^{m}\sum_{k=0}^{\gamma-1}\sum_{n=1}^{\infty}\alpha_n^{m-\gamma}|p_{\gamma kn}| < +\infty \tag{2}$$

is satisfied. Here $(\alpha_n)_{n=1}^{\infty}$ is an increasing sequence of positive numbers with $\alpha_n \to +\infty$ and the set $\{\alpha_n : n \in \mathbb{N}\}$ is an additive semigroup.

The operator L_λ is defined in the domain

$$\begin{aligned} D(L_\lambda) = \bigl\{&y(x)|y(x), y'(x), \dots, y^{(m-1)}(x) \in \mathrm{AC}[a,b] \text{ for all } a,b \in \mathbb{R}, \\ &y(x), \ell_\lambda(y) \in L_2(\mathbb{R})\bigr\}. \end{aligned}$$

If at least one of the functions $p_{\gamma k}(x)$, $\gamma = 1,2,\dots,m$, $k = 0,1,\dots,\gamma-1$, is not zero, then the operator L_λ is non-self-adjoint for each $\lambda \in \mathbb{C}$.

Let AP^+ be a class of Bohr almost-periodic functions $q(x) = \sum_{n=1}^{\infty} q_n e^{i\alpha_n x}$, where $\|q\| = \sum_{n=1}^{\infty}|q_n| < +\infty$. In the case $\alpha_n = n$, $n \in \mathbb{N}$, we denote this class by Q^+. It is clear that AP^+ is a normed space and (2) means that $p_{\gamma k}^{(\nu)}(x) \in AP^+$ for $\gamma = 1,2,\dots,m$, $k = 0,1,\dots,\gamma-1$, $\nu = 0,1,\dots,m-\gamma$.

Under the assumed conditions, coefficients $p_\gamma(x,\lambda)$ can be represented as

$$\begin{aligned} p_\gamma(x,\lambda) &= p_{\gamma\gamma}\lambda^\gamma + \sum_{k=0}^{\gamma-1}\lambda^k\sum_{n=1}^{\infty}p_{\gamma kn}e^{i\alpha_n x} = p_{\gamma\gamma}\lambda^\gamma + \sum_{n=1}^{\infty}\left(\sum_{k=0}^{\gamma-1}\lambda^k p_{\gamma kn}\right)e^{i\alpha_n x} \\ &= p_{\gamma\gamma}\lambda^\gamma + \sum_{n=1}^{\infty}\tilde{p}_{\gamma n}(\lambda)e^{i\alpha_n x}, \quad \gamma = 1,2,\dots,m. \end{aligned}$$

Here, $\tilde{p}_{\gamma n}(\lambda)$ is an algebraic polynomial whose degree does not exceed $\gamma - 1$. Moreover, according to (2) the series $\sum_{n=1}^{\infty}|\tilde{p}_{\gamma n}(\lambda)|$ is majorized in every compact set $S \subseteq \mathbb{C}$, *i.e.* for $\tilde{p}_{\gamma n} = \sup_{\lambda\in S}|\tilde{p}_{\gamma n}(\lambda)|$, $\gamma = 1,2,\dots,m$, $n \in \mathbb{N}$, the series $\sum_{n=1}^{\infty}\tilde{p}_{\gamma n}$ converges.

Let ω_k, $k = 1,2,\dots,m$, denote the roots of the characteristic polynomial

$$\phi(z) = z^m + p_{11}z^{m-1} + p_{22}z^{m-2} + \cdots + p_{m-1,m-1}z + p_{mm}$$

corresponding to the linear differential expression $\ell_\lambda(y)$, numbered as $0 \leq \arg\omega_1 \leq \arg\omega_2 \leq \cdots \leq \arg\omega_m < 2\pi$ and $\lambda_{sjn} = i\alpha_n(\omega_j - \omega_s)^{-1}$ for $n \in \mathbb{N}$, $j,s = 1,2,\dots,m$, $j \neq s$.

Let $\Lambda = \{\lambda_{sjn} : s,j = 1,2,\dots,m,\ j \neq s, n \in \mathbb{N}\}$ and $\Lambda_0 = \Lambda \cup \{0\}$. It is obvious that the roots are different from zero according to the condition $p_{mm} \neq 0$. Below we shall assume that these roots are different or all coincide and any three of these roots are not on the same line in the complex plane. Under these conditions, for each constant s the numbers $\lambda_{sjn} = i\alpha_n(\omega_j - \omega_s)^{-1}$, $n \in \mathbb{N}$, $j = 1,2,\dots,m$, $j \neq s$, are located on the $m-1$ rays from the origin. Moreover, since $\lambda_{sjn} = -\lambda_{jsn}$, the set Λ is symmetric with respect to the origin.

The lines $l_k = \{\lambda : \lambda \in \mathbb{C}, \mathrm{Re}(\lambda\omega_k) = 0\}$, $k = 1,2,\dots,m$, divide the complex λ-plane into $2m_0$ open sectors S_k, $k = 1,2,\dots,2m_0$. Let us assume that beginning from sector S_1 whose closure contains positive numbers, these sectors are numbered $S_1, S_2, \dots, S_{2m_0}$ $(m_0 \leq m)$ counterclockwise successively. It is clear that if there are different roots ω_k, ω_j such that $\omega_k/\omega_j \in \mathbb{R}$, then the lines l_k and l_j coincide. Therefore, the number of sectors S_k may be less than $2m$. In the case $\phi(z) = (z - \omega_0)^m$ the line $l_0 = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Re} \lambda\omega_0 = 0\}$ divides the complex λ-plane into two half planes $S^+ = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Re} \lambda\omega_0 > 0\}$, $S^- = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Re} \lambda\omega_0 < 0\}$. In the sequel we shall see that the resolvent set of the operator L_λ consists of the above defined sectors.

The interest of the investigation of the spectral properties of the differential operators with coefficients belonging to class AP^+ has been increased after the study of [1]. In [1] the differential operator $L(y) = -y'' + q(x)y$ with periodic potential $q(x) \in Q^+$ (case $m = 2$) has been investigated in the space $L_2(\mathbb{R})$. In this study the spectral data $\{s_n\}_{n\in\mathbb{N}}$ has been determined and sufficient conditions have been obtained for solvability of the inverse problem according to the spectral data. Afterward, in [2] were found the necessary and sufficient conditions for a set $\{s_n\}_{n\in\mathbb{N}}$ to be the spectral data of the operator $L(y) = -y'' + q(x)y$ with periodic potential $q(x) \in L_2(0, 2\pi)$. In [3], the results of [2] are generalized for almost-periodic potential $q(x)$ having only positive Fourier exponents. The spectral properties of ordinary differential operators of high order with coefficients from AP^+ have been investigated in [4, 5]. The spectrum and the resolvent of the bundle of $2n$ order and second order differential operators with coefficients from AP^+ and from Q^+ have been examined in [6–9] and in [10], respectively. In all of these studies the examined operators in the space $L_2(\mathbb{R})$ have a pure continuous spectrum which consists of a half-line or a union of lines passing from the origin. Moreover, there may be at most a countable number of spectral singularities on the continuous spectrum of the examined operators.

In the present paper the spectrum and the resolvent of the class of a pencil of m order differential operators, with coefficients from AP^+, have been investigated under more general conditions. It has been proved that the operator L_λ has a pure continuous spectrum. If the characteristic polynomial $\phi(z)$ has only simple roots ω_k, $k = 1, 2, \ldots, m$, the continuous spectrum consists of the lines $\operatorname{Re}(\lambda\omega_k) = 0$, $k = 1, 2, \ldots, m$. Moreover, there may be spectral singularities (in the sense of Naimark [11]) on the continuous spectrum which coincide with numbers of the form $\lambda = 0$, $\lambda_{sjn} = i\alpha_n(\omega_j - \omega_s)^{-1}$, $s, j = 1, 2, \ldots, m$, $j \neq s$, $n \in \mathbb{N}$. If the characteristic polynomial $\phi(z)$ has one multiple root ω_0 then the continuous spectrum consists of the line $\operatorname{Re}(\lambda\omega_0) = 0$ and a spectral singularity does not exist. The resolvent operator L_λ^{-1} is an integral operator in $L_2(\mathbb{R})$ with the kernel of Karleman type for any $\lambda \in \rho(L_\lambda)$. Under weakened conditions, the obtained results of this paper generalize all results of [5, 10] and some parts of results of [7–9].

2 Floquet solutions of the equation $\ell_\lambda(y) = 0$

Here, we will show the existence of the Floquet solutions of the equation $\ell_\lambda(y) = 0$, which plays an important role in the investigation of the spectrum of the operator L_λ. If the characteristic polynomial has more than one multiple root, then there may arise various cases to obtain the fundamental system of solutions. Below, we consider the cases when there exist simple roots or one multiple root.

Case I. The characteristic polynomial $\phi(z)$ has different simple roots $\omega_1, \omega_2, \ldots, \omega_m$.

Theorem 1 *If* (1), (2) *hold and* ω *is any root of* $\phi(z) = 0$ *then for each* $\lambda \in \mathbb{C}$, $\lambda \neq i\alpha_n(\omega_j - \omega)^{-1}$, $j = 1, 2, \ldots, m$, $\omega_j \neq \omega$, $n \in \mathbb{N}$, *the differential equation*

$$y^{(m)} + \sum_{\gamma=1}^{m} p_\gamma(x, \lambda) y^{(m-\gamma)} = 0 \tag{3}$$

has the solution

$$f(x,\lambda) = e^{\omega\lambda x}\left(1 + \sum_{n=1}^{\infty} U_n(\lambda)e^{i\alpha_n x}\right), \tag{4}$$

where

$$U_n(\lambda) = U_n + \sum_{k=1}^{n}\sum_{\substack{j=1\\ \omega_j\neq\omega}}^{m} \frac{U_{jkn}}{[i\alpha_k + (\omega-\omega_j)\lambda]}, \quad \forall n\in\mathbb{N},$$

with $U_n, U_{jkn} \in \mathbb{C}$ *and series*

$$\sum_{n=1}^{\infty} \alpha_n^{\gamma}\left|U_n(\lambda)\right|, \quad \gamma = 0,1,\ldots,m, \tag{5}$$

is majorized in each compact set $S \subseteq \mathbb{C}$ *which does not contain the numbers* $\lambda = i\alpha_n(\omega_j - \omega)^{-1}$ *for* $j = 1,2,\ldots,m$, $\omega_j \neq \omega$, $n \in \mathbb{N}$.

Proof Let ω be any root of the characteristic polynomial $\phi(\omega)$. If we assume the existence of the solution of equation (3) represented as (4) with convergent series (5), then we can find the derivatives of $f(x,\lambda)$ with respect to x as

$$f^{(\gamma)}(x,\lambda) = e^{\omega\lambda x}\left((\omega\lambda)^{\gamma} + \sum_{n=1}^{\infty}(i\alpha_n + \omega\lambda)^{\gamma} U_n(\lambda)e^{i\alpha_n x}\right), \quad \gamma = 0,1,\ldots,m. \tag{6}$$

If we substitute these derivatives in (3) and divide both sides by $e^{\omega\lambda x}$, then we get

$$\begin{aligned}
&(\omega\lambda)^m + \sum_{n=1}^{\infty}(i\alpha_n + \omega\lambda)^m U_n(\lambda)e^{i\alpha_n x} + \sum_{\gamma=1}^{m}(\omega\lambda)^{m-\gamma} p_{\gamma\gamma}\lambda^{\gamma} \\
&\quad + \sum_{\gamma=1}^{m} p_{\gamma\gamma}\lambda^{\gamma}\sum_{n=1}^{\infty}(i\alpha_n + \omega\lambda)^{m-\gamma} U_n(\lambda)e^{i\alpha_n x} \\
&\quad + \sum_{\gamma=1}^{m}(\omega\lambda)^{m-\gamma}\sum_{n=1}^{\infty}\tilde{p}_{\gamma n}(\lambda)e^{i\alpha_n x} \\
&\quad + \sum_{\gamma=1}^{m}\sum_{n=1}^{\infty}\tilde{p}_{\gamma n}(\lambda)e^{i\alpha_n x}\sum_{n=1}^{\infty}(i\alpha_n + \omega\lambda)^{m-\gamma} U_n(\lambda)e^{i\alpha_n x} = 0
\end{aligned}$$

or

$$\begin{aligned}
&\lambda^m\phi(\omega) + \sum_{n=1}^{\infty}\left[(i\alpha_n + \omega\lambda)^m + \sum_{\gamma=1}^{m} p_{\gamma\gamma}\lambda^{\gamma}(i\alpha_n + \omega\lambda)^{m-\gamma}\right] U_n(\lambda)e^{i\alpha_n x} \\
&\quad + \sum_{n=1}^{\infty}\sum_{\gamma=1}^{m}(\omega\lambda)^{m-\gamma}\tilde{p}_{\gamma n}(\lambda)e^{i\alpha_n x} \\
&\quad + \sum_{\gamma=1}^{m}\sum_{n=2}^{\infty}\left(\sum_{\alpha_r+\alpha_s=\alpha_n}\tilde{p}_{\gamma s}(\lambda)(i\alpha_r + \omega\lambda)^{m-\gamma} U_r(\lambda)\right)e^{i\alpha_n x} = 0.
\end{aligned}$$

Taking into account the uniqueness theorem for almost-periodic functions we have

$$\left[(i\alpha_n+\omega\lambda)^m+\sum_{\gamma=1}^{m}p_{\gamma\gamma}\lambda^\gamma(i\alpha_n+\omega\lambda)^{m-\gamma}\right]U_n(\lambda)+\sum_{\gamma=1}^{m}(\omega\lambda)^{m-\gamma}\tilde{p}_{\gamma n}(\lambda)$$
$$+\sum_{\gamma=1}^{m}\sum_{\alpha_r+\alpha_s=\alpha_n}\tilde{p}_{\gamma s}(\lambda)(i\alpha_r+\omega\lambda)^{m-\gamma}U_r(\lambda)=0,\quad n\in\mathbb{N}.$$

Using the expansion

$$\begin{aligned}&(i\alpha_n+\omega\lambda)^m+\sum_{\gamma=1}^{m}p_{\gamma\gamma}\lambda^\gamma(i\alpha_n+\omega\lambda)^{m-\gamma}\\&\quad=\lambda^m\left[\left(\frac{i\alpha_n}{\lambda}+\omega\right)^m+\sum_{\gamma=1}^{m}p_{\gamma\gamma}\left(\frac{i\alpha_n}{\lambda}+\omega\right)^{m-\gamma}\right]=\lambda^m\phi\left(\frac{i\alpha_n}{\lambda}+\omega\right)\\&\quad=\left[i\alpha_n+\lambda(\omega-\omega_1)\right]\cdot\left[i\alpha_n+\lambda(\omega-\omega_2)\right]\cdot\ldots\cdot\left[i\alpha_n+\lambda(\omega-\omega_m)\right],\end{aligned}$$

we obtain

$$U_n(\lambda)=-\frac{\sum_{\gamma=1}^{m}(\lambda\omega)^{m-\gamma}\tilde{p}_{\gamma n}(\lambda)+\sum_{\gamma=1}^{m}\sum_{\alpha_r+\alpha_s=\alpha_n}(i\alpha_r+\omega\lambda)^{m-\gamma}\tilde{p}_{\gamma s}(\lambda)U_r(\lambda)}{[i\alpha_n+\lambda(\omega-\omega_1)]\cdot[i\alpha_n+\lambda(\omega-\omega_2)]\cdot\ldots\cdot[i\alpha_n+\lambda(\omega-\omega_m)]}\tag{7}$$

for $\lambda\in\mathbb{C}$, $\lambda\neq i\alpha_n(\omega_j-\omega)^{-1}$, $j=1,2,\ldots,m$, $\omega_j\neq\omega$, $n\in\mathbb{N}$.

On the contrary, if $\{U_n(\lambda)\}$ satisfies the system of equations (7) and the series (5) converges, then it can be shown that $f(x,\lambda)$ determined by (4) is a solution of (3). Therefore, the solvability of (7) and the convergence of the series (5) are sufficient to prove the theorem.

From (7), $\{U_n(\lambda)\}$ is determined by the recurrent manner uniquely. It is possible to see that $U_n(\lambda)$ is the rational function which can have simple poles $\lambda=i\alpha_k(\omega_j-\omega)^{-1}$, $j=1,2,\ldots,m$, $\omega_j\neq\omega$, $k=1,2,\ldots,n$, and therefore it can be uniquely written as

$$U_n(\lambda)=U_n+\sum_{k=1}^{n}\sum_{\substack{j=1\\ \omega_j\neq\omega}}^{m}\frac{U_{jkn}}{[i\alpha_k+(\omega-\omega_j)\lambda]},\quad\forall n\in\mathbb{N},$$

where $U_n,U_{jkn}\in\mathbb{C}$. Let $S\subseteq\mathbb{C}$ be a compact set which does not contain the points $\lambda=i\alpha_n(\omega_j-\omega)^{-1}$ for $j=1,2,\ldots,m$, $\omega_j\neq\omega$, $n\in\mathbb{N}$. Let us show that the series (5) is majorized in S for $\{U_n(\lambda)\}$ which is determined from (7).

It is obvious that there exist $c_0>0$, $q>1$ such that

$$c_0\alpha_n^m\leq\left|\left[i\alpha_n+(\omega-\omega_1)\lambda\right]\cdot\left[i\alpha_n+(\omega-\omega_2)\lambda\right]\cdot\ldots\cdot\left[i\alpha_n+(\omega-\omega_m)\lambda\right]\right|$$

and $|\omega\lambda|\leq q$ for $\forall n\in\mathbb{N}$, $\forall\lambda\in S$. Then from (7), we have

$$\begin{aligned}c_0\alpha_n^m\left|U_n(\lambda)\right|&\leq\sum_{\gamma=1}^{m}|\omega\lambda|^{m-\gamma}\left|\tilde{p}_{\gamma n}(\lambda)\right|+\sum_{\gamma=1}^{m}\sum_{\alpha_r+\alpha_s=\alpha_n}\left|U_r(\lambda)\right|\cdot|i\alpha_r+\omega\lambda|^{m-\gamma}\left|\tilde{p}_{\gamma s}(\lambda)\right|\\&\leq\sum_{\gamma=1}^{m}q^{m-\gamma}\tilde{p}_{\gamma n}+\sum_{\gamma=1}^{m}\sum_{\alpha_r+\alpha_s=\alpha_n}\left|U_r(\lambda)\right|\left(|\alpha_r|+q\right)^{m-\gamma}\tilde{p}_{\gamma s}\end{aligned}$$

$$\le q^{m-1}\sum_{\gamma=1}^{m}\tilde{p}_{\gamma n}+\sum_{\gamma=1}^{m}\sum_{\alpha_r+\alpha_s=\alpha_n}\left|U_r(\lambda)\right|\alpha_r^{m-\gamma}\left(1+\frac{q}{\alpha_1}\right)^{m-\gamma}\tilde{p}_{\gamma s}$$

$$\le q^{m-1}\sum_{\gamma=1}^{m}\tilde{p}_{\gamma n}+\left(1+\frac{q}{\alpha_1}\right)^{m-1}\sum_{\alpha_r+\alpha_s=\alpha_n}\left|U_r(\lambda)\right|\alpha_r^{m-1}\cdot\sum_{\gamma=1}^{m}\tilde{p}_{\gamma s}$$

for $\forall\lambda\in S$, $\forall n\in\mathbb{N}$ and $\tilde{p}_{\gamma n}=\sup_{\lambda\in S}|\tilde{p}_{\gamma n}(\lambda)|$, $\gamma=1,2,\dots,m$.

Let

$$A_n=\frac{q^{m-1}}{c_0}\sum_{\gamma=1}^{m}\tilde{p}_{\gamma n},$$

$$B_n=\frac{1}{c_0}\left(1+\frac{q}{\alpha_1}\right)^{m-1}\sum_{\gamma=1}^{m}\tilde{p}_{\gamma n}.$$

Then from the last inequality we obtain

$$\alpha_n^m\left|U_n(\lambda)\right|\le A_n+\sum_{\alpha_r+\alpha_s=\alpha_n}\alpha_r^{m-1}\left|U_r(\lambda)\right|B_s,\quad n\in\mathbb{N}.$$

If $u_n=\sup_{\lambda\in S}|U_n(\lambda)|$, then we have

$$\alpha_n^m u_n\le A_n+\sum_{\alpha_r+\alpha_s=\alpha_n}\alpha_r^{m-1}u_rB_s,\quad n\in\mathbb{N}$$

or

$$\sum_{n=1}^{t}\alpha_n^m u_n\le\sum_{n=1}^{t}A_n+\sum_{n=1}^{t}\sum_{\alpha_r+\alpha_s=\alpha_n}\alpha_r^{m-1}u_rB_s$$

$$\le A+\sum_{r=1}^{t-1}\alpha_r^{m-1}u_r\sum_{s=1}^{t-1}B_s\le A+B\sum_{n=1}^{t-1}\alpha_n^{m-1}u_n,\quad n\in\mathbb{N}.$$

From (2) it is clear that $A=\sum_{n=1}^{\infty}A_n<+\infty$ and $B=\sum_{n=1}^{\infty}B_n<+\infty$.

Therefore, for all $t\in\mathbb{N}$, $\sum_{n=1}^{t}\alpha_n^m u_n\le A+B\sum_{n=1}^{t-1}\alpha_n^{m-1}u_n$ is satisfied. By using this inequality and $\alpha_n\to+\infty$, we can easily show the convergence of the series $\sum_{n=1}^{+\infty}\alpha_n^m u_n$ according to the lemma in [12] (see [12], pp.21-22). In this case, $\sum_{n=1}^{+\infty}\alpha_n^m|U_n(\lambda)|$ is a majorized series in S. According to the Weierstrass theorem, the series (5) is uniform convergent in S. Since $S\subseteq\mathbb{C}$ is an arbitrarily chosen compact set, the series (5) is convergent for all $\lambda\ne i\alpha_n(\omega_j-\omega)^{-1}$, $j=1,2,\dots,m$, $\omega_j\ne\omega$, and $n\in\mathbb{N}$. Thus $f(x,\lambda)$ is a solution of equation (3). The theorem is proved. □

It is clear that $\lambda_{jn}=i\alpha_n(\omega_j-\omega)^{-1}$ may be a singular point of $f(x,\lambda)$ for any $j=1,2,\dots,m$, $\omega_j\ne\omega$, $n\in\mathbb{N}$. Actually, according to Theorem 1, the functional series in the representation

$$\left[i\alpha_n+(\omega-\omega_j)\right]f(x,\lambda)=e^{\omega\lambda x}\left(1+\sum_{r=1}^{\infty}\left[i\alpha_n+(\omega-\omega_j)\lambda\right]U_r(\lambda)e^{i\alpha_r x}\right)$$

and the obtained series by m times term by term differentiation are absolutely and uniformly convergent with respect to λ in the closed disk with a small radius centered in point λ_{jn}. Therefore, the finite limits

$$\lim_{\lambda\to\lambda_{jn}}\bigl[i\alpha_n+(\omega-\omega_j)\lambda\bigr]\frac{\partial^s f(x,\lambda)}{\partial x^s}=e^{\omega\lambda_{jn}x}\sum_{r=n}^{\infty}U_{jnr}(\omega\lambda_{jn}+i\alpha_r)^s e^{i\alpha_r x},\quad s=0,1,2,\dots,m,$$

exist. Moreover, the series $\sum_{r=n}^{\infty}|U_{jnr}|\alpha_r^m$ is convergent. If this limit is not zero, then it means that the point λ_{jn} is a simple pole of the functions $\frac{\partial^s f(x,\lambda)}{\partial x^s}$, $s=0,1,\dots,m$.

Corollary 1 *For* $\forall x\in R$, *the functions* $\frac{\partial^s f(x,\lambda)}{\partial x^s}$, $s=0,1,\dots,m$, *are meromorphic functions with respect to* λ *and they can have only simple poles* $\lambda_{jn}=i\alpha_n(\omega_j-\omega)^{-1}$, $j=1,2,\dots,m$, $\omega_j\neq\omega$, $n\in\mathbb{N}$. *Moreover, these functions are also continuous functions of the pair* (x,λ) *for all* $(x,\lambda)\in\mathbb{R}\times\mathbb{C}$, $\lambda\neq i\alpha_n(\omega_j-\omega)^{-1}$, $j=1,2,\dots,m$, $\omega_j\neq\omega$, $n\in\mathbb{N}$.

Corollary 2 *For* $\forall\lambda\neq\lambda_{sjn}$, $s,j=1,2,\dots,m$, $j\neq s$, $n\in\mathbb{N}$, *equation* (3) *has the Floquet solutions represented as*

$$f_s(x,\lambda)=e^{\omega_s\lambda x}\left(1+\sum_{n=1}^{\infty}U_n^{(s)}(\lambda)e^{i\alpha_n x}\right),$$

where

$$U_n^{(s)}(\lambda)=U_n^{(s)}+\sum_{k=1}^{n}\sum_{\substack{j=1\\ j\neq s}}^{m}\frac{U_{jkn}^{(s)}}{i\alpha_k+(\omega_s-\omega_j)\lambda}.$$

The Wronskian of the functions $f_1(x,\lambda),f_2(x,\lambda),\dots,f_m(x,\lambda)$ *for* $\lambda\in\mathbb{C}\backslash\Lambda$ *is found as*

$$W[f_1,f_2,\dots,f_m]=W(x,\lambda)=\lambda^{\frac{m(m-1)}{2}}W_m e^{-\lambda p_{11}x-\sum_{n=1}^{\infty}\frac{p_{10n}}{i\alpha_n}e^{i\alpha_n x}}.\tag{8}$$

Here W_m *is the Vandermonde determinant of the numbers* $\omega_1,\omega_2,\dots,\omega_m$. *The functions* $f_1(x,\lambda),f_2(x,\lambda),\dots,f_m(x,\lambda)$ *form the fundamental system of solutions of equation* (3) *in the interval* $(-\infty,+\infty)$ *for* $\forall\lambda\in\mathbb{C}\backslash\Lambda_0$.

It is clear that the existence of the solutions $f_1(x,\lambda),f_2(x,\lambda),\dots,f_m(x,\lambda)$ follows from Theorem 1 for $\omega=\omega_s$, $s=1,2,\dots,m$. Since every $q(x)\in AP^+$ can be extended to the upper semi-plane as an analytic function of x and $\lim_{\operatorname{Im}x\to+\infty}q(x)=0$, by passing to the limit as $\operatorname{Im}x\to+\infty$ on both sides of the equation

$$W(x,\lambda)e^{\lambda p_{11}x}=W(0,\lambda)e^{-\int_0^x p_{10}(t)\,dt}$$

we get equation (8). From (8) it follows that if $\lambda\neq0$, then $W(x,\lambda)\neq0$. Thus the system of functions is independent.

Note that the solutions of type $f_s(x,\lambda)$, $s=1,2,\dots,m$, are obtained in [6–9] under the different conditions and in various forms of the representation.

According to Corollary 1, it is obvious that the function

$$f_{sjn}(x)=\lim_{\lambda\to\lambda_{sjn}}f_s(x,\lambda)\bigl[i\alpha_n+(\omega_s-\omega_j)\lambda\bigr]=e^{\omega_s\lambda_{sjn}x}\sum_{k=n}^{\infty}U_{jnk}^{(s)}e^{i\alpha_k x}$$

is a solution of equation (3) for $\lambda = \lambda_{sjn}$, where the series $\sum_{k=n}^{\infty} \alpha_k^m |U_{jnk}^{(s)}|$ is convergent. As in [4], writing the equation which is satisfied by the coefficients $U_{jnk}^{(s)}$ (see [4], p.778), it is seen easily that $U_{jnk}^{(s)} = 0$ for every $k \geq n$ if $U_{jnn}^{(s)} = 0$. Therefore, $f_{sjn}(x) \equiv 0$ if and only if $U_{jnn}^{(s)} = 0$. In this case, $f_s(x, \lambda)$ is regular at point $\lambda = \lambda_{sjn}$ and $f_s(x, \lambda_{sjn})$ is a solution of equation (3). It can be shown that the functions $f_{sjn}(x)$ and $f_j(x, \lambda_{sjn})$ are linearly dependent. Moreover, $f_{sjn}(x) = U_{jnn}^{(s)} f_j(x, \lambda_{sjn})$ for any $s = 1, 2, \ldots, m$, $s \neq j = 1, 2, \ldots, m$, $n \in \mathbb{N}$, is valid which is important for establishing the fundamental system of solutions of equation (3) for $\lambda = \lambda_{sjn}$.

Let s, j, n be fixed and $f_s(x, \lambda), f_{s_1}(x, \lambda), f_{s_2}(x, \lambda), \ldots, f_{s_\mu}(x, \lambda)$ be all functions which have a pole at the point $\lambda = \lambda_{sjn}$. It is only possible when the equality $\lambda_{sjn} = \lambda_{s_\beta j_\beta n_\beta}$, $\beta = 1, 2, \ldots, \mu$, is valid for some different indices $n_1, n_2, \ldots, n_\mu \in \mathbb{N}$, and $1 \leq j_1, j_2, \ldots, j_\mu \leq m$. Then all other functions $f_j(x, \lambda), f_{j_1}(x, \lambda), f_{j_2}(x, \lambda), \ldots, f_{j_\mu}(x, \lambda), f_{j_{\mu+1}}(x, \lambda), \ldots, f_{j_\nu}(x, \lambda)$, $\mu + \nu + 2 = m$, are regular at the point $\lambda = \lambda_{sjn}$.

If we define the functions $f_{kjn}(x) = \lim_{\lambda \to \lambda_{sjn}} f_k(x, \lambda)[i\alpha_n + (\omega_s - \omega_j)\lambda]$, $k = s_1, s_2, \ldots, s_\mu$, then it is obvious that the functions

$$\begin{aligned} &f_{sjn}(x), \quad f_{s_1 jn}(x), \quad f_{s_2 jn}(x), \quad \ldots, \\ &f_{s_\mu jn}(x), \quad f_k(x, \lambda_{sjn}), \quad k = j, j_1, j_2, \ldots, j_\nu, \end{aligned} \tag{9}$$

are solutions of equation (3) for $\lambda = \lambda_{sjn}$ and the functions of this system are linear dependent in $(-\infty, +\infty)$, since their Wronskian is equal to zero. Moreover, any three of the numbers $\mathrm{Re}(\lambda_{sjn}\omega_k)$, $k = 1, 2, \ldots, m$, can not be equal and there are some equal pairs between them. These equal pairs are $\mathrm{Re}(\lambda_{sjn}\omega_s) = \mathrm{Re}(\lambda_{sjn}\omega_j)$, $\mathrm{Re}(\lambda_{sjn}\omega_{s_\beta}) = \mathrm{Re}(\lambda_{sjn}\omega_{j_\beta})$, $\beta = 1, 2, \ldots, \mu$. Then taking into our account the behaviors as $x \to \pm\infty$ of the functions belonging to the system (9), as it is shown in [12] (see pp.43-45), we have the existence of some constants b_k, $k = s, s_1, s_2, \ldots, s_\mu$ such that

$$f_{sjn}(x) = b_s f_j(x, \lambda_{sjn}), \qquad f_{s_\beta jn}(x) = b_{s_\beta} f_{j_\beta}(x, \lambda_{sjn}), \quad \beta = 1, 2, \ldots, \mu.$$

From the equality $f_{sjn}(x) = b_s f_j(x, \lambda_{sjn})$, according to the uniqueness theorem for almost-periodic functions, it is seen that $b_s = U_{jnn}^{(s)}$. Using the equalities

$$f_{sjn}(x) = b_s f_j(x, \lambda_{sjn}), \qquad f_{s_\beta jn}(x) = b_{s_\beta} f_{j_\beta}(x, \lambda_{sjn}), \quad \beta = 1, 2, \ldots, \mu,$$

the system of linearly independent solutions of equation (3) corresponding to $\lambda = \lambda_{sjn}$ can be established.

Since the functions $f_k(x, \lambda)$, $k = j, j_1, j_2, \ldots, j_\mu$, are regular at $\lambda = \lambda_{sjn}$, the functions

$$\begin{aligned} \tilde{f}_{sjn}(x) &= \lim_{\lambda \to \lambda_{sjn}} \left(f_s(x, \lambda) - \frac{b_s f_j(x, \lambda)}{i\alpha_n + (\omega_s - \omega_j)\lambda} \right), \\ \tilde{f}_{s_\beta jn}(x) &= \lim_{\lambda \to \lambda_{sjn}} \left(f_{s_\beta}(x, \lambda) - \frac{b_{s_\beta} f_{j_\beta}(x, \lambda)}{i\alpha_n + (\omega_s - \omega_j)\lambda} \right), \quad \beta = 1, 2, \ldots, \mu, \end{aligned}$$

are also solutions of equation (3) corresponding to $\lambda = \lambda_{sjn}$. According to the expressions of the functions $f_k(x, \lambda)$, $k = s, s_1, s_2, \ldots, s_\mu, j, j_1, j_2, \ldots, j_\mu$, we conclude that $\tilde{f}_{kjn}(x) = e^{i\omega_k \lambda_{sjn} x}(\psi_{kjn}(x) + x\phi_{kjn}(x))$, where $\psi_{kjn}(x)$ and $\phi_{kjn}(x)$ are Bohr almost-periodic functions for $k = s, s_1, s_2, \ldots, s_\mu$. From the explicit form of the functions $\tilde{f}_{sjn}(x), \tilde{f}_{s_1 jn}(x), \tilde{f}_{s_2 jn}(x), \ldots, \tilde{f}_{s_\mu jn}(x)$,

$f_k(x, \lambda_{sjn})$, $k = j, j_1, j_2, \ldots, j_\nu$, it is seen that these functions are linearly independent on $(-\infty, +\infty)$. Therefore, these functions form a fundamental system of solutions of equation (3) for $\lambda = \lambda_{sjn}$.

Now let us construct the linearly independent solutions of equation (3) for $\lambda = 0$.

Note that, since the Wronskian of the solutions $f_s(x, \lambda)$, $s = 1, 2, \ldots, m$, is equal to zero for $\lambda = 0$, they are linearly dependent. Linearly independent solutions of equation (3) corresponding to $\lambda = 0$ are established according to Theorem 1. It is clear that solutions of the equation

$$y^{(m)} + \sum_{\gamma=1}^{m} p_{\gamma 0}(x) y^{(m-\gamma)} = \lambda^m y \tag{10}$$

corresponding to $\lambda = 0$ are also solutions of equation (3) for $\lambda = 0$. By Theorem 1, equation (10) has the solution

$$\tilde{f}(x, \lambda) = e^{\lambda x} \left(1 + \sum_{n=1}^{\infty} \widetilde{U}_n(\lambda) e^{i\alpha_n x} \right),$$

which is analytic with respect to λ in some small neighborhood of $\lambda = 0$. By putting $\tilde{f}(x, \lambda)$ in (10) and by differentiating equation (10) with respect to λ, it is sure that functions $\tilde{f}_s(x) = \frac{\partial^s \tilde{f}(x,\lambda)}{\partial \lambda^s}|_{\lambda=0}$, $s = 0, 1, \ldots, m-1$, are also solutions of (10) and (3) corresponding to $\lambda = 0$. We can see easily that $\tilde{f}_0(x) = \alpha_{00}(x)$, $\tilde{f}_1(x) = x\alpha_{11}(x) + \alpha_{10}(x), \ldots, \tilde{f}_{m-1}(x) = x^{m-1}\alpha_{m-1,m-1}(x) + x^{m-2}\alpha_{m-1,m-2}(x) + \cdots + \alpha_{m-1,0}(x)$, where $\alpha_{sj}(x)$, $s = 0, 1, \ldots, m-1$, $j = 0, 1, \ldots, s$, are Bohr almost-periodic functions and $\alpha_{ss}(x)$, $s = 0, 1, \ldots, m-1$, are nonzero. The linear independence of $\tilde{f}_s(x)$, $s = 0, 1, \ldots, m-1$, in $(-\infty, +\infty)$ is seen from their open form.

Case II. The characteristic polynomial has a unique multiple root ω_0, *i.e.* $\phi(z) = (z - \omega_0)^m$.

In this case, to find the particular solutions of equation (3) we will use the following theorem.

Theorem 2 *If the characteristic polynomial has a unique multiple root* ω_0, *then for each function* $g(x, \lambda) = e^{\omega_0 \lambda x} \sum_{n=1}^{\infty} g_n(\lambda) e^{i\alpha_n x}$ *such that* $g_n(\lambda)$, $n \in \mathbb{N}$, *are polynomials whose degree does not exceed* $n(m-1)$ *and the series* $\sum_{n=1}^{\infty} |g_n(\lambda)|$ *is majorized in any compact set* $S \subseteq \mathbb{C}$, *the equation*

$$y^{(m)} + \sum_{\gamma=1}^{m} p_\gamma(x, \lambda) y^{(m-\gamma)} = g(x, \lambda) \tag{11}$$

has a solution

$$h(x, \lambda) = e^{\omega_0 \lambda x} \left(1 + \sum_{n=1}^{\infty} h_n(\lambda) e^{i\alpha_n x} \right) \tag{12}$$

in $(-\infty, +\infty)$ *for every* $\lambda \in \mathbb{C}$. *Here the coefficients* $h_n(\lambda)$, $n \in \mathbb{N}$, *are polynomials whose degrees do not exceed* $n(m-1)$, *and the series* $\sum_{n=1}^{\infty} |h_n(\lambda)| \alpha_n^m$ *is majorized in each compact set* $S \subseteq \mathbb{C}$.

Proof If we substitute the function (12) in (11), to find the coefficients sequence $\{h_n(\lambda)\}$ as in the proof of Theorem 1, we obtain a system of equations,

$$\left[(i\alpha_n+\omega\lambda)^m+\sum_{\gamma=1}^m p_{\gamma\gamma}\lambda^\gamma(i\alpha_n+\omega\lambda)^{m-\gamma}\right]h_n(\lambda)+\sum_{\gamma=1}^m(\omega\lambda)^{m-\gamma}\tilde p_{\gamma n}(\lambda)$$
$$+\sum_{\gamma=1}^m\sum_{\alpha_r+\alpha_s=\alpha_n}\tilde p_{\gamma s}(\lambda)(i\alpha_r+\omega\lambda)^{m-\gamma}h_r(\lambda)=g_n(\lambda),\quad n\in\mathbb{N},\lambda\in\mathbb{C}$$

or

$$(i\alpha_n)^m h_n(\lambda)+\sum_{\gamma=1}^m(\omega\lambda)^{m-\gamma}\tilde p_{\gamma n}(\lambda)$$
$$+\sum_{\gamma=1}^m\sum_{\alpha_r+\alpha_s=\alpha_n}\tilde p_{\gamma s}(\lambda)(i\alpha_r+\omega\lambda)^{m-\gamma}h_r(\lambda)=g_n(\lambda),\quad n\in\mathbb{N},\lambda\in\mathbb{C},$$

or

$$h_n(\lambda)=-\frac{\sum_{\gamma=1}^m(\lambda\omega)^{m-\gamma}\tilde p_{\gamma n}(\lambda)+\sum_{\gamma=1}^m\sum_{\alpha_r+\alpha_s=\alpha_n}(i\alpha_r+\omega\lambda)^{m-\gamma}\tilde p_{\gamma s}(\lambda)h_r(\lambda)-g_n(\lambda)}{(i\alpha_n)^m},$$
$$n\in\mathbb{N},\lambda\in\mathbb{C}. \tag{13}$$

The coefficients $h_n(\lambda)$ are found uniquely from equation (13). In fact, the degree of the polynomial $h_1(\lambda)=-\frac{\sum_{\gamma=1}^m(\lambda\omega)^{m-\gamma}\tilde p_{\gamma 1}(\lambda)-g_1(\lambda)}{(i\alpha_1)^m}$ for $n=1$ does not exceed $m-1$. Subsequently, for $n=2$ the degree of the polynomial $h_2(\lambda)$ does not exceed $2(m-1)$ and, for each n, $h_n(\lambda)$ is found as a polynomial whose degree does not exceed $n(m-1)$. If for the obtained coefficients $h_n(\lambda)$, $n\in\mathbb{N}$, $h_n=\sup_{\lambda\in S}|h_n(\lambda)|$, then convergence of the series $\sum_{n=1}^{+\infty}\alpha_n^m h_n$ and so majorization of the series $\sum_{n=1}^{+\infty}\alpha_n^m|h_n(\lambda)|$ in the set $S\subseteq\mathbb{C}$ easily can be shown as in the proof of Theorem 1. Therefore, for each $\lambda\in S$ the function $h(x,\lambda)=e^{\omega\lambda x}(1+\sum_{n=1}^\infty h_n(\lambda)e^{i\alpha_n x})$ is the solution of equation (11) in $(-\infty,+\infty)$. The theorem is proved. □

Corollary 3 *If the characteristic polynomial $\phi(z)$ has a unique multiple root ω_0, then equation* (3) *has a solution $\hat f(x,\lambda)=e^{\omega\lambda x}q(x,\lambda)$ in $(-\infty,+\infty)$ for every $\lambda\in\mathbb{C}$. Here $q(x,\lambda)=1+\sum_{n=1}^\infty q_n(\lambda)e^{i\alpha_n x}$ is a Bohr almost-periodic function. The $q_n(\lambda)$, $n\in\mathbb{N}$, are polynomials whose degree does not exceed $n(m-1)$, the series $\sum_{n=1}^{+\infty}\alpha_n^m|q_n(\lambda)|$ is majorized in each compact set $S\subseteq\mathbb{C}$. $\hat f(x,\lambda)$, $\frac{\partial\hat f(x,\lambda)}{\partial x},\ldots,\frac{\partial^m\hat f(x,\lambda)}{\partial x^m}$ are continuous functions in $\mathbb{R}\times\mathbb{C}$ with respect to the ordered pair (x,λ) and they are an entire function of λ.*

To prove Corollary 3, it is enough to take $g(x,\lambda)=0$ in Theorem 2 and to see m times differentiability term by term of the series in the expression of the obtained solution $\hat f(x,\lambda)$ with respect to x. Here the obtained series are uniformly convergent in each bounded set of the ordered pairs (x,λ), therefore functions $\hat f(x,\lambda)$, $\frac{\partial\hat f(x,\lambda)}{\partial x},\ldots,\frac{\partial^m\hat f(x,\lambda)}{\partial x^m}$ are continuous functions of the ordered pairs (x,λ) and they are entire functions of λ.

Theorem 3 *If $\phi(z)=(z-\omega_0)^m$, then equation* (3) *has Floquet solutions in the interval $(-\infty,+\infty)$ as*

$$\hat f_1(x,\lambda)=e^{\omega\lambda x}q_1(x,\lambda),\qquad \hat f_2(x,\lambda)=e^{\omega\lambda x}\bigl[xq_1(x,\lambda)+q_2(x,\lambda)\bigr],\qquad\ldots,$$

$$\hat{f}_m(x,\lambda) = e^{\omega\lambda x}\left[\frac{x^{m-1}}{(m-1)!}q_1(x,\lambda) + \cdots + xq_{m-1}(x,\lambda) + q_m(x,\lambda)\right],$$

where the functions $q_1(x,\lambda), q_2(x,\lambda), \dots, q_m(x,\lambda)$ *are almost-periodic function*:

$$q_s(x,\lambda) = 1 + \sum_{n=1}^{\infty} q_{sn}(\lambda)e^{i\alpha_n x}, \quad s = 1,2,\dots,m$$

for $\forall\lambda \in \mathbb{C}$. *Here* $q_{sn}(\lambda)$, $s = 1,2,\dots,m$, $n \in \mathbb{N}$, *are polynomials whose degrees do not exceed* $n(m-1)$ *and the series* $\sum_{n=1}^{\infty} \alpha_n^m |q_{sn}(\lambda)|$ *are majorized in each compact set* $S \subseteq \mathbb{C}$.

Proof When $\phi(z) = (z-\omega_0)^m$, equation (3) has a solution $\hat{f}_1(x,\lambda) = e^{\omega\lambda x}q_1(x,\lambda)$ according to Corollary 3. In order to obtain other solutions which form a fundamental system of solutions of equation (3) together with $\hat{f}_1(x,\lambda)$, let us use the properties of the linear differential operator $L : C^m(\mathbb{R}) \to C(\mathbb{R})$, which is defined as

$$L(y) = p_0(x)y^{(m)} + p_1(x)y^{(m-1)} + p_2(x)y^{(m-2)} + \cdots + p_{m-1}(x)y' + p_m(x)y, \tag{14}$$

where $p_j(x) \in C(\mathbb{R})$, $j = 0,1,\dots,m$. Let us define the operators $L^{(k)}(y) = \sum_{\gamma=0}^{m-k} A_m^k p_\gamma(x) \times y^{(m-\gamma-k)}$, $L^{(k)} : C^m(\mathbb{R}) \to C(\mathbb{R})$, $k = 1,2,\dots,m$. Here, $A_m^k = m(m-1)\cdot\ldots\cdot(m-k+1)$, $k = 1,2,\dots,m$, and $A_m^0 = 1$. For any system of functions $y_1(x), y_2(x), \dots, y_m(x) \in C^m(\mathbb{R})$ it is not difficult to show that the identities

$$\begin{aligned}
&L(xy_1 + y_2) = L(y_2) + L^{(1)}(y_1) + xL(y_1),\\
&L\left(\frac{x^2}{2!}y_1 + xy_2 + y_3\right) = L(y_3) + L^{(1)}(y_2) + \frac{1}{2!}L^{(2)}(y_1) + x\bigl[L(y_2) + L^{(1)}(y_1)\bigr] + \frac{x^2}{2!}L(y_1),\\
&\dots,\\
&L\left(\frac{x^{s-1}}{(s-1)!}y_1 + \frac{x^{s-2}}{(s-2)!}y_2 + \cdots + \frac{x}{1!}y_{s-1} + y_s\right)\\
&\quad = L(y_s) + L^{(1)}(y_{s-1}) + \frac{1}{2!}L^{(2)}(y_{s-2}) + \cdots + \frac{1}{(s-1)!}L^{(s-1)}(y_1)\\
&\qquad + x\left[L(y_{s-1}) + \frac{1}{1!}L^{(1)}(y_{s-2}) + \cdots + \frac{1}{(s-2)!}L^{(s-2)}(y_1)\right] + \cdots\\
&\qquad + \frac{x^{s-2}}{(s-2)!}\bigl[L(y_2) + L^{(1)}(y_1)\bigr] + \frac{x^{s-1}}{(s-1)!}L(y_1), \quad s = 2,3,\dots,m, \forall x \in \mathbb{R},
\end{aligned}$$

hold. Therefore, when the equations

$$\begin{aligned}
&L(y_1) = 0,\\
&L(y_2) + L^{(1)}(y_1) = 0,\\
&L(y_3) + L^{(1)}(y_2) + \frac{1}{2!}L^{(2)}(y_1) = 0,\\
&\dots,\\
&L(y_m) + L^{(1)}(y_{m-1}) + \frac{1}{2!}L^{(2)}(y_{m-2}) + \cdots + \frac{1}{(m-1)!}L^{(m-1)}(y_1) = 0,
\end{aligned} \tag{15}$$

are satisfied, the functions

$$\tilde{y}_1 = y_1, \qquad \tilde{y}_j = \frac{x^{j-1}}{(j-1)!}y_1 + \frac{x^{j-2}}{(j-2)!}y_2 + \cdots + \frac{x}{1!}y_{j-1} + y_j, \quad j = 2,3,\ldots,m, \tag{16}$$

are solutions of the equation $L(y) = 0$.

Let us show the existence of functions $y_s = e^{\omega\lambda x}q_s(x,\lambda)$, $s = 1,2,\ldots,m$, satisfying the system of equations (15) for the operators $L = L_\lambda$, $L^{(k)} = L_\lambda^{(k)}$, $k = 1,2,\ldots,m$. If we set in these equations $L = L_\lambda$ and $y_1 = \hat{f}_1(x,\lambda)$, the solution $y_2 = e^{\omega\lambda x}q_2(x,\lambda)$, which satisfies the equation

$$L_\lambda(y_2) + L_\lambda^{(1)}(y_1) = 0,$$

exists according to Theorem 2 for $g(x,\lambda) = -L_\lambda^{(1)}(y_1)$. It is not difficult to verify that the conditions of Theorem 2 are satisfied. In the same manner, when the functions $y_s = e^{\omega\lambda x}p_s(x,\lambda)$, $s = 1,2,\ldots,k-1$, were found, the existence of the function $y_k = e^{\omega\lambda x}q_k(x,\lambda)$ which satisfies the equation

$$L_\lambda(y_k) = -L_\lambda^{(1)}(y_{k-1}) - \frac{1}{2!}L_\lambda^{(2)}(y_{k-2}) - \cdots - \frac{1}{(k-1)!}L_\lambda^{(k-1)}(y_1), \quad k = 2,3,\ldots,m,$$

is obtained according to Theorem 2 by induction for $g(x,\lambda) = -L_\lambda^{(1)}(y_{k-1}) - \frac{1}{2!}L_\lambda^{(2)}(y_{k-2}) - \cdots - \frac{1}{(k-1)!}L_\lambda^{(k-1)}(y_1)$. Consequently, according to (15), (16) the functions

$$\begin{aligned}
&\hat{f}_1(x,\lambda) = e^{\omega\lambda x}q_1(x,\lambda), \qquad \hat{f}_2(x,\lambda) = e^{\omega\lambda x}\bigl[xq_1(x,\lambda) + q_2(x,\lambda)\bigr], \\
&\ldots, \\
&\hat{f}_m(x,\lambda) = e^{\omega\lambda x}\left[\frac{x^{m-1}}{(m-1)!}q_1(x,\lambda) + \cdots + xq_{m-1}(x,\lambda) + q_m(x,\lambda)\right]
\end{aligned}$$

are solutions of equation (3) in $(-\infty,+\infty)$ for $\lambda \in \mathbb{C}$. The theorem is proved. □

Note that the solutions of type $\hat{f}_s(x,\lambda)$, $s = 1,2,\ldots,m$, are obtained in [7] under the different conditions and in various form of the representation.

Corollary 4 *When $\phi(z) = (z - \omega_0)^m$, for each $\lambda \in \mathbb{C}$, $x \in \mathbb{R}$ Wronskian of functions $\hat{f}_1(x,\lambda), \hat{f}_2(x,\lambda), \ldots, \hat{f}_m(x,\lambda)$ is found as*

$$\widehat{W}(x,\lambda) = e^{m\omega_0\lambda x - \sum_{n=1}^{\infty}\frac{p_{10n}}{i\alpha_n}e^{i\alpha_n x}} \neq 0 \tag{17}$$

and hence for each $\lambda \in \mathbb{C}$, the functions $\hat{f}_1(x,\lambda), \hat{f}_2(x,\lambda), \ldots, \hat{f}_m(x,\lambda)$ form the fundamental system of solutions of equation (3) *in the interval* $(-\infty,+\infty)$.

3 The spectrum and resolvent of the operator L_λ

Here we investigate the structure of the spectrum of the operator L_λ and the resolvent operator L_λ^{-1}.

Theorem 4 *The operator L_λ does not have eigenvalues, i.e. $\sigma_p(L_\lambda) = \varnothing$.*

Proof Let us show that equation $L_\lambda y = 0$ has only a trivial solution which belongs to $L_2(\mathbb{R})$ for $\forall\lambda \in \mathbb{C}$. In the case of simple roots of characteristic polynomial for $\lambda \neq 0$, $\lambda \neq \lambda_{sjn}$, $s,j = 1,2,\ldots,m$, $j \neq s$, $n \in \mathbb{N}$, it follows from the properties of the solutions $f_k(x,\lambda)$, $k = 1,2,\ldots,m$. Really, the solution $y(x,\lambda) = c_1 f_1(x,\lambda) + c_2 f_2(x,\lambda) + \cdots + c_m f_m(x,\lambda)$ is in $L_2(\mathbb{R})$ if and only if $c_1 = c_2 = \cdots = c_m = 0$. If we take linearly independent solutions of (3) according to $\lambda = 0$ or $\lambda = \lambda_{sjn}$, then a similar result is also valid. Hence $\sigma_p(L_\lambda) = \varnothing$. The theorem is proved. □

Theorem 5 *The residual spectrum of the operator L_λ is an empty set, i.e. $\sigma_r(L_\lambda) = \varnothing$.*

Proof Since $\sigma_p(L_\lambda) = \varnothing$ and for every $\lambda \in \mathbb{C}$ the operator L_λ is one to one, $\lambda \in \sigma_r(L_\lambda)$ if and only if the range $R(L_\lambda)$ is not dense in $L_2(\mathbb{R})$. It means the equation $L_\lambda^*(z) = 0$ has a nontrivial solution $z(x,\lambda) \in L_2(\mathbb{R})$, in other words $\overline{z(x,\lambda)}$ satisfies the conjugate equation

$$(-1)^m z^{(m)} + \sum_{\gamma=1}^{m} (-1)^{(m-\gamma)} \big[p_\gamma(x,\lambda) z\big]^{(m-\gamma)} = 0. \tag{18}$$

Since (18) can be written as

$$z^{(m)} + \sum_{\gamma=1}^{m} p_\gamma^*(x,\lambda) z^{(m-\gamma)} = 0 \tag{19}$$

which is in type of equation (3), equation (18) does not have a nontrivial solution which belongs to $L_2(\mathbb{R})$. Therefore $\overline{z(x,\lambda)} \equiv 0$ and $\sigma_p(L_\lambda^*) = \varnothing$, which means $\overline{R(L_\lambda)} = L_2(\mathbb{R})$, or $\sigma_r(L_\lambda) = \varnothing$. The theorem is proved. □

From Theorem 4 and Theorem 5 it follows that $\sigma(L_\lambda) = \sigma_c(L_\lambda)$ and L_λ^{-1} is defined in a dense set in $L_2(\mathbb{R})$ for each$\lambda \in \mathbb{C}$.

In order to find $L_\lambda^{-1}$and the resolvent set $\rho(L_\lambda)$, let us investigate the existence of the solution $y(x,\lambda) \in L_2(\mathbb{R})$ of the equation

$$y^{(m)} + \sum_{\gamma=1}^{m} p_\gamma(x,\lambda) y^{(m-\gamma)} = f(x) \tag{20}$$

when $f(x) \in L_2(\mathbb{R})$.

Let us consider cases I and II separately.

In case I if we apply the *Lagrange* method by using the Floquet solutions $f_s(x,\lambda)$, $s = 1,2,\ldots,m$, of equation (3) and linearly independent solutions (see [13], pp.208-210)

$$z_s(x,\lambda) = (-1)^{m+s} \frac{W[f_1,f_2,\ldots,f_{s-1},f_{s+1},\ldots,f_m](x,\lambda)}{W[f_1,f_2,\ldots,f_m](x,\lambda)}, \quad s = 1,2,\ldots,m,$$

of (18) for $\lambda \in \mathbb{C}\backslash\Lambda_0$, then we find the solution of (20) as

$$y(x,\lambda) = \int_{-\infty}^{+\infty} G(x,t,\lambda) f(t)\,dt,$$

where the expression of $G(x,t,\lambda)$ can be written explicitly via $f_s(x,\lambda)$, $z_s(x,\lambda)$, $s=1,2,\dots,m$. Using the properties of the functions $f_s(x,\lambda)$ we can show that

$$\begin{aligned}
&W[f_1,f_2,\dots,f_{s-1},f_{s+1},\dots,f_m](x,\lambda)\\
&=\begin{vmatrix}
f_1(x,\lambda) & \cdots & f_{s-1}(x,\lambda) & f_{s+1}(x,\lambda) & \cdots & f_m(x,\lambda)\\
f_1'(x,\lambda) & \cdots & f_{s-1}'(x,\lambda) & f_{s+1}'(x,\lambda) & \cdots & f_m'(x,\lambda)\\
f_1''(x,\lambda) & \cdots & f_{s-1}''(x,\lambda) & f_{s+1}''(x,\lambda) & \cdots & f_m''(x,\lambda)\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
f_1^{(m-2)}(x,\lambda) & \cdots & f_{s-1}^{(m-2)}(x,\lambda) & f_{s+1}^{(m-2)}(x,\lambda) & \cdots & f_m^{(m-2)}(x,\lambda)
\end{vmatrix}\\
&=e^{-(p_{11}+\omega_s)\lambda x}\left(A^{(s)}(\lambda)+\sum_{n=1}^{\infty}A_n^{(s)}(\lambda)e^{i\alpha_n x}\right),
\end{aligned}$$

where $A^{(s)}(\lambda)$, $A_n^{(s)}(\lambda)$ are complex valued functions of λ for which the series $\sum_{n=1}^{\infty}|A_n^{(s)}(\lambda)|$ is convergent and

$$\begin{aligned}
(-1)^{m+s}A^{(s)}(\lambda)&=(-1)^{m+s}\begin{vmatrix}
1 & \cdots & 1 & 1 & \cdots & 1\\
\omega_1\lambda & \cdots & \omega_{s-1}\lambda & \omega_{s+1}\lambda & \cdots & \omega_m\lambda\\
(\omega_1\lambda)^2 & \cdots & (\omega_{s-1}\lambda)^2 & (\omega_{s+1}\lambda)^2 & \cdots & (\omega_m\lambda)^2\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
(\omega_1\lambda)^{m-2} & \cdots & (\omega_{s-1}\lambda)^{m-2} & (\omega_{s+1}\lambda)^{m-2} & \cdots & (\omega_m\lambda)^{m-2}
\end{vmatrix}\\
&=(-1)^{m+s}\lambda^{\frac{(m-2)(m-1)}{2}}\begin{vmatrix}
1 & \cdots & 1 & 1 & \cdots & 1\\
\omega_1 & \cdots & \omega_{s-1} & \omega_{s+1} & \cdots & \omega_m\\
\omega_1^2 & \cdots & \omega_{s-1}^2 & \omega_{s+1}^2 & \cdots & \omega_m^2\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\omega_1^{m-2} & \cdots & \omega_{s-1}^{m-2} & \omega_{s+1}^{m-2} & \cdots & \omega_m^{m-2}
\end{vmatrix}\\
&=\lambda^{\frac{(m-2)(m-1)}{2}}W_{ms},
\end{aligned}$$

where W_{ms}, $s=1,2,\dots,m$, are cofactors of elements of mth row of the *Vandermonde* determinant W_m.

Thus we find that

$$\begin{aligned}
z_s(x,\lambda)&=(-1)^{m+s}\frac{e^{-(p_{11}+\omega_s)\lambda x}(A^{(s)}(\lambda)+\sum_{n=1}^{\infty}A_n^{(s)}(\lambda)e^{i\alpha_n x})}{\lambda^{\frac{m(m-1)}{2}}W_m}e^{p_{11}\lambda x+\sum_{n=1}^{\infty}\frac{p_{10n}}{i\alpha_n}e^{i\alpha_n x}}\\
&=(-1)^{m+s}\frac{e^{-i\omega_s\lambda x}(A^{(s)}(\lambda)+\sum_{n=1}^{\infty}A_n^{(s)}(\lambda)e^{i\alpha_n x})}{\lambda^{\frac{m(m-1)}{2}}W_m}e^{\sum_{n=1}^{\infty}\frac{p_{10n}}{i\alpha_n}e^{i\alpha_n x}},\quad s=1,2,\dots,m.
\end{aligned}$$

On the other hand, it can be seen that equation (18) which is equivalent to (19) is in the type of equation (3) and, moreover, the characteristic polynomial corresponding to the expression

$$\ell_\lambda^*(\lambda)=z^{(m)}+\sum_{\gamma=1}^{m}p_\gamma^*(x,\lambda)z^{(m-\gamma)}$$

is in the form of $\phi^*(\lambda)=(-1)^m\phi(-\lambda)$. Then, if $\omega_1,\omega_2,\dots,\omega_m$ are roots of $\phi(\lambda)$, then $-\omega_1,-\omega_2,\dots,-\omega_m$ will be roots of $\phi^*(\lambda)$. According to Theorem 1, for all $\lambda\in\mathbb{C}\backslash\Lambda_0$ equation (19) has solutions as

$$\varphi_s(x,\lambda) = e^{-\omega_s\lambda x}\left(1+\sum_{n=1}^{\infty} V_n^{(s)}(\lambda)e^{i\alpha_n x}\right), \quad s=1,2,\dots,m,$$

where

$$V_n^{(s)}(\lambda) = V_n^{(s)} + \sum_{k=1}^{n}\sum_{\substack{j=1\\ j\neq s}}^{m} \frac{V_{jkn}^{(s)}}{i\alpha_k-(\omega_s-\omega_j)\lambda}, \quad n\in\mathbb{N}.$$

Taking into consideration that for every point $\lambda\neq\lambda_{sjn}$, $s,j=1,2,\dots,m$, $j\neq s$, each function $z_s(x,\lambda)$ is a linear combination of the functions $\varphi_1(x,\lambda),\varphi_2(x,\lambda),\dots,\varphi_m(x,\lambda)$ and by virtue of the behavior of the function $z_s(x,\lambda)$ as $x\to\pm\infty$, it is possible to show that for some constants $C_s(\lambda)$ the equality

$$z_s(x,\lambda) = C_s(\lambda)\varphi_s(x,\lambda), \quad s=1,2,\dots,m,$$

is satisfied.

This means that for every $x\in\mathbb{R}$ the equality

$$\begin{aligned}&(-1)^{m+s}\frac{e^{-\omega_s\lambda x}(A^{(s)}(\lambda)+\sum_{n=1}^{\infty}A_n^{(s)}(\lambda)e^{i\alpha_n x})}{\lambda^{\frac{m(m-1)}{2}}W_m}e^{\sum_{n=1}^{\infty}\frac{p_{10n}}{i\alpha_n}e^{i\alpha_n x}}\\ &\quad = C_s(\lambda)\varphi_s(x,\lambda), \quad s=1,2,\dots,m,\end{aligned}$$

holds and hence dividing this equality by $e^{-\omega_s\lambda x}$ we obtain

$$\begin{aligned}&(-1)^{m+s}\frac{(A^{(s)}(\lambda)+\sum_{n=1}^{\infty}A_n^{(s)}(\lambda)e^{i\alpha_n x})}{\lambda^{\frac{m(m-1)}{2}}W_m}e^{\sum_{n=1}^{\infty}\frac{p_{10n}}{i\alpha_n}e^{i\alpha_n x}}\\ &\quad = C_s(\lambda)\left(1+\sum_{n=1}^{\infty}V_n^{(s)}(\lambda)e^{i\alpha_n x}\right), \quad s=1,2,\dots,m.\end{aligned}$$

According to the uniqueness theorem of analytic functions, this equality is satisfied for the analytic continuations of the functions on the semi-plane $\operatorname{Im}x\geq 0$ with respect to x, which are on both sides of the given equality.

Thus, taking the limit as $\operatorname{Im}x\to+\infty$ we have

$$C_s(\lambda) = (-1)^{m+s}\frac{A^{(s)}(\lambda)}{\lambda^{\frac{m(m-1)}{2}}W_m} = \frac{\lambda^{\frac{(m-2)(m-1)}{2}}W_{ms}}{\lambda^{\frac{m(m-1)}{2}}W_m} = \frac{W_{ms}}{\lambda^{m-1}W_m}, \quad s=1,2,\dots,m.$$

Therefore, the equalities

$$z_s(x,\lambda) = \frac{W_{ms}}{\lambda^{m-1}W_m}\varphi_s(x,\lambda), \quad s=1,2,\dots,m,$$

are valid.

For all $j = 1, 2, \ldots, m$, $k = 1, 2, \ldots, 2m_0$, and $\forall\lambda \in S_k$ the value of $\operatorname{Re}(\lambda\omega_j)$ has a constant sign. Hence, there are $M_k \subseteq \{1, 2, \ldots, 2m\}$ and $M'_k = \{1, 2, \ldots, 2m\} \backslash M_k$ such that if $j \in M_k$, $\forall\lambda \in S_k$ then $\operatorname{Re}(\lambda\omega_j) < 0$, if $j \in M'_k$ then $\operatorname{Re}(\lambda\omega_j) > 0$. Thus, for any $k = 1, 2, \ldots, 2m_0$, $\forall a \in \mathbb{R}$, and $\forall\lambda \in S_k \backslash \Lambda$, if $l \in M_k$ then $f_l(x, \lambda) \in L_2(a, +\infty)$, $f_l(x, \lambda) \notin L_2(-\infty, a)$, and if $l \in M'_k$ then $f_l(x, \lambda) \in L_2(-\infty, a)$, $f_l(x, \lambda) \notin L_2(a, +\infty)$. In the same manner, for $\forall a \in \mathbb{R}$, $\forall\lambda \in S_k \backslash \Lambda$ and $l \in M_k$ then $\varphi_l(x, \lambda) \notin L_2(a, +\infty)$, $\varphi_l(x, \lambda) \in L_2(-\infty, a)$, if $l \in M'_k$ then $\varphi_l(x, \lambda) \notin L_2(-\infty, a)$, $\varphi_l(x, \lambda) \in L_2(a, +\infty)$. Taking into account all these properties, for $k = 1, 2, \ldots, 2m_0$, $\forall\lambda \in S_k \backslash \Lambda$, the kernel $G(x, t, \lambda)$ can be written as

$$G(x,t,\lambda) = \frac{1}{\lambda^{m-1} W_m} \begin{cases} \sum_{l \in M_k} W_{ml} f_l(x,\lambda)\varphi_l(t,\lambda), & t \leq x, \\ -\sum_{l \in M'_k} W_{ml} f_l(x,\lambda)\varphi_l(t,\lambda), & t > x. \end{cases} \tag{21}$$

From the expression of the functions $f_s(x, \lambda)$ and $\varphi_s(x, \lambda)$ and from (21) it follows that for every $x, t \in R$ and $\lambda \in S_k \backslash \Lambda$, $k = 1, 2, \ldots, 2m_0$,

$$\left|G(x,t,\lambda)\right| \leq C(\lambda)e^{-\tau|x-t|}, \tag{22}$$

where $C(\lambda) > 0$, $\tau = \min\{|\operatorname{Re}(\lambda\omega_s)| : s = 1, 2, \ldots, m\}$. From (22) we have

$$\int_{-\infty}^{+\infty} \left|G(x,t,\lambda)\right|^2 dt < +\infty \quad \text{and} \quad \int_{-\infty}^{+\infty} \left|G(x,t,\lambda)\right|^2 dx < +\infty.$$

Using (22) it can be proved by the standard method (see [11], pp.302-304) that the operator

$$L_\lambda^{-1} f(x) = \int_{-\infty}^{+\infty} G(x,t,\lambda) f(t)\, dt$$

as $L_\lambda^{-1} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is bounded for $\lambda \in S_1 \cup S_2 \cup \cdots \cup S_{2m_0}$ (it is $\lambda \in \rho(L_\lambda)$). In the case $\lambda \in \bigcup_{k=1}^m l_k$ the operator L_λ^{-1} is a closed operator defined on a dense proper subset $R(L_\lambda)$ of $L_2(\mathbb{R})$ and so L_λ^{-1} is an unbounded operator which means $\lambda \in \sigma_c(L_\lambda)$. On the other hand, since the functions $f_l(x, \lambda)$ and $\varphi_l(t, \lambda)$ do not have the same poles the points $\lambda = \lambda_{sjn}$ may be simple poles of $G(x, t, \lambda)$. If any $\lambda = \lambda_{sjn}$ belong to any set S_k then these points can be only eigenvalues of the operator L_λ. Since L_λ does not have eigenvalue, there is no singularity of operator L_λ^{-1} at these points. Therefore $\lambda_{sjn} \in \rho(L_\lambda)$ and $G(x, t, \lambda)$ is regular at these points too. So the lines $\operatorname{Re}(\lambda\omega_s) = 0$, $s = 1, 2, \ldots, m$, consist of a continuous spectrum of L_λ, *i.e.* $\sigma_c(L_\lambda) = \bigcup_{k=1}^m l_k$. Then the resolvent set of operator L_λ is $\rho(L_\lambda) = S_1 \cup S_2 \cup \cdots \cup S_{2m_0}$. An analytic continuation of the kernel $G(x, t, \lambda)$, with respect to λ, may have some poles at the points $\lambda \in \Lambda_0$, which belong to $\sigma_c(L_\lambda)$. These poles are called spectral singularities (in the sense of [11], p.306) of the operator L_λ.

Similarly, in case II the kernel $G(x, t, \lambda)$ can be written as

$$G(x,t,\lambda) = \begin{cases} \sum_{k=1}^m \hat{f}_k(x,\lambda)\hat{\varphi}_k(t,\lambda), & t < x, \\ 0, & t \geq x \end{cases}$$

for $\lambda \in S^-$ or as

$$G(x,t,\lambda) = \begin{cases} 0, & t < x, \\ -\sum_{k=1}^m \hat{f}_k(x,\lambda)\hat{\varphi}_k(t,\lambda), & t \geq x \end{cases}$$

for $\lambda \in S^+$, where $\hat{f}_k(x,\lambda)$, $k = 1,2,\dots,m$, are solutions of equation (3) and $\hat{\varphi}_k(x,\lambda)$ are solutions of (11) according to Theorem 4. In order to have $y(x,\lambda) = \int_{-\infty}^{+\infty} G(x,t,\lambda)f(t)\,dt \in L_2(\mathbb{R})$, for each $\lambda \in \mathbb{C}$, $\operatorname{Re}(\omega_0\lambda) \neq 0$ and $f(x) \in L_2(\mathbb{R})$ we use

$$\sum_{k=1}^{m} \hat{f}_k(x,\lambda)\hat{\varphi}_k(t,\lambda) = \frac{H(x,t,\lambda)}{W[\hat{f}_1,\hat{f}_2,\dots,\hat{f}_m](t)}.$$

Here

$$\begin{aligned}
&H(x,t,\lambda)\\
&= \begin{vmatrix}
\hat{f}_1(t,\lambda) & \hat{f}_2(t,\lambda) & \cdots & \hat{f}_m(t,\lambda)\\
\hat{f}_1'(t,\lambda) & \hat{f}_2'(x,\lambda) & \cdots & \hat{f}_m'(t,\lambda)\\
\cdots & \cdots & \cdots & \cdots\\
\hat{f}_1^{(m-2)}(t,\lambda) & \hat{f}_2^{(m-2)}(x,\lambda) & \cdots & \hat{f}_m^{(m-2)}(t,\lambda)\\
\hat{f}_1(x,\lambda) & \hat{f}_2(x,\lambda) & \cdots & \hat{f}_m(x,\lambda)
\end{vmatrix}\\
&= \begin{vmatrix}
e^{\omega\lambda t}q_1(t,\lambda) & e^{\omega\lambda t}[tq_1(t,\lambda)+q_2(t,\lambda)] & \cdots & e^{\omega\lambda t}\sum_{j=1}^{m}\frac{t^{m-j}}{(m-j)!}q_j(t,\lambda)\\
(e^{\omega\lambda t}q_1(t,\lambda))' & (e^{\omega\lambda t}[tq_1(t,\lambda)+q_2(t,\lambda)])' & \cdots & (e^{\omega\lambda t}\sum_{j=1}^{m}\frac{t^{m-j}}{(m-j)!}q_j(t,\lambda))'\\
\cdots & \cdots & \cdots & \cdots\\
(e^{\omega\lambda t}q_1(t,\lambda))^{(m-2)} & (e^{\omega\lambda t}[tq_1(t,\lambda)+q_2(t,\lambda)])^{(m-2)} & \cdots & (e^{\omega\lambda t}\sum_{j=1}^{m}\frac{t^{m-j}}{(m-j)!}q_j(t,\lambda))^{(m-2)}\\
e^{\omega\lambda x}q_1(x,\lambda) & e^{\omega\lambda x}[xq_1(x,\lambda)+q_2(x,\lambda)] & \cdots & e^{\omega\lambda x}\sum_{j=1}^{m}\frac{x^{m-j}}{(m-j)!}q_j(x,\lambda)
\end{vmatrix}.
\end{aligned}$$

When we make elementary operations on rows and columns of this determinant, we can transform this determinant as follows:

$$\begin{aligned}
&H(x,t,\lambda)\\
&= e^{\omega\lambda[(m-1)t+x]}\begin{vmatrix}
q_1(t,\lambda) & tq_1(t,\lambda)+q_2(t,\lambda) & \cdots & \sum_{j=1}^{m}\frac{t^{m-j}}{(m-j)!}q_j(t,\lambda)\\
q_1'(t,\lambda) & [tq_1(t,\lambda)+q_2(t,\lambda)]' & \cdots & (\sum_{j=1}^{m}\frac{t^{m-j}}{(m-j)!}q_j(t,\lambda))'\\
\cdots & \cdots & \cdots & \cdots\\
q_1^{(m-2)}(t,\lambda) & [tq_1(t,\lambda)+q_2(t,\lambda)]^{(m-2)} & \cdots & (\sum_{j=1}^{m}\frac{t^{m-j}}{(m-j)!}q_j(t,\lambda))^{(m-2)}\\
q_1(x,\lambda) & xq_1(x,\lambda)+q_2(x,\lambda) & \cdots & \sum_{j=1}^{m}\frac{x^{m-j}}{(m-j)!}q_j(x,\lambda)
\end{vmatrix}\\
&= e^{\omega\lambda[(m-1)t+x]}\begin{vmatrix}
q_1(t,\lambda) & q_2(t,\lambda) & \cdots & q_m(t,\lambda)\\
q_1'(t,\lambda) & q_2'(t,\lambda)+q_1(t,\lambda) & \cdots & q_m'(t,\lambda)+q_{m-1}(t,\lambda)\\
\cdots & \cdots & \cdots & \cdots\\
q_1^{(m-2)}(t,\lambda) & q_2^{(m-2)}(t,\lambda)+(m-2)q_1(t,\lambda) & \cdots & \sum_{j=0}^{m-2}C_{m-2}^{j}q_{m-j}^{(m-j-2)}(t,\lambda)\\
q_1(x,\lambda) & (x-t)q_1(x,\lambda)+q_2(x,\lambda) & \cdots & \sum_{j=1}^{m}\frac{(x-t)^{m-j}}{(m-j)!}q_j(x,\lambda)
\end{vmatrix}\\
&= e^{\omega\lambda[(m-1)t+x]}\left(c_1(x,t,\lambda)+(x-t)c_2(x,t,\lambda)+\cdots+\frac{(x-t)^{m-1}}{(m-1)!}c_m(x,t,\lambda)\right).
\end{aligned}\tag{23}$$

Here, $c_i(x,t,\lambda)$, $i = 1,2,\dots,m$, are continuous and bounded functions of (x,t) in $\mathbb{R}^2$ for each constant λ. According to equations (17), (23), for $\tau = \operatorname{Re}(\omega_0\lambda)$ and some constant $C'(\lambda) > 0$, we obtain the inequality

$$\bigl|G(x,t,\lambda)\bigr| \le C'(\lambda)e^{-\tau|x-t|}\bigl(1+|x-t|\bigr)^{m-1}, \quad \forall x,t \in R, \forall\lambda \in \mathbb{C}. \tag{24}$$

From (24) we have

$$\int_{-\infty}^{+\infty} \left|G(x,t,\lambda)\right|^2 dt < +\infty \quad \text{and} \quad \int_{-\infty}^{+\infty} \left|G(x,t,\lambda)\right|^2 dx < +\infty.$$

By considering inequality (24) it can be proved by the standard method (see [11], pp.302-304) that the operator

$$L_\lambda^{-1} f(x) = \int_{-\infty}^{+\infty} G(x,t,\lambda) f(t)\, dt$$

as $L_\lambda^{-1} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is bounded for $\lambda \in \mathbb{C}$, $\operatorname{Re}(\lambda\omega_0) \neq 0$ (it is $\lambda \in \rho(L_\lambda)$). If $\operatorname{Re}(\lambda\omega_0) = 0$, then, as in case I, the operator L_λ^{-1} is a closed operator defined on a dense proper subset $R(L_\lambda)$ of $L_2(\mathbb{R})$ and so L_λ^{-1} is an unbounded operator, which means $\lambda \in \sigma_c(L_\lambda)$. From the expression of $G(x,t,\lambda)$ we see that $G(x,t,\lambda)$ is a holomorphic function in S^+ and S^-. Analytic continuation of the function $G(x,t,\lambda)$ with respect to λ out of sectors S^+ and S^- does not have a singularity on the line $\operatorname{Re}(\lambda\omega_0) = 0$. Therefore, the line $\operatorname{Re}(\lambda\omega_0) = 0$ consists of a continuous spectrum of L_λ, *i.e.* $\sigma_c(L_\lambda) = l_0$ and does not have a spectral singularity of this operator. Then the resolvent set of the operator L_λ is $\rho(L_\lambda) = S^+ \cup S^-$. Thus, the following theorem is true.

Theorem 6 *The operator L_λ has a pure continuous spectrum $\sigma_c(L_\lambda)$. If the characteristic polynomial $\phi(z)$ has simple roots ω_s, $s = 1,2,\dots,m$, then it is made up of lines $\operatorname{Re}(\lambda\omega_s) = 0$, $s = 1,2,\dots,m$. The countable set of simple spectral singularities may exist at the points $\lambda = \lambda_{sjn} \in \sigma_c(L_\lambda)$ and the spectral singularity degree, which does not exceed $m-1$, may exist at the point $\lambda = 0$. If the characteristic polynomial $\phi(z)$ has a unique multiple root ω_0, then a continuous spectrum consists of the line $\operatorname{Re}(\lambda\omega_0) = 0$ and spectral singularities of L_λ do not exist. For any $\lambda \in \rho(L_\lambda)$ the resolvent L_λ^{-1} is an integral operator with a kernel of Karleman type.*

Competing interests
The author declares that they have no competing interests.

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