Fang *Advances in Difference Equations* 2014, 2014:267
http://www.advancesindifferenceequations.com/content/2014/1/267

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Applications of a generalized q-difference equation

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Abstract
In this paper, we establish one general q-exponential operator identity by solving one simple q-difference equation. Using this q-difference equation, we get some generalizations of Andrews-Askey and Askey-Wilson integral. In addition, we also discuss some properties of q-polynomials H_n.

MSC: 33D05; 33D45; 11B65; 33D60

Keywords: q-difference equation; Andrews-Askey integral; Askey-Wilson integral; q-exponential operator; q-polynomials

1 Introduction and notations

For decades, various families of q-polynomials and q-integral have been investigated rather widely and extensively due mainly to their having been found to be potentially useful in such wide variety of fields as theory of partitions, number theory, combinatorial analysis, finite vector spaces, Lie theory, *etc.* (*cf.* [1–27]). There are many techniques to achieve the ends; for instance, analysis methods (*cf.* [3–6, 14, 16]), combinatorics method (*cf.* [17]), and q-operator method (*cf.* [7, 9–11, 19, 24]) and so on. In resent years, the authors [8, 20–22] derived some formulas of q-polynomials and q-integral from studying the properties of solutions about some q-difference equations. Inspired by their work, in this paper, we will present one more generalized q-difference equation and give some applications of it.

We adopt the notations used by Gasper and Rahman [15]. Throughout the paper unless otherwise stated we assume that $0 < |q| < 1$. Let $\mathbb{N}$ denote the set of non-negative integer, $\mathbb{C}$ denote the set of complex numbers.

For any complex number a, the q-shifted factorial are defined as

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad n \in \mathbb{N}, \tag{1}$$

and we also adopt the following compact notation for the multiple q-shifted factorial:

$$(a_0,a_1,\ldots,a_m;q)_n = (a_0;q)_n(a_1;q)_n\cdots(a_m;q)_n, \quad m\in\mathbb{N}, n=\infty \text{ or } n\in\mathbb{N}. \tag{2}$$

The basic hypergeometric series ${}_s\Phi_t$ is given by

$${}_s\Phi_t\left(\begin{matrix}a_1,\ldots,a_s\\ b_1,\ldots,b_t\end{matrix};q,x\right) = \sum_{k=0}^{\infty}\frac{(a_1,a_2,\ldots,a_s;q)_k}{(q,b_1,\ldots,b_t;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+t-s}x^k, \quad s,t\in\mathbb{N}. \tag{3}$$

For any function $f(x)$ of one variable, the q-derivative of $f(x)$ with respect to x is defined as (*cf.* [7–11, 18–22])

$$D_{q,x}\{f(x)\} = \frac{f(x) - f(qx)}{x},$$

and we further defined $D^0_{q,x}\{f(x)\} = f(x)$, and for $n \geq 1$, $D^n_{q,x}\{f(x)\} = D_{q,x}\{D^{n-1}_{q,x}\{f(x)\}\}$.

The q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}, & \text{for } 0 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

For $a_0, a_1, \ldots, a_s, b_1, \ldots, b_s, b, c \in \mathbb{C}$, we define the following generalized q-operator:

$$F(a_0, \ldots, a_s; b_1, \ldots, b_s; cD_{q,b}) = {}_{s+1}\Phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_s \\ b_1, \ldots, b_s \end{matrix}; q, cD_{q,b} \right). \tag{5}$$

Some special cases of the above q-operator had been studied by many researchers. For instance, the authors [9, 19–21, 24] made a systematic study on $F(0; -; cD_{q,b})$. Some applications of $F(a_0; -; cD_{q,b})$ were given in [10, 11, 22]. Some properties and applications of $F(a_0, a_1; b_1; cD_{q,b})$ were discussed in [8, 11]. In this paper, we present the following more generalized q-difference equation for the above q-operator.

Theorem 1.1 *Let $f(a_0, \ldots, a_s, b_1, \ldots, b_s, b, c)$ be a $2s + 3$-variable analytic function in a neighborhood of $(a_0, \ldots, a_s, b_1, \ldots, b_s, b, c) = (0, 0, \ldots, 0) \in \mathbb{C}^{2s+3}$, $s \in \mathbb{N}$, satisfying the q-difference equation*

$$\begin{aligned} &b \sum_{j=0}^{s+1} \frac{(-1)^j B_j}{q^j} f(a_0, \ldots, b_1, \ldots, b_s, b, cq^j) - c \sum_{j=0}^{s+1} (-1)^j A_j \big[f(a_0, \ldots, b_1, \ldots, b_s, b, cq^j) \\ &\quad - f(a_0, \ldots, a_s, b_1, \ldots, b_s, bq, cq^j) \big] = 0, \end{aligned} \tag{6}$$

where

$$\begin{aligned} &b_0 = q, \qquad B_0 = A_0 = 1, \qquad B_1 = \sum_{i=0}^{s} b_i, \qquad B_2 = \sum_{0 \leq i < j \leq s} b_i b_j, \\ &B_3 = \sum_{0 \leq i < j < k \leq s} b_i b_j b_k, \ldots, \qquad B_{s+1} = b_0 b_1 \cdots b_s, \qquad A_1 = \sum_{i=0}^{s} a_i, \\ &A_2 = \sum_{0 \leq i < j \leq s} a_i a_j, \qquad A_3 = \sum_{0 \leq i < j < k \leq s} a_i a_j a_k, \ldots, \qquad A_{s+1} = a_0 a_1 \cdots a_s. \end{aligned} \tag{7}$$

Then we have

$$\begin{aligned} &f(a_0, \ldots, a_s, b_1, \ldots, b_s, b, c) \\ &\quad = F(a_0, \ldots, a_s; b_1, \ldots, b_s; cD_{q,b}) f(a_0, \ldots, a_s, b_1, \ldots, b_s, b, 0). \end{aligned} \tag{8}$$

Corollary 1.2 ([8], Eq. (1.12)) *Let $f(a_0,a_1,b_1,b,c)$ be a 5-variable analytic function in a neighborhood of $(a_0,a_1,b_1,b,c)=(0,0,0,0,0)\in\mathbb{C}^5$, satisfying the q-difference equation*

$$\begin{aligned}&b\bigl[f(a_0,a_1,b_1,b,c)-\bigl(1+q^{-1}b_1\bigr)f(a_0,a_1,b_1,b,cq)+q^{-1}b_1f\bigl(a_0,a_1,b_1,b,cq^2\bigr)\bigr]\\&\quad-c\bigl[\bigl(f(a_0,a_1,b_1,b,c)-f(a_0,a_1,b_1,bq,c)\bigr)-(a_0+a_1)\bigl(f(a_0,a_1,b_1,b,cq)\\&\quad-f(a_0,a_1,b_1,bq,cq)\bigr)-a_0a_1\bigl(f\bigl(a_0,a_1,b_1,b,cq^2\bigr)-f\bigl(a_0,a_1,b_1,bq,cq^2\bigr)\bigr)\bigr]=0,\end{aligned}\tag{9}$$

then

$$f(a_0,a_1,b_1,b,c)=F(a_0,a_1;b_1;cD_{q,b})f(a_0,a_1,b_1,b,0).\tag{10}$$

Remark 1.3 Letting $a_i=b_i=0$, $i=2,3,\dots,s$, Eq. (6) reduces to (9). Setting $a_i=b_i=0$, $i=0,1,\dots,s$, then replacing b, c by a, b respectively, Eq. (6) reduces to [20], Theorem 1. Putting $a_i=b_i=0$, $i=1,2,\dots,s$, then replacing a_0, b, c by a, c, b, respectively, Eq. (6) reduces to [22], Proposition 1.2.

Proof of Theorem 1.1 From the theory of several complex variables in [28] (or [25], p.28, Hartog's theorem), we assume that

$$f(a_0,\dots,a_s,b_1,\dots,b_s,b,c)=\sum_{n=0}^{\infty}W_n(a_0,\dots,a_s,b_1,\dots,b_s,b)c^n\tag{11}$$

and then substitute the above equation into (6) yielding

$$\begin{aligned}&b\sum_{j=0}^{s+1}\frac{(-1)^jB_j}{q^j}\sum_{n=0}^{\infty}W_n(a_0,\dots,a_s,b_1,\dots,b_s,b)c^nq^{jn}\\&\quad=c\sum_{j=0}^{s+1}(-1)^jA_j\Biggl[\sum_{n=0}^{\infty}W_n(a_0,\dots,a_s,b_1,\dots,b_s,b)c^nq^{jn}\\&\qquad-\sum_{n=0}^{\infty}W_n(a_0,\dots,a_s,b_1,\dots,b_s,bq)c^nq^{jn}\Biggr].\end{aligned}\tag{12}$$

Equating the coefficients of c^n, we have

$$\begin{aligned}&b\sum_{j=0}^{s+1}(-1)^jB_jq^{j(n-1)}W_n(a_0,\dots,a_s,b_1,\dots,b_s,b)\\&\quad=\sum_{j=0}^{s+1}(-1)^jq^{j(n-1)}A_j\bigl[W_{n-1}(a_0,\dots,a_s,b_1,\dots,b_s,b)\\&\qquad-W_{n-1}(a_0,\dots,a_s,b_1,\dots,b_s,bq)\bigr].\end{aligned}\tag{13}$$

For each $n\geq 1$, we get

$$\begin{aligned}&W_n(a_0,\dots,a_s,b_1,\dots,b_s,b)\\&\quad=\frac{(1-a_0q^{n-1})(1-a_1q^{n-1})\cdots(1-a_sq^{n-1})}{(1-b_0q^{n-1})(1-b_1q^{n-1})\cdots(1-b_sq^{n-1})}D_{q,b}\bigl\{W_{n-1}(a_0,\dots,a_s,b_1,\dots,b_s,b)\bigr\}.\end{aligned}\tag{14}$$

By iteration, we find that

$$\begin{aligned} &W_n(a_0,\ldots,a_s,b_1,\ldots,b_s,b) \\ &\quad = \frac{(a_0;q)_n(a_1;q)_n\cdots(a_s;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n} D_{q,b}^n\bigl\{W_0(a_0,\ldots,a_s,b_1,\ldots,b_s,b)\bigr\}. \end{aligned} \tag{15}$$

Putting $c = 0$ in (11), we get $W_0(a_0,\ldots,a_s,b_1,\ldots,b_s,b) = f(a_0,\ldots,a_s,b_1,\ldots,b_s,b,0)$. Substituting (15) into (11), we get (8). This completes the proof. □

Theorem 1.4 *If* $a_0 = q^{-G}$, $G \in \mathbb{N}$, $b,w,u,v,a_i,b_i \in \mathbb{C}$, $i = 1,2,\ldots,s$, *then*

$$\begin{aligned} &F(a_0,\ldots,a_s;b_1,\ldots,b_s;cD_{q,b})\left\{\frac{(bw;q)_\infty}{(bu,bv;q)_\infty}\right\} \\ &\quad = \frac{(bw;q)_\infty}{(bu,bv;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0,a_1,\ldots,a_s;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left(\frac{c}{b}\right)^n {}_3\Phi_2\left(\begin{matrix} q^{-n}, bu, bv \\ 0, bw \end{matrix}; q, q\right). \end{aligned} \tag{16}$$

Proof We use $f(a_0,\ldots,a_s,b_1,\ldots,b_s,b,c)$ to denote the right side of (16). We have

$$W_n = \frac{(a_0,a_1,\ldots,a_s;q)_n}{(q,b_1,\ldots,b_s;q)_n} \tag{17}$$

and A_i, B_i ($i = 0,1,\ldots,s$) are defined as (7), we have

$$\begin{aligned} &b\sum_{j=0}^{s+1}\frac{(-1)^j B_j}{q^j} f\bigl(a_0,\ldots,b_1,\ldots,b_s,b,cq^j\bigr) \\ &\quad = b\frac{(bw;q)_\infty}{(bu,v;q)_\infty}\sum_{n=0}^{\infty} W_n\left(\frac{c}{b}\right)^n\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} \\ &\qquad \times \frac{(bu,bv;q)_k}{(bw;q)_k}(-1)^k q^{\binom{k}{2}+k-nk}\bigl[1 - B_1q^{n-1} + \cdots + (-1)^{s+1}B_{s+1}q^{(s+1)(n-1)}\bigr] \\ &\quad = \frac{b(bwq;q)_\infty}{(buq,bvq;q)_\infty}\sum_{n=0}^{\infty} W_n\left(\frac{c}{b}\right)^n\sum_{k=0}^{n}\left(\begin{bmatrix} n-1 \\ k \end{bmatrix}q^k + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right) \\ &\qquad \times \frac{(buq,bvq;q)_{k-1}}{(bwq;q)_{k-1}}(-1)^k q^{\binom{k}{2}+k-nk}\bigl(1-b_0q^{n-1}\bigr)\cdots\bigl(1-b_sq^{n-1}\bigr) \\ &\quad = \frac{c(bwq;q)_\infty}{(buq,bvq;q)_\infty}\sum_{n=1}^{\infty} W_{n-1}\left(\frac{c}{b}\right)^{n-1}\sum_{k=0}^{n}\begin{bmatrix} n-1 \\ k \end{bmatrix}\frac{(buq,bvq;q)_{k-1}}{(bwq;q)_{k-1}}(-1)^k q^{\binom{k}{2}+2k-nk} \\ &\qquad \times \bigl(1-a_0q^{n-1}\bigr)\cdots\bigl(1-a_sq^{n-1}\bigr) - \frac{c(bwq;q)_\infty}{(buq,bvq;q)_\infty}\sum_{n=1}^{\infty} W_{n-1}\left(\frac{c}{b}\right)^{n-1}\sum_{k=0}^{n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \\ &\qquad \times \frac{(buq,bvq;q)_{k-1}}{(bwq;q)_{k-1}}(-1)^{k-1} q^{\binom{k}{2}+k-nk}\bigl(1-a_0q^{n-1}\bigr)\cdots\bigl(1-a_sq^{n-1}\bigr). \end{aligned} \tag{18}$$

Replacing $n-1$ by n, then applying (4), we find that the above equation is equal to

$$\begin{aligned} &\frac{c(bw;q)_\infty}{(bu,bv;q)_\infty}\sum_{n=0}^{\infty} W_n\left(\frac{c}{b}\right)^n\sum_{k=0}^{n+1}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(bu,bv;q)_k}{(bw;q)_k}(-1)^k q^{\binom{k}{2}+k-nk}\bigl(1-a_0q^n\bigr)\cdots\bigl(1-a_sq^n\bigr) \\ &\quad - \frac{c(bwq;q)_\infty}{(buq,bvq;q)_\infty}\sum_{n=0}^{\infty} W_n\left(\frac{c}{bq}\right)^n\sum_{k=0}^{n+1}\begin{bmatrix} n \\ k \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&\times \frac{(buq,bvq;q)_k}{(bwq;q)_k}(-1)^k q^{\binom{k}{2}+k-nk}\left(1-a_0q^n\right)\cdots\left(1-a_sq^n\right)\\
&= \frac{c(bw;q)_\infty}{(bu,bv;q)_\infty}\sum_{n=0}^{\infty} W_n\left(\frac{c}{b}\right)^n\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}\\
&\quad\times \frac{(bu,bv;q)_k}{(bw;q)_k}(-1)^k q^{\binom{k}{2}+k-nk}\left(1-a_0q^n\right)\cdots\left(1-a_sq^n\right)\\
&\quad- \frac{c(bwq;q)_\infty}{(buq,bvq;q)_\infty}\sum_{n=0}^{\infty} W_n\left(\frac{c}{bq}\right)^n\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}\\
&\quad\times \frac{(buq,bvq;q)_k}{(bwq;q)_k}(-1)^k q^{\binom{k}{2}+k-nk}\left(1-a_0q^n\right)\cdots\left(1-a_sq^n\right)\\
&= c\sum_{j=0}^{s+1}(-1)^j A_j\left[f\left(a_0,\ldots,b_1,\ldots,b_s,b,cq^j\right)-f\left(a_0,\ldots,a_s,b_1,\ldots,b_s,bq,cq^j\right)\right].
\end{aligned}\tag{19}$$

So $f(a_0,\ldots,a_s,b_1,\ldots,b_s,b,c)$ satisfies (6), applying (8), we complete the proof. □

Letting $e\to 0$, $b\to bu$, $c\to bv$, $d\to bw$ in Eq. (III.12) ([15], p.360), we have

$$\left(\frac{1}{b}\right)^n {}_3\Phi_2\left(\begin{matrix} q^{-n},bu,bv\\ 0,bw\end{matrix};q,q\right)=v^n{}_3\Phi_1\left(\begin{matrix} q^{-n},w/u,bv\\ bw\end{matrix};q,uq^n/v\right).\tag{20}$$

Combining the above identity and (16), we find the following generalized formula of [11], Lemma 2.3 (or [8], Eq. (3.4)).

Corollary 1.5 *If* $a_0=q^{-G}$, $G\in\mathbb{N}$, $b,w,u,v,a_i,b_i\in\mathbb{C}$, $i=1,2,\ldots,s$, *then*

$$\begin{aligned}
&F(a_0,\ldots,a_s;b_1,\ldots,b_s;cD_{q,b})\left\{\frac{(bw;q)_\infty}{(bu,bv;q)_\infty}\right\}\\
&= \frac{(bw;q)_\infty}{(bu,bv;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0,a_1,\ldots,a_s;q)_n}{(q,b_1,\ldots,b_s;q)_n}(cv)^n{}_3\Phi_1\left(\begin{matrix} q^{-n},w/u,bv\\ bw\end{matrix};q,uq^n/v\right).
\end{aligned}\tag{21}$$

Letting $w=0$ in (16), we find the following.

Corollary 1.6 *If* $a_0=q^{-G}$, $G\in\mathbb{N}$, $b,c,u,v,a_i,b_i\in\mathbb{C}$, $i=1,2,\ldots,s$, *then*

$$\begin{aligned}
&F(a_0,\ldots,a_s;b_1,\ldots,b_s;cD_{q,b})\left\{\frac{1}{(bu,bv;q)_\infty}\right\}\\
&= \frac{1}{(bu,bv;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0,a_1,\ldots,a_s;q)_n}{(q,b_1,\ldots,b_s;q)_n}\left(\frac{c}{b}\right)^n{}_3\Phi_2\left(\begin{matrix} q^{-n},bu,bv\\ 0,0\end{matrix};q,q\right).
\end{aligned}\tag{22}$$

Letting $w=v=0$ in (16), we find the following.

Corollary 1.7 *If* $\max\{|bu|,|cu|\}<1$, $u,a_i,b_i\in\mathbb{C}$, $i=1,2,\ldots,s$, *then*

$$F(a_0,\ldots,a_s;b_1,\ldots,b_s;cD_{q,b})\left\{\frac{1}{(bu;q)_\infty}\right\}=\frac{1}{(bu;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0,a_1,\ldots,a_s;q)_n}{(q,b_1,\ldots,b_s;q)_n}(cu)^n.\tag{23}$$

Letting $v = 0$ in (16), then applying q-Chu-Vandermonde summation ([15], p.354, Eq. (II.6))

$$ {}_2\Phi_1\left(\begin{matrix} q^{-n}, a \\ c \end{matrix}; q, q\right) = \frac{(c/a;q)_n}{(c;q)_n} a^n, \tag{24}$$

and we obtain the following.

Corollary 1.8 *If* $\max\{|bu|, |bw|, |cu|\} < 1$, $u, v, a_i, b_i \in \mathbb{C}$, $i = 1, 2, \ldots, s$, *then*

$$\begin{aligned} &F(a_0, \ldots, a_s; b_1, \ldots, b_s; cD_{q,b})\left\{\frac{(bw;q)_\infty}{(bu;q)_\infty}\right\} \\ &\quad = \frac{(bw;q)_\infty}{(bu;q)_\infty} \sum_{n=0}^{\infty} \frac{(a_0, a_1, \ldots, a_s, w/u;q)_n}{(q, b_1, \ldots, b_s, bw;q)_n} (cu)^n. \end{aligned} \tag{25}$$

Remark 1.9 It were difficult to distinguish analysis of the functions of the right side of (16) (or (21), (22)) if we would remove the condition $a_0 = q^{-G}$. But in (23) and (25), we do not need the condition $a_0 = q^{-G}$. Under $\max\{|bu|, |bv|, |bw|, |cu|\} < 1$, it is easy to verify that the right sides of (23) and (25) are analytic functions in a neighborhood of $(0, 0, \ldots, 0) \in \mathbb{C}^{2s+3}$. In this paper, the symbols W_n and U_n are frequently used. Here W_n is defined as (17), and U_n is equal to $(h_0, h_1, \ldots, h_t;q)_n/(q, g_1, \ldots, g_t;q)_n$.

The paper is organized in the following manner. In the next two sections we give some generalizations of Andrews-Askey and Askey-Wilson integrals by the q-difference equation. In Section 4, we discuss some properties of q-polynomials H_n. Several special cases and examples of our results are also pointed out, in the concluding section.

2 Generalizations of Andrews-Askey integrals

We have

$$\int_c^d \frac{(qx/c, qx/d;q)_\infty}{(ax, bx;q)_\infty} d_q x = \frac{d(1-q)(q, dq/c, c/d, abcd;q)_\infty}{(ac, ad, bc, bd;q)_\infty}, \tag{26}$$

provided that there are no zero factors in the denominator of the integral, which could be directly derived from Andrews-Askey integrals ([3], Eq. (2.1) or [5], Eq. (1.15)) after some simple replacing. In [8] (or [21, 26, 27]), some generalizations and applications of (26) are given. In this paper we give the following generalizations of the above identity.

Theorem 2.1 *If* $a_0 = q^{-N}$, $a_i, b_i \in \mathbb{C}$, $i = 1, 2, \ldots, s$, $N \in \mathbb{N}$, *then*

$$\begin{aligned} &\int_c^d \frac{(qx/c, qx/d;q)_\infty}{(ax, bx;q)_\infty} \sum_{n=0}^{\infty} W_n \left(\frac{e}{a}\right)^n {}_3\Phi_2\left(\begin{matrix} q^{-n}, ax, abcd \\ 0, ac \end{matrix}; q, q\right) d_q x \\ &\quad = \frac{d(1-q)(q, dq/c, c/d, abcd;q)_\infty}{(ac, ad, bc, bd;q)_\infty} \sum_{n=0}^{\infty} W_n (de)^n. \end{aligned} \tag{27}$$

Applying (20), we rewrite (27) as follows.

Corollary 2.2 *If* $a_0 = q^{-N}$, $a_i, b_i \in \mathbb{C}$, $i = 1, 2, \ldots, s$, $N \in \mathbb{N}$, *then*

$$\begin{aligned}
&\int_c^d \frac{(qx/c, qx/d;q)_\infty}{(ax, bx;q)_\infty} \sum_{n=0}^{\infty} W_n (bcde)^n {}_3\Phi_1 \left(\begin{matrix} q^{-n}, c/x, abcd \\ ac \end{matrix}; q, xq^n/bcd \right) d_q x \\
&\quad = \frac{d(1-q)(q, dq/c, c/d, abcd;q)_\infty}{(ac, ad, bc, bd;q)_\infty} \sum_{n=0}^{\infty} W_n (de)^n. \qquad (28)
\end{aligned}$$

If $a_i = b_i = 0$, $i = 2, 3, \ldots, s$, then letting $e = q/bcd$, the left-hand side of (28) is equal to

$$\begin{aligned}
&\int_c^d \frac{(qx/c, qx/d;q)_\infty}{(ax, bx;q)_\infty} \sum_{n=0}^{\infty} \frac{(a_0, a_1;q)_n}{(q, b_1;q)_n} q^n {}_3\Phi_1 \left(\begin{matrix} q^{-n}, c/x, abcd \\ ac \end{matrix}; q, xq^n/bcd \right) d_q x \\
&\quad = \int_c^d \frac{(qx/c, qx/d;q)_\infty}{(ax, bx;q)_\infty} \sum_{k=0}^{\infty} \frac{(a_0, a_1, c/x, abcd;q)_k}{(q, ac, b_1;q)_k} \\
&\qquad \times \left(\frac{qx}{bcd} \right)^k \sum_{n=0}^{\infty} \frac{(a_0 q^k, a_1 q^k;q)_n q^n}{(q, b_1 q^k;q)_n} d_q x. \qquad (29)
\end{aligned}$$

For $a_0 = q^{-N}$, the inner summation is equal to

$$\frac{(b_1/a_1;q)_{N-k} (a_1 q^k)^{N-k}}{(b_1 q^k;q)_{N-k}} = (-1)^k q^{-\binom{k}{2}} a_1^N \frac{(b_1;q)_k b_1^{-k}}{(qa_0 a_1/b_1;q)_k} \frac{(b_1/a_1;q)_N}{(b_1;q)_N}. \qquad (30)$$

Substituting the above identity into (29), we have the following.

Corollary 2.3 ([8], Theorem 14) *If* $a_0 = q^{-N}$, $a_1, b_1 \in \mathbb{C}$, $N \in \mathbb{N}$, *then*

$$\begin{aligned}
&\int_c^d \frac{(qx/c, qx/d;q)_\infty}{(ax, bx;q)_\infty} {}_4\Phi_2 \left(\begin{matrix} a_0, a_1, c/x, abcd \\ ac, qa_0 a_1/b_1 \end{matrix}; q, xq/bcd \right) d_q x \\
&\quad = \frac{d(1-q)(q, dq/c, c/d, abcd;q)_\infty (b_1;q)_N}{(ac, ad, bc, bd;q)_\infty (b_1/a_1;q)_N a_1^N} {}_2\Phi_1 \left(\begin{matrix} a_0, a_1 \\ b_1 \end{matrix}; q, q/bc \right). \qquad (31)
\end{aligned}$$

Remark 2.4 For $a_0 = q^{-N}$, we find that

$$\frac{(b_1;q)_N}{(b_1/a_1;q)_N a_1^N} = \frac{(qa_1/b_1, qa_0/b_1;q)_\infty}{(qa_0 a_1/b_1, q/b_1;q)_\infty}. \qquad (32)$$

So we see that the identity (31) is the same as Theorem 14 in [8] after replacing (a_0, a_1, b_1) by (r, w, v), respectively.

Proof of Theorem 2.1 We rewrite (27) as follows:

$$\begin{aligned}
&\int_c^d \frac{(qx/c, qx/d;q)_\infty}{(bx;q)_\infty} \frac{(ac;q)_\infty}{(ax, abcd;q)_\infty} \sum_{n=0}^{\infty} W_n \left(\frac{e}{a} \right)^n {}_3\Phi_2 \left(\begin{matrix} q^{-n}, ax, abcd \\ 0, ac \end{matrix}; q, q \right) d_q x \\
&\quad = \frac{d(1-q)(q, dq/c, c/d;q)_\infty}{(bc, bd;q)_\infty} \frac{1}{(ad;q)_\infty} \sum_{n=0}^{\infty} W_n (de)^n. \qquad (33)
\end{aligned}$$

If we use $f_L = f_L(a_0,\ldots,a_s,b_1,\ldots,b_s,a,e)$ we have

$$\frac{(ac;q)_\infty}{(ax,abcd;q)_\infty}\sum_{n=0}^{\infty}W_n\left(\frac{e}{a}\right)^n{}_3\Phi_2\left(\begin{matrix}q^{-n},ax,abcd\\0,ac\end{matrix};q,q\right). \tag{34}$$

In the same way as proving (16), we can verify f_L satisfies (6), so we have

$$\begin{aligned}f_L &= F(a_0,\ldots,a_s;b_1,\ldots,b_s,eD_{q,a})\{f_L(a_0,\ldots,a_s,b_1,\ldots,b_s,a,0)\}\\ &= F(a_0,\ldots,a_s;b_1,\ldots,b_s,eD_{q,a})\left\{\frac{(ac;q)_\infty}{(ax,abcd;q)_\infty}\right\}.\end{aligned} \tag{35}$$

We use $f_R = f_R(a_0,\ldots,a_s,b_1,\ldots,b_s,a,e)$ and we have

$$\frac{1}{(ad;q)_\infty}\sum_{n=0}^{\infty}W_n(de)^n. \tag{36}$$

It is easy to prove f_R satisfies (6), so we find that

$$\begin{aligned}f_R &= F(a_0,\ldots,a_s;b_1,\ldots,b_s,eD_{q,a})\{f_R(a_0,\ldots,a_s,b_1,\ldots,b_s,a,0)\}\\ &= F(a_0,\ldots,a_s;b_1,\ldots,b_s,eD_{q,a})\left\{\frac{1}{(ad;q)_\infty}\right\}.\end{aligned} \tag{37}$$

Combining the above identity and (26), we complete the proof of (27). □

Theorem 2.5 *If* $a_0,a_i,b_i\in\mathbb{C}$, $\max\{|ax|,|ex|,|ac|,|abcd|,|ad|,|de|\}<1$, $i=1,2,\ldots,s$, $N\in\mathbb{N}$, *then*

$$\begin{aligned}&\int_c^d\frac{(qx/c,qx/d;q)_\infty}{(ax,bx;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0,a_1,\ldots,a_s,c/x;q)_n}{(q,b_1,\ldots,b_s,ac;q)_n}(ex)^n\,d_qx\\ &\quad=\frac{d(1-q)(q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0,a_1,\ldots,a_s,bc;q)_n}{(q,b_1,\ldots,b_s,abcd;q)_n}(de)^n.\end{aligned} \tag{38}$$

Proof We rewrite (38) as follows:

$$\begin{aligned}&\int_c^d\frac{(qx/c,qx/d;q)_\infty}{(bx;q)_\infty}\frac{(ac;q)_\infty}{(ax;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0,a_1,\ldots,a_s,c/x;q)_n}{(q,b_1,\ldots,b_s,ac;q)_n}(ex)^n\,d_qx\\ &\quad=\frac{d(1-q)(q,dq/c,c/d;q)_\infty}{(bc,bd;q)_\infty}\frac{(abcd;q)_\infty}{(ad;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0,a_1,\ldots,a_s,bc;q)_n}{(q,b_1,\ldots,b_s,abcd;q)_n}(de)^n.\end{aligned} \tag{39}$$

Setting $f_L = f_L(a_0,a_1,\ldots,a_s,b_1,\ldots,b_s,a,e)$ and $f_R = f_R(a_0,a_1,\ldots,a_s,b_1,\ldots,b_s,a,e)$ denoting the left-hand and the right-hand side of (39), respectively, and taking $\nu=0$ in proving of Theorem 1.4, we can verify both f_L and f_R satisfy (6). Letting $F = F(a_0,a_1,\ldots,a_s;b_1,\ldots,b_s,eD_{q,a})$, from (25), we get

$$\begin{aligned}f_L &= F\{f_L(a_0,a_1,\ldots,a_s,b_1,\ldots,b_s,a,0)\}\\ &= F\left\{\int_c^d\frac{(qx/c,qx/d;q)_\infty}{(bx;q)_\infty}\frac{(ac;q)_\infty}{(ax;q)_\infty}\,d_qx\right\}\end{aligned}$$

$$= F\left\{ \frac{d(1-q)(q,dq/c,c/d;q)_\infty}{(bc,bd;q)_\infty} \frac{(abcd;q)_\infty}{(ad;q)_\infty} \right\}$$
$$= F\left\{f_R(a_0,a_1,\ldots,a_s,b_1,\ldots,b_s,a,0)\right\} = f_R. \tag{40}$$

This completes the proof. □

Theorem 2.6 *If* $a_0 = q^{-N}$, $a_i, b_i \in \mathbb{C}$, $i = 1,2,\ldots,s$, $s,N \in \mathbb{N}$, *then*

$$\int_c^d \frac{(qx/c,qx/d;q)_\infty}{(ax,bx;q)_\infty} {}_{s+1}\Phi_s\left(\begin{matrix} a_0,a_1,\ldots,a_s \\ b_1,\ldots,b_s \end{matrix};q,ex\right) d_qx$$
$$= \frac{d(1-q)(q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty} \sum_{n=0}^{\infty} W_n \left(\frac{e}{a}\right)^n {}_3\Phi_2\left(\begin{matrix} q^{-n},ac,ad \\ 0,abcd \end{matrix};q,q\right). \tag{41}$$

Proof Letting

$$f_L = f_L(a_0,\ldots,a_s,b_1,\ldots,b_s,a,e) = \frac{1}{(ax;q)_\infty} {}_{s+1}\Phi_s\left(\begin{matrix} a_0,a_1,\ldots,a_s \\ b_1,\ldots,b_s \end{matrix};q,ex\right) \tag{42}$$

and

$$f_R = f_R(a_0,\ldots,a_s,b_1,\ldots,b_s,a,e)$$
$$= \frac{(abcd;q)_\infty}{(ac,ad;q)_\infty} \sum_{n=0}^{\infty} W_n \left(\frac{e}{a}\right)^n {}_3\Phi_2\left(\begin{matrix} q^{-n},ac,ad \\ 0,abcd \end{matrix};q,q\right), \tag{43}$$

we can easily verify both of the above identities satisfy (6), so we have

$$f_L = F(a_0,\ldots,a_s;b_1,\ldots,b_s,eD_{q,a})\left\{f_L(a_0,\ldots,a_s,b_1,\ldots,b_s,a,0)\right\}$$
$$= F(a_0,\ldots,a_s;b_1,\ldots,b_s,eD_{q,a})\left\{\frac{1}{(ax;q)_\infty}\right\} \tag{44}$$

and

$$f_R = F(a_0,\ldots,a_s;b_1,\ldots,b_s,eD_{q,a})\left\{f_R(a_0,\ldots,a_s,b_1,\ldots,b_s,a,0)\right\}$$
$$= F(a_0,\ldots,a_s;b_1,\ldots,b_s,eD_{q,a})\left\{\frac{(abcd;q)_\infty}{(ac,ad;q)_\infty}\right\}. \tag{45}$$

Combining (26), we complete the proof of (41). □

Interchanging a and b in (41), we get

$$\int_c^d \frac{(qx/c,qx/d;q)_\infty}{(ax,bx;q)_\infty} {}_{s+1}\Phi_s\left(\begin{matrix} a_0,a_1,\ldots,a_s \\ b_1,\ldots,b_s \end{matrix};q,ex\right) d_qx$$
$$= \frac{d(1-q)(q,dq/c,c/d,abcd;q)_\infty}{(ac,ad,bc,bd;q)_\infty} \sum_{n=0}^{\infty} W_n \left(\frac{e}{b}\right)^n {}_3\Phi_2\left(\begin{matrix} q^{-n},bc,bd \\ 0,abcd \end{matrix};q,q\right). \tag{46}$$

Combing the above identity and (41), then replacing $(bc,bd,abcd)$ by (b,c,d), respectively, we recover the special case for $e = 0$ in Eq. (III.11) ([15], p.360).

Corollary 2.7 ([15], p.360, Eq. (III.11)) *We have*

$$ {}_3\Phi_2\left(\begin{matrix} q^{-n}, b, c \\ 0, d \end{matrix}; q, q\right) = \left(\frac{bc}{d}\right)^n {}_3\Phi_2\left(\begin{matrix} q^{-n}, d/b, d/c \\ 0, d \end{matrix}; q, q\right). \tag{47} $$

Theorem 2.8 *If* $a_0 = q^{-N}$, $h_0 = q^{-G}$, $a_i, b_i, h_j, g_j \in \mathbb{C}$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, t$, $s, t, G, N \in \mathbb{N}$, *then*

$$ \begin{aligned} &\int_c^d \frac{(qx/c, qx/d; q)_\infty}{(ax, bx; q)_\infty} {}_{s+1}\Phi_s\left(\begin{matrix} a_0, a_1, \ldots, a_s \\ b_1, \ldots, b_s \end{matrix}; q, ex\right) {}_{t+1}\Phi_t\left(\begin{matrix} h_0, h_1, \ldots, h_t \\ g_1, \ldots, g_t \end{matrix}; q, fx\right) d_qx \\ &\quad = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty} \\ &\qquad \times \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{k=0}^{n} \sum_{j=0}^{l} W_n U_l \left(\frac{e}{a}\right)^n \left(\frac{f}{b}\right)^l \frac{(q^{-n}, ac, ad; q)_k}{(q, abcd; q)_k} \frac{(q^{-l}, bc, bd; q)_j}{(q, abcdq^k; q)_j} q^{k+j}. \end{aligned} \tag{48} $$

Proof Letting

$$ f_L = f_L(h_0, \ldots, h_t, g_1, \ldots, g_t, b, f) = \frac{1}{(bx; q)_\infty} {}_{s+1}\Phi_s\left(\begin{matrix} h_0, h_1, \ldots, h_t \\ g_1, \ldots, g_t \end{matrix}; q, fx\right) \tag{49} $$

and

$$ \begin{aligned} f_R &= f_R(h_0, \ldots, h_t, g_1, \ldots, g_t, b, f) \\ &= \frac{(abcdq^k; q)_\infty}{(bc, bd; q)_\infty} \sum_{l=0}^{\infty} \frac{(h_0, h_1, \ldots, h_t; q)_l}{(q, g_1, \ldots, g_t; q)_l} \left(\frac{f}{b}\right)^l {}_3\Phi_2\left(\begin{matrix} q^{-n}, bc, bd \\ 0, abcdq^k \end{matrix}; q, q\right), \end{aligned} \tag{50} $$

we can easily verify both of the above identities satisfy (6), so we have

$$ \begin{aligned} f_L &= F(h_0, \ldots, h_t; g_1, \ldots, g_t, fD_{q,b})\left\{f_L(h_0, \ldots, h_t, g_1, \ldots, g_t, b, 0)\right\} \\ &= F(h_0, \ldots, h_t; g_1, \ldots, g_t, fD_{q,b})\left\{\frac{1}{(bx; q)_\infty}\right\} \end{aligned} \tag{51} $$

and

$$ \begin{aligned} f_R &= F(h_0, \ldots, h_t; g_1, \ldots, g_t, fD_{q,b})\left\{f_R(h_0, \ldots, h_t, g_1, \ldots, g_t, b, 0)\right\} \\ &= F(h_0, \ldots, h_t; g_1, \ldots, g_t, fD_{q,b})\left\{\frac{(abcdq^k; q)_\infty}{(bc, bd; q)_\infty}\right\}. \end{aligned} \tag{52} $$

From (41), we conclude that

$$ \begin{aligned} &\int_c^d \frac{(qx/c, qx/d; q)_\infty}{(ax, bx; q)_\infty} {}_{s+1}\Phi_s\left(\begin{matrix} a_0, a_1, \ldots, a_s \\ b_1, \ldots, b_s \end{matrix}; q, ex\right) d_qx \\ &\quad = \frac{d(1-q)(q, dq/c, c/d; q)_\infty}{(ac, ad; q)_\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{n} W_n \left(\frac{e}{a}\right) \frac{(q^{-n}, ac, ad; q)_k q^k}{(q, abcd; q)_k} \frac{(abcdq^k; q)_\infty}{(bc, bd; q)_\infty}. \end{aligned} \tag{53} $$

By (51) and (52), we complete the proof. □

Interchanging a and b in (48), similar to (47), we find the following.

Corollary 2.9 *We have*

$$\begin{aligned} &a^{n-l}\sum_{k=0}^{n}\frac{(q^{-n},ac,ad;q)_k q^k}{(q,abcd;q)_k}{}_3\Phi_2\left(\begin{matrix}q^{-l},bc,bd\\0,abcdq^k\end{matrix};q,q\right)\\ &\quad=b^{n-l}\sum_{k=0}^{n}\frac{(q^{-n},bc,bd;q)_k q^k}{(q,abcd;q)_k}{}_3\Phi_2\left(\begin{matrix}q^{-l},ac,ad\\0,abcdq^k\end{matrix};q,q\right). \end{aligned}\tag{54}$$

Setting $a=d=q$, $b=c=-q$ in (54), then letting $q\to 1$, we have the following.

Corollary 2.10 *If* $n-l=1(\operatorname{mod}2)$, *then*

$$\sum_{k=0}^{n}\sum_{j=0}^{l}\binom{n}{k}\binom{l}{j}\frac{3\cdot 2^{k+1}}{(k+2)(k+3)}\frac{2^j(j+1)!}{(4+k)_j}(-1)^{k+j}=0,\tag{55}$$

where $(a)_0=1$, $(a)_j=a(a+1)\cdots(a+j-1)$.

3 Generalizations of Askey-Wilson integral

In [12], we had derived a new of q-contour integral formula from the following elegant Askey-Wilson integral formula (*cf.* [6], Theorem 2.1):

$$\frac{1}{2\pi i}\int_C\frac{(z^2,z^{-2};q)_\infty}{(az,a/z,bz,b/z,cz,c/z,dz,d/z;q)_\infty}\frac{dz}{z}=\frac{2(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty},\tag{56}$$

where the contour C is a deformation of unit circle so that the poles of $1/(az,bz,cz,dz;q)_\infty$ lie outside the contour and the origin and poles of $1/(a/z,b/z,c/z,d/z;q)_\infty$ lie inside the contour. In this section, we get the following generalizations of the above equation.

Theorem 3.1 *If* $a_0=q^{-N}$, $a_i,b_i\in\mathbb{C}$, $i=1,2,\ldots,s$, $N\in\mathbb{N}$, *then*

$$\begin{aligned} &\frac{1}{2\pi i}\int_C\frac{(z^2,z^{-2};q)_\infty}{(az,a/z,bz,b/z,cz,c/z,dz,d/z;q)_\infty}\\ &\qquad\times\sum_{n=0}^{\infty}W_n\left(\frac{e}{a}\right)^n{}_3\Phi_2\left(\begin{matrix}q^{-n},az,a/z\\0,ac\end{matrix};q,q\right)\frac{dz}{z}\\ &\quad=\frac{2(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}\sum_{n=0}^{\infty}W_n\left(\frac{e}{a}\right)^n{}_3\Phi_2\left(\begin{matrix}q^{-n},ab,ad\\0,abcd\end{matrix};q,q\right). \end{aligned}\tag{57}$$

Proof We rewrite (57) as follows:

$$\begin{aligned} &\frac{1}{2\pi i}\int_C\frac{(z^2,z^{-2};q)_\infty}{(bz,b/z,cz,c/z,dz,d/z;q)_\infty}\frac{(ac;q)_\infty}{(az,a/z;q)_\infty}\\ &\qquad\times\sum_{n=0}^{\infty}W_n\left(\frac{e}{a}\right)^n{}_3\Phi_2\left(\begin{matrix}q^{-n},az,a/z\\0,ac\end{matrix};q,q\right)\frac{dz}{z}\\ &\quad=\frac{2}{(q,bc,bd,cd;q)_\infty}\frac{(abcd;q)_\infty}{(ab,ad;q)_\infty}\sum_{n=0}^{\infty}W_n\left(\frac{e}{a}\right)^n{}_3\Phi_2\left(\begin{matrix}q^{-n},ab,ad\\0,abcd\end{matrix};q,q\right). \end{aligned}\tag{58}$$

We use $f_L = f_L(a_0,\ldots,a_s,b_1,\ldots,b_s,a,e)$ and $f_R = f_R(a_0,\ldots,a_s,b_1,\ldots,b_s,a,e)$ to denote the left-hand and the right-hand side of (58), respectively. By the same method as in Theorem 1.4, we can verify they both satisfy (6). Letting $F = F(a_0,\ldots,a_s;b_1,\ldots,b_s;eD_{q,a})$, we have

$$\begin{aligned} f_R &= F\bigl\{f_R(a_0,\ldots,a_s,b_1,\ldots,b_s,a,0)\bigr\} = F\left\{\frac{2}{(q,bc,bd,cd;q)_\infty}\frac{(abcd;q)_\infty}{(ab,ad;q)_\infty}\right\} \\ &= \frac{1}{2\pi i}\int_C \frac{(z^2,z^{-2};q)_\infty}{(bz,b/z,cz,c/z,dz,d/z;q)_\infty} F\left\{\frac{(ac;q)_\infty}{(az,a/z;q)_\infty}\right\}\frac{dz}{z}. \end{aligned} \tag{59}$$

Applying (16), the above identity is equal to the left side of (58). This completes the proof. □

Employing the above theorem, using q-operator $F = F(h_0,\ldots,h_t;g_1,\ldots,g_t;fD_{q,b})$, similar to the above proof, we conclude the following.

Theorem 3.2 *If $a_0 = q^{-N}$, $h_0 = q^{-G}$, $a_i,b_i,h_j,g_j \in \mathbb{C}$, $i = 1,\ldots,s$, $j = 1,\ldots,t$, $G,N \in \mathbb{N}$, then*

$$\begin{aligned} &\frac{1}{2\pi i}\int_C \frac{(z^2,z^{-2};q)_\infty}{(az,a/z,bz,b/z,cz,c/z,dz,d/z;q)_\infty}\sum_{n=0}^{\infty} W_n\left(\frac{e}{a}\right)^n {}_3\Phi_2\left(\begin{matrix} q^{-n},az,a/z \\ 0,ac \end{matrix};q,q\right) \\ &\quad\times\sum_{m=0}^{\infty} U_m\left(\frac{f}{b}\right)^n {}_3\Phi_2\left(\begin{matrix} q^{-m},bz,b/z \\ 0,bc \end{matrix};q,q\right)\frac{dz}{z} \\ &= \frac{2(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{m} W_n U_m\left(\frac{e}{a}\right)^n\left(\frac{f}{b}\right)^m \\ &\quad\times\frac{(q^{-n},ab,ad;q)_k}{(q,abcd;q)_k}\frac{(q^{-m},abq^k,ad;q)_l}{(q,abcdq^k;q)_l}q^{k+l}. \end{aligned} \tag{60}$$

4 Some properties of q-polynomials H_n

For $a_0,a_1,\ldots,a_s,b_1,\ldots,b_s,b,c \in \mathbb{C}$, $s \in \mathbb{N}$, we define

$$\begin{aligned} H_n &= H_n(a_0,a_1,\ldots,a_s;b_1,\ldots,b_s;b,c) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(a_0,a_1,\ldots,a_s;q)_k}{(b_1,\ldots,b_s;q)_k}c^k b^{n-k} \\ &= b^n{}_{s+2}\Phi_s\left(\begin{matrix} q^{-n},a_0,a_1,a_2,\ldots,a_s \\ b_1,b_2,\ldots,b_s \end{matrix};q,q^n c/b\right). \end{aligned} \tag{61}$$

We can get some famous polynomials from H_n, *e.g.*, letting $b = 1$, $a_i = b_i = 0$, $i = 1,2,\ldots,s$, the polynomials H_n reduce to the classical Al-Salam-Carlitz polynomials (*cf.* [1], Eq. (1.11)),

$$\Phi_n^{(a_0)}(c) = \Phi_n^{(a_0)}(c,q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(a_0;q)_k c^k. \tag{62}$$

Setting $b = 1$, $c = q/a_0q^n$ in (61), then letting $a_0 \to \infty$, we have

$$\lim_{a_0\to\infty} H_n\bigl(a_0,a_1,\ldots,a_s;b_1,\ldots,b_s;1,q/a_0q^n\bigr) = {}_{s+1}\Phi_s\left(\begin{matrix} q^{-n},a_1,a_2,\ldots,a_s \\ b_1,b_2,\ldots,b_s \end{matrix};q,q\right). \tag{63}$$

Taking $a_1 = abcdq^{n-1}$, $a_2 = ae^{i\theta}$, $a_3 = ae^{-i\theta}$, $b_1 = ab$, $b_2 = ac$, $b_3 = ad$, $a_i = b_i = 0$, $i = 4,\dots,s$ in (63), we have the Askey-Wilson polynomials ([6], Eq. (1.15))

$$\frac{a^n p_n(x;a,b,c,d|q)}{(ab,ac,ad;q)_n} = {}_4\Phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right). \tag{64}$$

Putting $a_1 = abq^{n+1}$, $a_2 = q^{-x}$, $a_3 = cq^{x-N}$, $b_1 = aq$, $b_2 = q^{-N}$, $b_3 = bcq$, $a_i = b_i = 0$, $i = 4,\dots,s$ in (63), we get the q-Racah polynomials ([15], Eq. (7.2.17)),

$$W_n(x;a,b,c,N|q) = {}_4\Phi_3\left(\begin{matrix} q^{-n}, abq^{n+1}, q^{-x}, cq^{x-N} \\ aq, q^{-N}, bcq \end{matrix}; q, q\right). \tag{65}$$

In this section, we will give some properties of q-polynomials H_n by q-difference equation. We now show the H_n satisfies the following q-difference equation.

Theorem 4.1 *If $a_0, a_1, \dots, a_s, b_1, \dots, b_s, b, c \in \mathbb{C}$, $s \in \mathbb{N}$, then*

$$\begin{aligned} &b\sum_{j=0}^{s+1} \frac{(-1)^j B_j}{q^j} H_n\bigl(a_0,\dots,b_1,\dots,b_s,b,cq^j\bigr) - c\sum_{j=0}^{s+1}(-1)^j A_j\bigl[H_n\bigl(a_0,\dots,b_1,\dots,b_s,b,cq^j\bigr) \\ &\quad - H_n\bigl(a_0,\dots,a_s,b_1,\dots,b_s,bq,cq^j\bigr)\bigr] = 0, \end{aligned} \tag{66}$$

where A_j, B_j are defined as (7).

Proof Letting W_n defined as (17), and denoting $W_n = W_n'/(q;q)_n$, we have

$$\begin{aligned} &b\sum_{j=0}^{s+1} \frac{(-1)^j B_j}{q^j} H_n\bigl(a_0,\dots,b_1,\dots,b_s,b,cq^j\bigr) \\ &\quad = b\sum_{k=0}^{n} \frac{b^{n-k}c^k(q;q)_n}{(q;q)_{n-k}} W_k\bigl[1 - B_1 q^{k-1} + \cdots + (-1)^{s+1}B_{s+1}q^{(s+1)(k-1)}\bigr] \\ &\quad = c\sum_{k=0}^{n} \frac{b^{n-(k-1)}c^{k-1}(q;q)_{n-1}(1-q^n)}{(q;q)_{n-k}} W_{k-1}\bigl(1-a_0q^{k-1}\bigr)\cdots\bigl(1-a_sq^{k-1}\bigr) \\ &\quad = c\bigl(1-q^n\bigr)\sum_{k=0}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} b^{n-(k-1)}c^{k-1}W'_{k-1}\bigl(1-a_0q^{k-1}\bigr)\cdots\bigl(1-a_sq^{k-1}\bigr). \end{aligned} \tag{67}$$

Replacing $k-1$ by k, we find that the above equation is equal to

$$c\bigl(1-q^n\bigr)\sum_{k=0}^{n} \begin{bmatrix} n-1 \\ k \end{bmatrix} b^{n-k}c^k W'_k\bigl(1-a_0q^k\bigr)\cdots\bigl(1-a_sq^k\bigr). \tag{68}$$

On the other hand

$$\begin{aligned} &c\sum_{j=0}^{s+1}(-1)^j A_j\bigl[H_n\bigl(a_0,\dots,b_1,\dots,b_s,b,cq^j\bigr) - H_n\bigl(a_0,\dots,a_s,b_1,\dots,b_s,bq,cq^j\bigr)\bigr] \\ &\quad = c\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} b^{n-k}\bigl(1-q^{n-k}\bigr)c^k W'_k\bigl(1-a_0q^k\bigr)\cdots\bigl(1-a_sq^k\bigr) \end{aligned}$$

$$= c\sum_{k=0}^{n}\left(\begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} q^{n-k}\right) b^{n-k}c^k W_k'\big(1-a_0q^k\big)\cdots\big(1-a_sq^k\big)$$

$$- c\sum_{k=0}^{n}\left(\begin{bmatrix} n-1 \\ k \end{bmatrix} q^k + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right) (bq)^{n-k}c^k W_k'\big(1-a_0q^k\big)\cdots\big(1-a_sq^k\big)$$

$$= c\big(1-q^n\big)\sum_{k=0}^{n}\begin{bmatrix} n-1 \\ k \end{bmatrix} b^{n-k}c^k W_k'\big(1-a_0q^k\big)\cdots\big(1-a_sq^k\big). \tag{69}$$

This completes the proof. □

For H_n satisfies (6), applying (8), we find the following.

Corollary 4.2 *If $H_n(a_0, a_1, \dots, a_s; b_1, \dots, b_s; b, c)$ is defined as (61), then*

$$H_n(a_0, a_1, \dots, a_s; b_1, \dots, b_s; b, c) = F(a_0, a_1, \dots, a_s; b_1, \dots, b_s; cD_{q,b})\big\{b^n\big\}. \tag{70}$$

Combining the above equation and (23), we obtain the following generating functions for H_n.

Theorem 4.3 *If* $\max\{|bu|, |cu|\} < 1$, *then*

$$\sum_{n=0}^{\infty} H_n \frac{u^n}{(q;q)_n} = \frac{1}{(bu;q)_\infty}\sum_{n=0}^{\infty} W_n (cu)^n. \tag{71}$$

Setting $b = 1$, $a_i = b_i = 0$, $i = 1, 2, \dots, s$ in (71), we conclude the following.

Corollary 4.4 ([1], Eq. (1.13)) *If* $\max\{|u|, |cu|\} < 1$, *then*

$$\sum_{n=0}^{\infty} \Phi_n^{(a)}(c)\frac{u^n}{(q;q)_n} = \frac{(a_0cu;q)_\infty}{(u, cu;q)_\infty}. \tag{72}$$

Theorem 4.5 *If* $a_0 = q^{-N}$, $h_0 = q^{-G}$, $G, N \in \mathbb{N}$, $\max\{|bev|, |cev|, |bfv|\} < 1$, *then*

$$\sum_{n=0}^{\infty} H_n(a_0, a_1, \dots, a_s; b_1, \dots, b_s; b, c) H_n(h_0, h_1, \dots, h_t; g_1, \dots, g_t; e, f)\frac{v^n}{(q;q)_n}$$

$$= \frac{1}{(bev;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix} W_m U_n \frac{(q, bev;q)_k}{(bev)^k}(bfv)^n(cev)^m. \tag{73}$$

To prove the above theorem, we need the following lemma.

Lemma 4.6 *If* $a_0 = q^{-N}$, $v, u, a_i, b_i \in \mathbb{C}$, $i = 1, 2, \dots, s$, $s, N \in \mathbb{N}$, *then*

$$F(a_0, a_1, \dots, a_s; b_1, \dots, b_s; cD_{q,b})\left\{\frac{(bv)^n}{(bu;q)_\infty}\right\}$$

$$= \frac{(bv)^n}{(bu;q)_\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix} W_m (cu)^m \frac{(q, bu;q)_k}{(bu)^k}. \tag{74}$$

Proof Letting $f(a_0,a_1,\ldots,a_s;b_1,\ldots,b_s;b,c)$ denoting the right-hand side of (74), similar to the proof of Theorem 1.4, we see that the functions $f(a_0,a_1,\ldots,a_s;b_1,\ldots,b_s;b,c)$ satisfies (6). Applying (8), we complete the proof. □

Proof of Theorem 4.5 The left-hand side of (73) is equal to

$$\begin{aligned}
&\sum_{n=0}^{\infty} F(a_0,a_1,\ldots,a_s;b_1,\ldots,b_s;cD_{q,b})\{b^n\}H_n(a_0,a_1,\ldots,a_s;b_1,\ldots,b_s;e,f)\frac{v^n}{(q;q)_n}\\
&\quad= F(a_0,a_1,\ldots,a_s;b_1,\ldots,b_s;cD_{q,b})\left\{\sum_{n=0}^{\infty} H_n(a_0,a_1,\ldots,a_s;b_1,\ldots,b_s;e,f)\frac{(bv)^n}{(q;q)_n}\right\}\\
&\quad= F(a_0,a_1,\ldots,a_s;b_1,\ldots,b_s;cD_{q,b})\left\{\frac{1}{(bev;q)_\infty}\sum_{n=0}^{\infty} W_n(bfv)^n\right\}. \qquad (75)
\end{aligned}$$

Using Lemma 4.6, we complete the proof. □

Theorem 4.7 *If* $a_0 = q^{-N}$, $a_i, b_i \in \mathbb{C}$, $i = 1,2,\ldots,s$, $s,N \in \mathbb{N}$, $\max\{|bu|,|bw|,|bv|\} < 1$, *then*

$$\begin{aligned}
&\sum_{m,n,k=0}^{\infty} H_{m+n+k}\frac{u^m v^n w^k(-1)^k q^{\binom{k}{2}}}{(q;q)_m(q;q)_n(q;q)_k}\\
&\quad= \frac{(bw;q)_\infty}{(bu,bv;q)_\infty}\sum_{n=0}^{\infty} W_n\left(\frac{c}{b}\right)^n {}_3\Phi_2\left(\begin{matrix} q^{-n},bu,bv\\ 0,bw\end{matrix};q,q\right). \qquad (76)
\end{aligned}$$

Proof The left-hand side of (76) is equal to

$$\begin{aligned}
&F(a_0,a_1,\ldots,a_s;b_1,\ldots,b_s;cD_{q,b})\left\{\sum_{m,n,k=0}^{\infty}\frac{(bu)^m(bv)^n(bw)^k(-1)^k q^{\binom{k}{2}}}{(q;q)_m(q;q)_n(q;q)_k}\right\}\\
&\quad= F(a_0,a_1,\ldots,a_s;b_1,\ldots,b_s;cD_{q,b})\left\{\frac{(bw;q)_\infty}{(bu,bv;q)_\infty}\right\}. \qquad (77)
\end{aligned}$$

By Theorem 1.4, the proof is complete. □

Letting $w = 0$ in (76), we have the following.

Corollary 4.8 *If* $a_0 = q^{-N}$, $a_i, b_i \in \mathbb{C}$, $i = 1,2,\ldots,s$, $s,N \in \mathbb{N}$, $\max\{|bu|,|bv|\} < 1$, *then*

$$\sum_{m,n=0}^{\infty} H_{m+n}\frac{u^m v^n}{(q;q)_m(q;q)_n} = \frac{1}{(bu,bv;q)_\infty}\sum_{n=0}^{\infty} W_n\left(\frac{c}{b}\right)^n {}_3\Phi_2\left(\begin{matrix} q^{-n},bu,bv\\ 0,0\end{matrix};q,q\right). \qquad (78)$$

Setting $v = 0$ in (76), then applying (24), we find the following.

Corollary 4.9 *If* $\max\{|bu|,|bw|,|cu|\} < 1$, *then*

$$\sum_{m,k=0}^{\infty} H_{m+k}\frac{u^m w^k(-1)^k q^{\binom{k}{2}}}{(q;q)_m(q;q)_k} = \frac{(bw;q)_\infty}{(bu;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0,a_1,\ldots,a_s,w/u;q)_n}{(q,b_1,\ldots,b_s,bw;q)_n}(cu)^n. \qquad (79)$$

5 Some special cases

In this section, we briefly consider some consequences and special cases of the results derived in Section 2. If we take $e = q/d$, $a_i = b_i = 0$, $i = 2, 3, \ldots, s$ in (27), applying (24), we obtain the following.

Proposition 5.1 *If* $a_0 = q^{-N}$, $a_1, b_1 \in \mathbb{C}$, $N \in \mathbb{N}$, *then*

$$\begin{aligned} &\int_c^d \frac{(qx/c, qx/d;q)_\infty}{(ax, bx;q)_\infty} \sum_{n=0}^{N} \frac{(q^{-N}, a_1;q)_n}{(q, b_1;q)_n} \left(\frac{q}{ad}\right)^n {}_3\Phi_2 \left(\begin{matrix} q^{-n}, ax, abcd \\ 0, ac \end{matrix}; q, q\right) d_q x \\ &\quad = \frac{d(1-q)(q, dq/c, c/d, abcd;q)_\infty (b_1/a_1;q)_N a_1{}^N}{(ac, ad, bc, bd;q)_\infty (b_1;q)_N}. \end{aligned} \tag{80}$$

If we take $e = b_1/a_0c$, $a_1 = ac$, $a_i = b_i = 0$, $i = 2, 3, \ldots, s$ in (38), applying the q-Gauss summation ([15], p.354, Eq. (II.8))

$${}_2\Phi_1 \left(\begin{matrix} a, b \\ c \end{matrix}; q, c/ab\right) = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty}, \tag{81}$$

then replacing (b_1, a_0) by $(b_1a_0c, a_0/b_1)$, respectively, we get

$$\begin{aligned} &\int_c^d \frac{(qx/c, qx/d, a_0x;q)_\infty}{(ax, bx, b_1x;q)_\infty} d_q x \\ &\quad = \frac{d(1-q)(q, dq/c, c/d, abcd, a_0c;q)_\infty}{(ac, ad, bc, bd, b_1c;q)_\infty} {}_3\Phi_2 \left(\begin{matrix} ac, a_0/b_1, bc \\ a_0c, abcd \end{matrix}; q, db_1\right). \end{aligned} \tag{82}$$

For ${}_3\Phi_2$ series, using Hall's transformation ([15], p.359, Eq. (III.10))

$${}_3\Phi_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, de/abc\right) = \frac{(b, de/ab, de/bc;q)_\infty}{(d, e, de/abc;q)_\infty} {}_3\Phi_2 \left(\begin{matrix} d/b, e/b, de/abc \\ de/ab, de/bc \end{matrix}; q, b\right), \tag{83}$$

we find the following.

Proposition 5.2 ([20], Theorem 9) *We have*

$$\begin{aligned} &\int_c^d \frac{(qx/c, qx/d, a_0x;q)_\infty}{(ax, bx, b_1x;q)_\infty} d_q x \\ &\quad = \frac{d(1-q)(q, dq/c, c/d, a_0/b_1, acdb_1, bcdb_1;q)_\infty}{(ac, ad, bc, bd, b_1c, b_1d;q)_\infty} \\ &\qquad \times {}_3\Phi_2 \left(\begin{matrix} b_1c, b_1d, abcdb_1/a_0 \\ acdb_1, bcdb_1 \end{matrix}; q, a_0/b_1\right). \end{aligned} \tag{84}$$

Setting $a_0 = abcdb_1$ in the above identity, we obtain the following.

Proposition 5.3 ([20], Theorem 8) *We have*

$$\int_c^d \frac{(qx/c, qx/d, abcdb_1x;q)_\infty}{(ax, bx, b_1x;q)_\infty} d_q x = \frac{d(1-q)(q, dq/c, c/d, abcd, acdb_1, bcdb_1;q)_\infty}{(ac, ad, bc, bd, b_1c, b_1d;q)_\infty}. \tag{85}$$

Noting

$$\int_c^d f(x)\,d_qx = d(1-q)\sum_{m=0}^{\infty} f\bigl(dq^m\bigr)q^m - c(1-q)\sum_{m=0}^{\infty} f\bigl(cq^m\bigr)q^m, \tag{86}$$

then letting $a = q$, $b = -q$, $c = -1$, $d = 1$ in (41), we get the following.

Proposition 5.4 *If* $G \in \mathbb{N}$, *we get*

$$\begin{aligned}&\sum_{k=0}^{G} \begin{bmatrix} G \\ k \end{bmatrix} e^k q^{\binom{k}{2}-Gk} \frac{(a_1,\ldots,a_s;q)_k}{(b_1,\ldots,b_s;q)_k} \frac{(1-q)(1+(-1)^k)}{1-q^{k+1}} \\ &\quad = 2\sum_{n=0}^{G}\sum_{k=0}^{n} \begin{bmatrix} G \\ n \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} \left(-\frac{e}{q}\right)^n q^{\binom{k+1}{2}+\binom{n}{2}-Gn-nk} \frac{(a_1,\ldots,a_s;q)_n}{(b_1,\ldots,b_s;q)_n}(-1)^k \frac{(q^2;q^2)_k}{(q^2;q)_k}.\end{aligned} \tag{87}$$

Taking $e = -q$, $a_i = q$, $b_i = q^2$, $i = 1,2,\ldots,2$ in (87), then letting $q \to 1$ yields the following.

Corollary 5.5 *If* $s, G \in \mathbb{N}$, *then*

$$\sum_{n=0}^{[G/2]} \binom{G}{2n} \frac{1}{(2n+1)^{s+1}} = \sum_{n=0}^{G}\sum_{k=0}^{n} \binom{G}{n}\binom{n}{k} \frac{1}{(n+1)^s} \frac{(-1)^k 2^k}{k+1}. \tag{88}$$

If let $e = q$, $a_i = q$, $b_i = q^2$, $i = 1,2,\ldots,2$ in (87), and setting $q \to 1$, we have the following.

Corollary 5.6 *If* $s, G \in \mathbb{N}$, *then*

$$\sum_{n=0}^{[G/2]} \binom{G}{2n} \frac{1}{(2n+1)^{s+1}} = \sum_{n=0}^{G}\sum_{k=0}^{n} \binom{G}{n}\binom{n}{k} \frac{(-1)^n}{(n+1)^s} \frac{(-1)^k 2^k}{k+1}. \tag{89}$$

Combining with the above two identities, we obtain the following.

Corollary 5.7 *If* $s, G \in \mathbb{N}$, *then*

$$\sum_{m=0}^{[G-1/2]}\sum_{k=0}^{2m+1} \binom{G}{2m+1}\binom{2m+1}{k} \frac{1}{(2m+2)^s} \frac{(-1)^k 2^k}{k+1} = 0. \tag{90}$$

Taking $e = -q$, $a_i = q^2$, $b_i = q$, $i = 1,2,\ldots,2$ in (87), then letting $q \to 1$ yields the following.

Corollary 5.8 *If* $s, G \in \mathbb{N}$, *then*

$$\sum_{n=0}^{[G/2]} \binom{G}{2n} (2n+1)^{s-1} = \sum_{n=0}^{G}\sum_{k=0}^{n} \binom{G}{n}\binom{n}{k} (n+1)^s \frac{(-1)^k 2^k}{k+1}. \tag{91}$$

Setting $e = q$, $a_i = q^2$, $b_i = q$, $i = 1,2,\ldots,2$ in (87), then letting $q \to 1$ yields the following.

Corollary 5.9 *If* $s, G \in \mathbb{N}$, *then*

$$\sum_{n=0}^{[G/2]} \binom{G}{2n} (2n+1)^{s-1} = \sum_{n=0}^{G}\sum_{k=0}^{n} \binom{G}{n}\binom{n}{k} (-1)^n (n+1)^s \frac{(-1)^k 2^k}{k+1}. \tag{92}$$

Combining with the above two identities, we obtain the following.

Corollary 5.10 *If* $s, G \in \mathbb{N}$, *then*

$$\sum_{m=0}^{[G-1/2]} \sum_{k=0}^{2m+1} \binom{G}{2m+1} \binom{2m+1}{k} (2m+2)^s \frac{(-1)^k 2^k}{k+1} = 0. \tag{93}$$

Remark 5.11 The symbol $[x]$ denotes the largest integer $\leq x$.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author read and approved the final manuscript.

Acknowledgements
The author would like to thank the editors and the anonymous referees for their careful reading of the paper. The author is supported by National Natural Science Foundation of China (No. 11471138) and National Natural Science Foundation of China (No. 11371163). The author is also supported by Jiangsu Overseas Research and Training Program for University Prominent Young and Middle-Aged Teachers and Presidents, Universities Natural Science Foundation of Jiangsu (No. 14KJB110002) and SRF for ROCS, SEM.

Received: 14 July 2014 Accepted: 30 September 2014 Published: 14 Oct 2014

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10.1186/1687-1847-2014-267
Cite this article as: Fang: Applications of a generalized q-difference equation. *Advances in Difference Equations* 2014, 2014:267