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*Research article*

# Remarks on an elliptic problem arising in weighted energy estimates for wave equations with space-dependent damping term in an exterior domain

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**Abstract:** This paper is concerned with weighted energy estimates and diffusion phenomena for the initial-boundary problem of the wave equation with space-dependent damping term in an exterior domain. In this analysis, an elliptic problem was introduced by Todorova and Yordanov. This attempt was quite useful when the coefficient of the damping term is radially symmetric. In this paper, by modifying their elliptic problem, we establish weighted energy estimates and diffusion phenomena even when the coefficient of the damping term is not radially symmetric.

**Keywords:** Damped wave equation; elliptic problem; exterior domain; weighted energy estimates; diffusion phenomena

## 1. Introduction

Let $N \geq 2$. We consider the wave equation with space-dependent damping term in an exterior domain $\Omega \subset \mathbb{R}^N$ with a smooth boundary:

$$\begin{cases} u_{tt} - \Delta u + a(x)u_t = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ (u, u_t)(x, 0) = (u_0, u_1)(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where we denote by $\Delta$ the usual Laplacian in $\mathbb{R}^N$ and by $u_t$ and $u_{tt}$ the first and second derivative of $u$ with respect to the variable $t$, and $u = u(x,t)$ is a real-valued unknown function. The coefficient of the damping term $a(x)$ satisfies $a \in C^2(\overline{\Omega})$, $a(x) > 0$ on $\overline{\Omega}$ and

$$\lim_{|x|\to\infty} \left(\langle x\rangle^{\alpha} a(x)\right) = a_0 \tag{1.2}$$

with some constants $\alpha \in [0, 1)$ and $a_0 \in (0, \infty)$, where $\langle y\rangle = (1 + |y|^2)^{\frac{1}{2}}$ for $y \in \mathbb{R}^N$. In this moment, the initial data $(u_0, u_1)$ are assumed to have compact supports in $\Omega$ and to satisfy the compatibility condition of order $k \geq 1$:

$$(u_{\ell-1}, u_\ell) \in (H^2 \cap H^1_0(\Omega)) \times H^1_0(\Omega), \quad \text{for all } \ell = 1, \ldots, k \tag{1.3}$$

where $u_\ell$ is successively defined by $u_\ell = \Delta u_{\ell-2} - a(x)u_{\ell-1}$ ($\ell = 2, \ldots, k$). We note that existence and uniqueness of solution to the problem (1.1) have been discussed (see e.g., Ikawa [2, Theorem 2]).

It is proved in Matsumura [4] that if $\Omega = \mathbb{R}^N$ and $a(x) \equiv 1$, then the solution $u$ of (1.1) satisfies the energy decay estimate

$$\int_{\mathbb{R}^N} (|\nabla u(x, t)|^2 + |u_t(x, t)|^2)\, dx \leq C(1 + t)^{-\frac{N}{2}-1} \|(u_0, u_1)\|^2_{H^1\times L^2},$$

where the constant $C$ depends on the size of the supprot of initial data. Moreover, it is shown in Nishihara [7] that $u$ has the same asymptotic behavior as the one of the problem

$$\begin{cases} v_t - \Delta v = 0, & x \in \mathbb{R}^N,\ t > 0, \\ v(x, 0) = u_0(x) + u_1(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.4}$$

In particular, we have

$$\|u(\cdot, t) - v(\cdot, t)\|_{L^2} = o(t^{-\frac{N}{4}})$$

as $t \to \infty$. Energy decay properties of solutions to (1.1) for general cases with $a(x) \geq \langle x\rangle^{-\alpha}$ $(0 \leq \alpha \leq 1)$ have been dealt with by Matsumura [5]. On the other hand, Mochizuki [6] proved that if $0 \leq a(x) \leq C\langle x\rangle^{-\alpha}$ for some $\alpha > 1$, then the energy of the solution to (1.1) does not vanish as $t \to \infty$ for suitable initial data. (The solution has an asymptotic behavior similar to the solution of the usual wave equation without damping). Therefore one can expect that diffusion phenomena occur only when $a(x) \geq C\langle x\rangle^{-\alpha}$ for $\alpha \leq 1$.

In this paper, we discuss precise decay rates of the weighted energy

$$\int_{\mathbb{R}^N} (|\nabla u(x, t)|^2 + |u_t(x, t)|^2)\Phi(x, t)\, dx$$

with a special weight function

$$\Phi(x, t) = \exp\left(\beta \frac{A(x)}{1 + t}\right)$$

(for some $A \in C^2(\mathbb{R}^N)$ and $\beta > 0$) which is introduced by Todorova and Yordanov [12] based on the ideas in [11] and in [3]. They proved weighted energy estimates

$$\int_{\mathbb{R}^N} a(x)|u(x, t)|^2\Phi(x, t)\, dx \leq C(1 + t)^{-\frac{N-\alpha}{2-\alpha}+\varepsilon},$$

$$\int_{\mathbb{R}^N} (|\nabla u(x, t)|^2 + |u_t(x, t)|^2)\Phi(x, t)\, dx \leq C(1 + t)^{-\frac{N-\alpha}{2-\alpha}-1+\varepsilon}$$

when $a(x)$ is radially symmetric and satisfies (1.2). After that, Radu, Todorova and Yordanov [8] extended it to higher-order derivatives. In [13], the second author proved diffusion phenomena for

(1.1) with $\Omega = \mathbb{R}^N$ and $a(x) = \langle x\rangle^{-\alpha}$ ($\alpha \in [0,1)$) by comparing the solution of the following problem

$$\begin{cases} a(x)v_t - \Delta v = 0, & x \in \mathbb{R}^N,\ t > 0, \\ v(x,0) = u_0(x) + \dfrac{1}{a(x)}u_1(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.5}$$

In [10], diffusion phenomena for (1.1) with an exterior domain and for general radially symmetric damping term are obtained. However, the weighted energy estimates and diffusion phenomena for (1.1) with **non-radially symmetric damping** are still remaining open. The difficulty seems to come from the choice of auxiliary function $A$ in the weighted energy, which strongly depends on the existence of positive solution to the Poisson equation $\Delta A(x) = a(x)$. In fact, an example of non-existence of positive solution to $\Delta A = a$ for non-radial $a(x)$ is shown in [10]. Radu, Todorova and Yordanov [9] considered the case $\Omega = \mathbb{R}^N$ and used a solution $A_*(x)$ of $\Delta A_* = a_1(1 + |x|)^{-\alpha}$ with $a_1 > 0$ satisfying $a_1(1 + |x|)^{-\alpha} \geq a(x)$ for $x \in \mathbb{R}^N$, that is, $A_*(x)$ is a subsolution of the equation $\Delta A = a$. In general one cannot obtain the optimal decay estimate via this choice because of the luck of the precise behavior of $a(x)$ at the spatial infinity which can be expected to determine the precise decay late of weighted energy estimates. Our main idea to overcome this difficulty is to weaken the equality $\Delta A = a$ and consider the inequality $(1-\varepsilon)a \leq \Delta A \leq (1+\varepsilon)a$, and to construct a solution having appropriate behavior, we employ a cut-off argument.

The aim of this paper is to give a proof of Ikehata–Todorova–Yordanov type weighted energy estimates for (1.1) with non-radially symmetric damping and to obtain diffusion phenomena for (1.1) under the compatibility condition of order 1 and the condition (1.2) (without any restriction).

This paper is originated as follows. In Section 2, we discuss related elliptic and parabolic problems. The weighted energy estimates for (1.1) are established in Section 3 (Proposition 3.5). Section 4 is devoted to show diffusion phenomena (Proposition 4.1).

## 2. Related elliptic and parabolic problems

### *2.1. An elliptic problem for weighted energy estimates*

As we mentioned above, in general, existence of positive solutions to the Poisson equation $\Delta A(x) = a(x)$ is false for non-radial $a(x)$. Thus, we weaken this equation and consider the following inequality

$$(1-\varepsilon)a(x) \leq \Delta A(x) \leq (1+\varepsilon)a(x), \quad x \in \Omega, \tag{2.1}$$

where $\varepsilon \in (0,1)$ is a parameter. Here we construct a positive solution $A$ of (2.1) satisfying

$$A_{1\varepsilon}\langle x\rangle^{2-\alpha} \leq A(x) \leq A_{2\varepsilon}\langle x\rangle^{2-\alpha}, \tag{2.2}$$

$$\frac{|\nabla A(x)|^2}{a(x)A(x)} \leq \frac{2-\alpha}{N-\alpha} + \varepsilon \tag{2.3}$$

for some constants $A_{1\varepsilon}, A_{2\varepsilon} > 0$.

**Lemma 2.1.** *For every $\varepsilon \in (0,1)$, there exists $A_\varepsilon \in C^2(\overline{\Omega})$ such that $A_\varepsilon$ satisfies* (2.1)–(2.3).

*Proof.* Firstly, we extend $a(x)$ as a positive function in $C^2(\mathbb{R}^N)$; note that this is possible by virtue of the smoothness of $\partial\Omega$. To simplify the notation, we use the same symbol $a(x)$ as a function defined on

$\mathbb{R}^N$. We construct a solution of approximated equation

$$\Delta A_\varepsilon(x) = a_\varepsilon(x), \quad x \in \mathbb{R}^N$$

for some $a_\varepsilon \in C^2(\mathbb{R}^N)$ satisfying

$$(1-\varepsilon)a(x) \le a_\varepsilon(x) \le (1+\varepsilon)a(x), \quad x \in \mathbb{R}^N. \tag{2.4}$$

Noting (1.2), we divide $a(x)$ as $a(x) = b_1(x) + b_2(x)$ with

$$\begin{aligned}
b_1(x) &= \Delta\left(\frac{a_0}{(N-\alpha)(2-\alpha)}\langle x\rangle^{2-\alpha}\right) = a_0\langle x\rangle^{-\alpha} + \frac{a_0\alpha}{N-\alpha}\langle x\rangle^{-\alpha-2},\\
b_2(x) &= a(x) - a_0\langle x\rangle^{-\alpha} - \frac{a_0\alpha}{N-\alpha}\langle x\rangle^{-\alpha-2}.
\end{aligned}$$

Then we have

$$\lim_{|x|\to\infty}\left(\frac{b_2(x)}{a(x)}\right) = \lim_{|x|\to\infty}\left[\frac{1}{\langle x\rangle^\alpha a(x)}\left(\langle x\rangle^\alpha a(x) - a_0 - \frac{a_0\alpha}{N-\alpha}\langle x\rangle^{-2}\right)\right] = 0. \tag{2.5}$$

Let $\varepsilon \in (0,1)$ be fixed. Then by (2.5) there exists a constant $R_\varepsilon > 0$ such that $|b_2(x)| \le \varepsilon a(x)$ for $x \in \mathbb{R}^N \setminus B(0,R_\varepsilon)$. Here we introduce a cut-off function $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^N,[0,1])$ such that $\eta_\varepsilon \equiv 1$ on $B(0,R_\varepsilon)$. Define

$$a_\varepsilon(x) := b_1(x) + \eta_\varepsilon(x)b_2(x) = a(x) - (1-\eta_\varepsilon(x))b_2(x), \quad x \in \mathbb{R}^N.$$

Then $a_\varepsilon(x) = a(x)$ on $B(0,R_\varepsilon)$ and for $x \in \mathbb{R}^N \setminus B(0,R_\varepsilon)$,

$$\left|\frac{a_\varepsilon(x)}{a(x)} - 1\right| = (1-\eta_\varepsilon(x))\frac{|b_2(x)|}{a(x)} \le \varepsilon$$

and therefore (2.4) is verified.

Next we define

$$\begin{aligned}
B_{1\varepsilon}(x) &:= \frac{a_0}{(N-\alpha)(2-\alpha)}\langle x\rangle^{2-\alpha}, \quad x \in \mathbb{R}^N,\\
B_{2\varepsilon}(x) &:= -\int_{\mathbb{R}^N}\mathcal{N}(x-y)\eta_\varepsilon(y)b_2(y)\,dy, \quad x \in \mathbb{R}^N,
\end{aligned}$$

where $\mathcal{N}$ is the Newton potential given by

$$\mathcal{N}(x) = \begin{cases} \dfrac{1}{2\pi}\log\dfrac{1}{|x|} & \text{if } N = 2,\\[2mm] \dfrac{\Gamma(\frac{N}{2}+1)}{N(N-2)\pi^{\frac{N}{2}}}|x|^{2-N} & \text{if } N \ge 3.\end{cases}$$

Then we easily see that $\Delta B_{1\varepsilon}(x) = b_1(x)$ and $\Delta B_{2\varepsilon} = \eta_\varepsilon(x)b_2(x)$. Moreover, noting that $\operatorname{supp}(\eta_\varepsilon b_2)$ is compact, we see from a direct calculation that there exist a constant $M_\varepsilon > 0$ such that

$$|B_{2\varepsilon}(x)| \le \begin{cases} M_\varepsilon(1+\log\langle x\rangle) & \text{if } N = 2,\\ M_\varepsilon\langle x\rangle^{2-N} & \text{if } N \ge 3,\end{cases} \qquad |\nabla B_{2\varepsilon}(x)| \le M_\varepsilon\langle x\rangle^{1-N}, \quad x \in \mathbb{R}^N.$$

This yields that $B_\varepsilon := B_{1\varepsilon} + B_{2\varepsilon}$ is bounded from below and positive for $x \in \mathbb{R}^N$ with sufficiently large $|x|$. Moreover, we have

$$\lim_{|x|\to\infty}\Big(\langle x\rangle^{\alpha-2}B_\varepsilon(x)\Big) = \frac{a_0}{(N-\alpha)(2-\alpha)}$$

and

$$\begin{aligned}\lim_{|x|\to\infty}\left(\frac{|\nabla B_\varepsilon(x)|^2}{a(x)B_\varepsilon(x)}\right) &= \lim_{|x|\to\infty}\left(\frac{1}{\langle x\rangle^{\alpha}a(x)}\cdot\frac{1}{\langle x\rangle^{\alpha-2}B_\varepsilon(x)}\left|\frac{a_0}{N-\alpha}\langle x\rangle^{-1}x + \langle x\rangle^{\alpha-1}\nabla B_{2\varepsilon}(x)\right|^2\right)\\ &= \frac{2-\alpha}{N-\alpha}.\end{aligned}$$

Using the same argument as in the proof of [10, Lemma 3.1], we can see that there exists a constant $\lambda_\varepsilon \geq 0$ such that $A_\varepsilon(x) := \lambda_\varepsilon + B_\varepsilon(x)$ satisfies (2.1)-(2.3). □

### *2.2. A parabolic problem for diffusion phenomena*

Here we consider $L^p$-$L^q$ type estimates for solutions to the initial-boundary value problem of the following parabolic equation

$$\begin{cases} a(x)w_t - \Delta w = 0, & x \in \Omega,\ t > 0,\\ w(x,t) = 0, & x \in \partial\Omega,\ t > 0,\\ w(x,0) = f(x), & x \in \Omega.\end{cases} \tag{2.6}$$

Here we introduce a weighted $L^p$-spaces

$$L^p_{d\mu} := \left\{ f \in L^p_{\mathrm{loc}}(\Omega)\ ;\ \|f\|_{L^p_{d\mu}} := \left(\int_\Omega |f(x)|^p a(x)\,dx\right)^{\frac{1}{p}} < \infty\right\},\quad 1 \leq p < \infty$$

which is quite reasonable because the corresponding elliptic operator $a(x)^{-1}\Delta$ can be regarded as a symmetric operator in $L^2_{d\mu}$.

The $L^p$-$L^q$ type estimates for the semigroup associated with the Friedrichs' extension $-L_*$ (in $L^2_{d\mu}$) of $-a(x)^{-1}\Delta$ are stated in [10]. The proof is based on Beurling–Deny's criterion and Gagliardo–Nirenberg inequality.

**Proposition 2.2** ([10, Proposition 2.6])**.** *Let $e^{tL_*}$ be a semigroup generated by $L_*$. For every $f \in L^1_{d\mu} \cap L^2_{d\mu}$, we have*

$$\|e^{tL_*}f\|_{L^2_{d\mu}} \leq Ct^{-\frac{N-\alpha}{2(2-\alpha)}}\|f\|_{L^1_{d\mu}} \tag{2.7}$$

*and*

$$\|L_*e^{tL_*}f\|_{L^2_{d\mu}} \leq Ct^{-\frac{N-\alpha}{2(2-\alpha)}-1}\|f\|_{L^1_{d\mu}}. \tag{2.8}$$

## 3. Weighted energy estimates

In this section we establish weighted energy estimates for solutions of (1.1) by introducing Ikehata–Todorova–Yordanov type weight function with an auxiliary function $A_\varepsilon$ constructed in Subsection 2.1.

To begin with, let us recall the finite speed propagation property of the wave equation (see [2]).

**Lemma 3.1** (Finite speed of propagation)**.** *Let $u$ be the solution of* (1.1) *with the initial data* $(u_0, u_1)$ *satisfying* $\operatorname{supp}(u_0, u_1) \subset \overline{B}(0, R_0) = \{x \in \Omega; |x| \le R_0\}$. *Then, one has*

$$\operatorname{supp} u(\cdot, t) \subset \{x \in \Omega \ ; \ |x| \le R_0 + t\}$$

*and therefore* $|x|/(R_0 + 1 + t) \le 1$ *for* $t \ge 0$ *and* $x \in \operatorname{supp} u(\cdot, t)$.

Before introducing a weight function, we also recall two identities for partial energy functionals proved in [10].

**Lemma 3.2** ([10, Lemma 3.7])**.** *Let* $\Phi \in C^2(\overline{\Omega} \times [0, \infty))$ *satisfy* $\Phi > 0$ *and* $\partial_t\Phi < 0$ *and let $u$ be a solution of* (1.1). *Then*

$$\begin{aligned}\frac{d}{dt}\left[\int_\Omega \Big(|\nabla u|^2 + |u_t|^2\Big)\Phi \, dx\right] &= \int_\Omega (\partial_t\Phi)^{-1}\Big|\partial_t\Phi\nabla u - u_t\nabla\Phi\Big|^2 \, dx \\ &\quad + \int_\Omega \Big( - 2a(x)\Phi + \partial_t\Phi - (\partial_t\Phi)^{-1}|\nabla\Phi|^2\Big)|u_t|^2 \, dx.\end{aligned}$$

**Lemma 3.3** ([10, Lemma 3.9])**.** *Let* $\Phi \in C^2(\overline{\Omega} \times [0, \infty))$ *satisfy* $\Phi > 0$ *and* $\partial_t\Phi < 0$ *and let $u$ be a solution to* (1.1). *Then, we have*

$$\begin{aligned}\frac{d}{dt}\left[\int_\Omega \Big(2uu_t + a(x)|u|^2\Big)\Phi \, dx\right] &= 2\int_\Omega uu_t(\partial_t\Phi) \, dx + 2\int_\Omega |u_t|^2\Phi \, dx - 2\int_\Omega |\nabla u|^2\Phi \, dx \\ &\quad + \int_\Omega (a(x)\partial_t\Phi + \Delta\Phi)|u|^2 \, dx.\end{aligned}$$

Here we introduce a weight function for weighted energy estimates, which is a modification of the one in Todorova-Yordanov [12].

**Definition 3.4.** *Define* $h := \frac{2-\alpha}{N-\alpha}$ *and for* $\varepsilon \in (0, 1)$,

$$\Phi_\varepsilon(x, t) = \exp\left(\frac{1}{h + 2\varepsilon}\,\frac{A_\varepsilon(x)}{1 + t}\right), \tag{3.1}$$

*where $A_\varepsilon$ is given in Lemma 2.1. And define for $t \ge 0$,*

$$E_{\partial x}(t; u) := \int_\Omega |\nabla u|^2\Phi_\varepsilon \, dx, \quad E_{\partial t}(t; u) := \int_\Omega |u_t|^2\Phi_\varepsilon \, dx, \tag{3.2}$$

$$E_a(t; u) := \int_\Omega a(x)|u|^2\Phi_\varepsilon \, dx, \quad E_*(t; u) := 2\int_\Omega uu_t\Phi_\varepsilon \, dx, \tag{3.3}$$

*and also define* $E_1(t; u) := E_{\partial x}(t; u) + E_{\partial t}(t; u)$ *and* $E_2(t; u) := E_*(t; u) + E_a(t; u)$.

Now we are in a position to state our main result for weighted energy estimates for solutions of (1.1).

**Proposition 3.5.** *Assume that* $(u_0, u_1)$ *satisfies* $\operatorname{supp}(u_0, u_1) \subset \overline{B}(0, R_0)$ *and the compatibility condition of order* $k_0 \ge 1$. *Let $u$ be a solution of the problem* (1.1). *For every* $\delta > 0$ *and* $0 \le k \le k_0 - 1$, *there exist* $\varepsilon > 0$ *and* $M_{\delta,k,R_0} > 0$ *such that for every* $t \ge 0$,

$$(1 + t)^{\frac{N-\alpha}{2-\alpha}+2k+1-\delta}\Big(E_{\partial x}(t; \partial_t^k u) + E_{\partial t}(t; \partial_t^k u)\Big) + (1 + t)^{\frac{N-\alpha}{2-\alpha}+2k-\delta}E_a(t; \partial_t^k u) \le M_{\delta,k,R_0}\|(u_0, u_1)\|^2_{H^{k+1}\times H^k(\Omega)}.$$

To prove, this, we prepare the following two lemmas.

**Lemma 3.6.** *For $t \geq 0$, we have*

$$\frac{1-\varepsilon}{h+2\varepsilon}\frac{1}{1+t}E_a(t;u) \leq E_{\partial x}(t;u). \tag{3.4}$$

*Proof.* As in the proof of [10, Lemma 3.6], by integration by parts we have

$$\int_\Omega \Delta(\log \Phi_\varepsilon)|u|^2\Phi_\varepsilon\,dx = \int_\Omega \left(\Delta\Phi_\varepsilon - \frac{|\nabla\Phi_\varepsilon|^2}{\Phi_\varepsilon}\right)|u|^2\,dx \leq \int_\Omega |\nabla u|^2\,\Phi_\varepsilon\,dx.$$

Noting that

$$\Delta(\log \Phi_\varepsilon(x)) = \frac{1}{h+2\varepsilon}\frac{\Delta A_\varepsilon(x)}{1+t} \geq \frac{1-\varepsilon}{h+2\varepsilon}\frac{a(x)}{1+t},$$

we have (3.4). □

In order to clarify the effect of the finite propagation property, we now put

$$a_1 := \inf_{x\in\Omega}\Big(\langle x\rangle^\alpha a(x)\Big).$$

Then

**Lemma 3.7.** *For $t \geq 0$, we have*

$$E_{\partial t}(t;u) \leq \frac{1}{a_1}(R_0+1+t)^\alpha E_a(t;\partial_t u), \tag{3.5}$$

$$\int_\Omega \frac{A_\varepsilon(x)}{a(x)}|u_t|^2\Phi_\varepsilon\,dx \leq \frac{A_{2\varepsilon}}{a_1}(R_0+1+t)^2 E_{\partial t}(t;u), \tag{3.6}$$

$$|E_*(t;u)| \leq \frac{2}{\sqrt{a_1}}(R_0+1+t)^{\frac{\alpha}{2}}\sqrt{E_a(t;u)E_{\partial t}(t;u)}. \tag{3.7}$$

*Proof.* By $a(x)^{-1} \leq a_1^{-1}\langle x\rangle^\alpha \leq a_1^{-1}(1+|x|)^\alpha$ and the finite propagation property we have

$$\int_\Omega |u_t|^2\Phi_\varepsilon\,dx = \int_\Omega \frac{a(x)}{a(x)}|u_t|^2\Phi_\varepsilon\,dx \leq \frac{1}{a_1}(R_0+1+t)^\alpha E_a(t;\partial_t u).$$

Using the Cauchy-Schwarz inequality and the above inequality yields (3.6):

$$\begin{aligned}\left|\int_\Omega uu_t\Phi_\varepsilon\,dx\right|^2 &\leq \left(\int_\Omega |u|^2\Phi_\varepsilon\,dx\right)\left(\int_\Omega |u_t|^2\Phi_\varepsilon\,dx\right)\\ &\leq \frac{(R_0+1+t)^\alpha}{a_1}\left(\int_\Omega a(x)|u|^2\Phi_\varepsilon\,dx\right)E_{\partial t}(t;u)\\ &\leq \frac{(R_0+1+t)^\alpha}{a_1}E_a(t;u)E_{\partial t}(t;u).\end{aligned}$$

We can prove (3.7) in a similar way. □

**Lemma 3.8.** **(i)** *For every $t \geq 0$, we have*

$$\frac{d}{dt}E_1(t;u) \leq -E_a(t;\partial_t u). \tag{3.8}$$

**(ii)** *For every $\varepsilon \in (0, \frac{1}{3})$ and $t \geq 0$,*

$$\frac{d}{dt}E_2(t;u) \leq -\frac{1-3\varepsilon}{1-\varepsilon}E_{\partial x}(t;u) + \left(\frac{2}{a_1} + \frac{A_{2\varepsilon}(R_0+1)^2}{\varepsilon a_1^2}\right)(R_0+1+t)^{\alpha}E_a(t;\partial_t u). \tag{3.9}$$

*Proof.* Noting (2.3), we have

$$\begin{aligned}
-2a(x)\Phi_\varepsilon + \partial_t\Phi_\varepsilon - (\partial_t\Phi_\varepsilon)^{-1}|\nabla\Phi_\varepsilon|^2 &= \left(-2a(x) - \frac{A_\varepsilon(x)}{(h+2\varepsilon)(1+t)^2} + \frac{1}{h+2\varepsilon}\frac{|\nabla A_\varepsilon(x)|^2}{A_\varepsilon(x)}\right)\Phi_\varepsilon \\
&\leq \left(-2a(x) + \frac{h+\varepsilon}{h+2\varepsilon}a(x)\right)\Phi_\varepsilon \\
&\leq -a(x)\Phi_\varepsilon.
\end{aligned}$$

This implies (3.8). On the other hand, from (2.3) and (2.1) we see

$$\begin{aligned}
a(x)\partial_t\Phi_\varepsilon + \Delta\Phi_\varepsilon &= \frac{1}{h+2\varepsilon}\left(-\frac{a(x)A_\varepsilon(x)}{(1+t)^2} + \frac{|\nabla A_\varepsilon(x)|^2}{(h+2\varepsilon)(1+t)^2} + \frac{\Delta A_\varepsilon(x)}{1+t}\right)\Phi_\varepsilon \\
&\leq \frac{1}{h+2\varepsilon}\left(-\frac{a(x)A_\varepsilon(x)}{(1+t)^2} + \frac{(h+\varepsilon)a(x)A_\varepsilon(x)}{(h+2\varepsilon)(1+t)^2} + \frac{(1+\varepsilon)a(x)}{1+t}\right)\Phi_\varepsilon \\
&\leq \left(-\frac{\varepsilon}{(h+2\varepsilon)^2}\frac{a(x)A_\varepsilon(x)}{(1+t)^2} + \frac{1+\varepsilon}{h+2\varepsilon}\frac{a(x)}{1+t}\right)\Phi_\varepsilon.
\end{aligned}$$

Therefore combining it with Lemma 3.6, we have

$$\int_\Omega (a(x)\partial_t\Phi_\varepsilon + \Delta\Phi_\varepsilon)|u|^2\,dx \leq \frac{1+\varepsilon}{1-\varepsilon}\int_\Omega |\nabla u|^2\Phi_\varepsilon\,dx - \frac{\varepsilon}{(h+2\varepsilon)^2}\frac{1}{(1+t)^2}\int_\Omega a(x)A_\varepsilon(x)|u|^2\Phi_\varepsilon\,dx.$$

Using (3.6), we have

$$\begin{aligned}
2\int_\Omega uu_t(\partial_t\Phi_\varepsilon)\,dx &= -\frac{2}{h+2\varepsilon}\frac{1}{(1+t)^2}\int_\Omega uu_tA_\varepsilon(x)\Phi_\varepsilon\,dx \\
&\leq \frac{2}{h+2\varepsilon}\frac{1}{(1+t)^2}\left(\int_\Omega a(x)A_\varepsilon(x)|u|^2\Phi_\varepsilon\,dx\right)^{\frac{1}{2}}\left(\int_\Omega \frac{A_\varepsilon(x)}{a(x)}|u_t|^2\Phi_\varepsilon\,dx\right)^{\frac{1}{2}} \\
&\leq \frac{2(R_0+1)}{h+2\varepsilon}\frac{1}{1+t}\left(\int_\Omega a(x)A_\varepsilon(x)|u|^2\Phi_\varepsilon\,dx\right)^{\frac{1}{2}}\left(\frac{A_{2\varepsilon}}{a_1}E_{\partial t}(t;u)\right)^{\frac{1}{2}} \\
&\leq \frac{\varepsilon}{(h+2\varepsilon)^2}\frac{1}{(1+t)^2}\int_\Omega a(x)A_\varepsilon(x)|u|^2\Phi_\varepsilon\,dx + \frac{A_{2\varepsilon}(R_0+1)^2}{\varepsilon a_1}E_{\partial t}(t;u).
\end{aligned}$$

Applying (3.5), we obtain (3.9). □

**Lemma 3.9.** *The following assertions hold:*

**(i)** *Set* $t_*(R_0,\alpha,m) := \max\left\{\left(\frac{2m}{a_1}\right)^{\frac{1}{1-\alpha}}, R_0+1\right\}$. *Then for every* $t, m \geq 0$ *and* $t_1 \geq t_*(R_0,\alpha,m)$,

$$\frac{d}{dt}\Big((t_1+t)^m E_1(t;u)\Big) \leq m(t_1+t)^{m-1}E_{\partial x}(t;u) - \frac{1}{2}(t_1+t)^m E_a(t;\partial_t u). \tag{3.10}$$

**(ii)** *for every* $t, \lambda \geq 0$ *and* $t_2 \geq R_0+1$,

$$\begin{aligned}\frac{d}{dt}\Big((t_2+t)^\lambda E_2(t;u)\Big) &\leq \lambda(1+\varepsilon)(t_2+t)^{\lambda-1}E_a(t;u) - \frac{1-3\varepsilon}{1-\varepsilon}(t_2+t)^\lambda E_{\partial x}(t;u)\\ &\quad + \left(\frac{2}{a_1} + \frac{A_{2\varepsilon}(R_0+1)^2}{\varepsilon a_1^2} + \frac{\lambda}{2\varepsilon a_1^2 t_2^{1-\alpha}}\right)(t_2+t)^{\lambda+\alpha}E_a(t;\partial_t u).\end{aligned} \tag{3.11}$$

**(iii)** *In particular, setting*

$$\begin{aligned}\nu &:= \frac{4}{a_1} + \frac{2A_{2\varepsilon}(R_0+1)^2}{\varepsilon a_1^2} + \frac{1}{4\varepsilon a_1},\\ t_{**}(\varepsilon,R_0,\alpha,\lambda) &:= \max\left\{\left(\frac{(1-\varepsilon)(\lambda+\alpha)\nu}{\varepsilon}\right)^{\frac{1}{1-\alpha}}, \left(\frac{2(\lambda+\alpha)}{a_1}\right)^{\frac{1}{1-\alpha}}, R_0+1\right\},\end{aligned}$$

*one has that for* $t, \lambda \geq 0$ *and* $t_3 \geq t_{**}(\varepsilon,R_0,\alpha,\lambda)$,

$$\begin{aligned}&\frac{d}{dt}\Big(\nu(t_3+t)^{\lambda+\alpha}E_1(t;u) + (t_3+t)^\lambda E_2(t;u)\Big)\\ &\leq -\frac{1-4\varepsilon}{1-\varepsilon}(t_3+t)^\lambda E_{\partial x}(t;u) + \lambda(1+\varepsilon)(t_3+t)^{\lambda-1}E_a(t;u).\end{aligned} \tag{3.12}$$

*Proof.* **(i)** Let $m \geq 0$ be fixed and let $t_1 \geq t_*(R_0,\alpha,m)$. Using (3.8) and (3.5), we have

$$\begin{aligned}(t_1+t)^{-m}\frac{d}{dt}\Big((t_1+t)^m E_1(t;u)\Big) &\leq \frac{m}{t_1+t}E_{\partial x}(t;u) + \frac{m}{t_1+t}E_{\partial t}(t;u) + \frac{d}{dt}E_1(t;u)\\ &\leq \frac{m}{t_1+t}E_{\partial x}(t;u) + \frac{m}{t_1+t}E_{\partial t}(t;u) - E_a(t;\partial_t u)\\ &\leq \frac{m}{t_1+t}E_{\partial x}(t;u) + \left(\frac{m(R_0+1+t)^\alpha}{a_1(t_1+t)} - 1\right)E_a(t;\partial_t u).\end{aligned}$$

Therefore we obtain (3.10).

**(ii)** For $t \geq 0$, and $t \geq R_0+1$,

$$\begin{aligned}&(t_2+t)^{-\lambda}\frac{d}{dt}\Big((t_2+t)^\lambda E_2(t;u)\Big)\\ &\leq \frac{\lambda}{t_2+t}E_*(t;u) + \frac{\lambda}{t_2+t}E_a(t;u) + \frac{d}{dt}E_2(t;u)\\ &\leq \frac{\lambda}{t_2+t}E_*(t;u) + \frac{\lambda}{t_2+t}E_a(t;u) - \frac{1-3\varepsilon}{1-\varepsilon}E_{\partial x}(t;u) + \left(\frac{2}{a_1} + \frac{A_{2\varepsilon}(R_0+1)^2}{\varepsilon a_1^2}\right)(R_0+1+t)^\alpha E_a(t;\partial_t u).\end{aligned}$$

Noting that by (3.7) and (3.5),

$$\begin{aligned}\frac{\lambda}{t_2+t}E_*(t;u) &\le \frac{2\lambda(R_0+1+t)^{\alpha}}{a_1(t_2+t)}\sqrt{E_a(t;u)E_a(t;\partial_t u)}\\ &\le \frac{\lambda\varepsilon}{t_2+t}E_a(t;u)+\frac{\lambda}{\varepsilon a_1^2}\frac{(R_0+1+t)^{2\alpha}}{t_2+t}E_a(t;\partial_t u)\\ &\le \frac{\lambda\varepsilon}{t_2+t}E_a(t;u)+\frac{\lambda}{\varepsilon a_1^2 t_2^{1-\alpha}}(t_2+t)^{\alpha}E_a(t;\partial_t u),\end{aligned}$$

we deduce (3.11).

**(iii)** Combining (3.10) with $m=\lambda+\alpha$ and (3.11), we have for $t_3\ge t_{**}(\varepsilon,R_0,\alpha,\lambda)$ and $t\ge 0$,

$$\begin{aligned}&\frac{d}{dt}\Big(\nu(t_3+t)^{\lambda+\alpha}E_1(t;u)+(t_3+t)^{\lambda}E_2(t;u)\Big)\\ &\le\left(\nu(\lambda+\alpha)(t_3+t)^{\alpha-1}-\frac{1-3\varepsilon}{1-\varepsilon}\right)(t_3+t)^{\lambda}E_{\partial x}(t;u)+\lambda(1+\varepsilon)(t_3+t)^{\lambda-1}E_a(t;u)\\ &\quad+\left(\frac{2}{a_1}+\frac{A_{2\varepsilon}(R_0+1)^2}{\varepsilon a_1^2}+\frac{\lambda}{2\varepsilon a_1^2 t_3^{1-\alpha}}-\frac{\nu}{2}\right)(t_3+t)^{\lambda+\alpha}E_a(t;\partial_t u)\\ &\le -\frac{1-4\varepsilon}{1-\varepsilon}(t_3+t)^{\lambda}E_{\partial x}(t;u)+\lambda(1+\varepsilon)(t_3+t)^{\lambda-1}E_a(t;u).\end{aligned}$$

This proves the assertion. □

*Proof of Proposition 3.5.* Firstly, by (3.7) we observe that

$$\begin{aligned}\nu(t_3+t)^{\alpha}E_1(t;u)+E_2(t;u) &\ge \frac{4}{a_1}(t_3+t)^{\alpha}E_1(t;u)-|E_*(t;u)|+E_a(t;u)\\ &\ge \frac{4}{a_1}(t_3+t)^{\alpha}E_{\partial t}(t;u)-\frac{2}{\sqrt{a_1}}(t_3+t)^{\frac{\alpha}{2}}\sqrt{E_a(t;u)E_{\partial t}(t;u)}+E_a(t;u)\\ &\ge \frac{3}{4}E_a(t;u).\end{aligned}$$

By using the above estimate, we prove the assertion via mathematical induction.

**Step 1 ($k=0$).** By (3.12) using Lemma 3.6 implies that

$$\frac{d}{dt}\Big(\nu(t_3+t)^{\lambda+\alpha}E_1(t;u)+(t_3+t)^{\lambda}E_2(t;u)\Big)\le\left(-\frac{1-4\varepsilon}{1-\varepsilon}+\frac{\lambda(1+\varepsilon)(h+2\varepsilon)}{1-\varepsilon}\right)(t_3+t)^{\lambda}E_{\partial x}(t;u).$$

Therefore taking $\lambda_0=\frac{(1-\varepsilon)(1-4\varepsilon)}{(1+\varepsilon)(h+2\varepsilon)}$, ($\lambda_0\uparrow h^{-1}$ as $\varepsilon\downarrow 0$) we have

$$\frac{d}{dt}\Big(\nu(t_3+t)^{\lambda_0+\alpha}E_1(t;u)+(t_3+t)^{\lambda_0}E_2(t;u)\Big)\le -\frac{\varepsilon(1-4\varepsilon)}{1-\varepsilon}(t_3+t)^{\lambda_0}E_{\partial x}(t;u).$$

Integrating over $(0,t)$ with respect to $t$, we see

$$\frac{3}{4}(t_3+t)^{\lambda_0}E_a(t;u)+\frac{\varepsilon(1-4\varepsilon)}{1-\varepsilon}\int_0^t(t_3+s)^{\lambda_0}E_{\partial x}(s;u)\,ds$$

$$\leq \nu(t_3+t)^{\lambda_0+\alpha}E_1(t;u)+(t_3+t)^{\lambda_0}E_2(t;u)+\frac{\varepsilon(1-4\varepsilon)}{1-\varepsilon}\int_0^t(t_3+s)^{\lambda_0}E_{\partial x}(s;u)\,ds$$
$$\leq \nu t_3^{\lambda_0+\alpha}E_1(0;u)+t_3^{\lambda_0}E_2(0;u).$$

Using (3.10) with $m=\lambda_0+1$ and integrating over $(0,t)$, we obtain

$$(t_3+t)^{\lambda_0+1}E_1(t;u)+\frac{1}{2}\int_0^t(t_3+s)^{\lambda_0+1}E_a(s;\partial_t u)\,ds$$
$$\leq t_3^{\lambda_0+1}E_1(0;u)+(\lambda_0+1)\int_0^t(t_3+s)^{\lambda_0}E_{\partial x}(s;u)\,ds$$
$$\leq t_3^{\lambda_0+1}E_1(0;u)+\frac{(\lambda_0+1)(1-\varepsilon)}{\varepsilon(1-4\varepsilon)}\Big(\nu t_3^{\lambda_0+\alpha}E_1(0;u)+t_3^{\lambda_0}E_2(0;u)\Big).$$

This proves the desired assertion with $k=0$ and also the integrability of $(t_3+s)^{\lambda_0+1}E_a(s;\partial_t u)$.

**Step 2 ($1<k\leq k_0-1$).** Suppose that for every $t\geq 0$,

$$(1+t)^{\lambda_0+2k-1}E_1(t;\partial_t^{k-1}u)+(1+t)^{\lambda_0+2k-2}E_a(t;\partial_t^{k-1}u)\leq M_{\varepsilon,k-1}\|(u_0,u_1)\|^2_{H^k\times H^{k-1}(\Omega)}$$

and additionally,

$$\int_0^t(1+s)^{\lambda_0+2k-1}E_a(s;\partial_t^k u)\,ds\leq M'_{\varepsilon,k-1}\|(u_0,u_1)\|^2_{H^k\times H^{k-1}(\Omega)}.$$

Since the initial value $(u_0,u_1)$ satisfies the compatibility condition of order $k$, $\partial_t^k u$ is also a solution of (1.1) with replaced $(u_0,u_1)$ with $(u_{k-1},u_k)$. Applying (3.12) with $\lambda=\lambda_0+2k$, putting $t_{3k}=t_{**}(\varepsilon,R_0,\alpha,\lambda_0+2k)$ (see Lemma 3.9 **(iii)**) and integrating over $(0,t)$, we have

$$\frac{3}{4}(t_{3k}+t)^{\lambda_0+2k}E_a(t;\partial_t^k u)+\frac{1-4\varepsilon}{1-\varepsilon}\int_0^t(t_{3k}+s)^{\lambda_0+2k}E_{\partial x}(s;\partial_t^k u)\,ds$$
$$\leq \nu(t_{3k}+t)^{\lambda_0+2k+\alpha}E_1(t;\partial_t^k u)+(t_{3k}+t)^{\lambda_0+2k}E_2(t;\partial_t^k u)+\frac{1-4\varepsilon}{1-\varepsilon}\int_0^t(t_{3k}+s)^{\lambda_0+2k}E_{\partial x}(s;\partial_t^k u)\,ds$$
$$\leq \nu t_{3k}^{\lambda_0+2k+\alpha}E_1(0;\partial_t^k u)+t_{3k}^{\lambda_0+2k}E_2(0;\partial_t^k u)+(\lambda_0+2k)(1+\varepsilon)\int_0^t(t_{3k}+s)^{\lambda_0+2k-1}E_a(s;\partial_t^k u)\,ds$$
$$\leq \nu t_{3k}^{\lambda_0+2k+\alpha}E_1(0;\partial_t^k u)+t_{3k}^{\lambda_0+2k-1}E_2(0;\partial_t^k u)+(\lambda_0+2k)(1+\varepsilon)M'_{\varepsilon,k-1}\|(u_0,u_1)\|^2_{H^k\times H^{k-1}(\Omega)}.$$

Moreover, from (3.10) with $m=\lambda_0+2k+1$ we have

$$(t_{3k}+t)^{\lambda_0+2k+1}E_1(t;\partial_t^k u)+\frac{1}{2}\int_0^t(t_{3k}+s)^{\lambda_0+2k+1}E_a(s;\partial_t^{k+1}u)\,ds$$
$$\leq t_{3k}^{\lambda_0+2k+1}E_1(0;\partial_t^k u)+(\lambda_0+2k+1)\int_0^t(t_{3k}+s)^{\lambda_0+2k}E_{\partial x}(s;\partial_t^k u)\,ds$$
$$\leq M''_{\varepsilon,k}\Big(E_1(0;\partial_t^k u)+E_2(0;\partial_t^k u)+\|(u_0,u_1)\|^2_{H^k\times H^{k-1}(\Omega)}\Big)$$

with some constant $M''_{\varepsilon,k}>0$. By induction we obtain the desired inequalities for all $k\leq k_0-1$. □

## 4. Diffusion phenomena as an application of weighted energy estimates

**Proposition 4.1.** *Assume that* $(u_0, u_1) \in (H^2 \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ *and suppose that* $\operatorname{supp}(u_0, u_1) \subset \overline{B}(0, R_0)$. *Let* $u$ *be the solution of* (1.1). *Then for every* $\varepsilon > 0$, *there exists a constant* $C_{\varepsilon,R_0} > 0$ *such that*

$$\left\| u(\cdot, t) - e^{tL_*}[u_0 + a(\cdot)^{-1}u_1] \right\|_{L^2_{d\mu}} \le C_{\varepsilon,R_0}(1+t)^{-\frac{N-\alpha}{2(2-\alpha)} - \frac{1-\alpha}{2-\alpha} + \varepsilon} \|(u_0, u_1)\|_{H^2 \times H^1}.$$

To prove Proposition 4.1 we use the following lemma stated in [10, Section 4].

**Lemma 4.2.** *Assume that* $(u_0, u_1) \in (H^2 \cap H_0^1(\Omega)) \times H_0^1(\Omega)$ *and suppose that* $\operatorname{supp}(u_0, u_1) \subset \{x \in \Omega; |x| \le R_0\}$. *Then for every* $t \ge 0$,

$$\begin{aligned} u(x,t) - e^{tL_*}[u_0 + a(\cdot)^{-1}u_1] = &- \int_{t/2}^{t} e^{(t-s)L_*}[a(\cdot)^{-1}u_{tt}(\cdot, s)]ds \\ &- e^{\frac{t}{2}L_*}[a(\cdot)^{-1}u_t(\cdot, t/2)] \\ &- \int_0^{t/2} L_* e^{(t-s)L_*}[a(\cdot)^{-1}u_t(\cdot, s)]ds, \end{aligned} \tag{4.1}$$

*where* $L_*$ *is the (negative) Friedrichs extension of* $-L = -a(x)^{-1}\Delta$ *in* $L^2_{d\mu}$.

*Proof of Proposition 4.1.* First we show the assertion for $(u_0, u_1)$ satisfying the compatibility condition of order 2. Taking $L^2_{d\mu}$-norm of both side, we have

$$\left\| u(x, \cdot) - e^{tL_*}[u_0 + a(\cdot)^{-1}u_1] \right\|_{L^2_{d\mu}} \le \mathcal{J}_1(t) + \mathcal{J}_2(t) + \mathcal{J}_3(t),$$

where

$$\begin{aligned} \mathcal{J}_1(t) &:= \int_{t/2}^{t} \left\| e^{(t-s)L_*}[a(\cdot)^{-1}u_{tt}(\cdot, s)] \right\|_{L^2_{d\mu}} ds, \\ \mathcal{J}_2(t) &:= \left\| e^{\frac{t}{2}L_*}[a(\cdot)^{-1}u_t(\cdot, t/2)] \right\|_{L^2_{d\mu}}, \\ \mathcal{J}_3(t) &:= \int_0^{t/2} \left\| L_* e^{(t-s)L_*}[a(\cdot)^{-1}u_t(\cdot, s)] \right\|_{L^2_{d\mu}} ds. \end{aligned}$$

Noting that for $x \in \Omega$,

$$a(x)^{-1}\Phi_\varepsilon(x,t)^{-1} \le \frac{1}{a_1}\langle x\rangle^\alpha \exp\left(-\frac{A_{1\varepsilon}}{h+2\varepsilon}\frac{\langle x\rangle^{2-\alpha}}{1+t}\right) \le \frac{1}{a_1}\left(\frac{\alpha(h+2\varepsilon)}{(2-\alpha)eA_{1\varepsilon}}\right)^{\frac{\alpha}{2-\alpha}}(1+t)^{\frac{\alpha}{2-\alpha}},$$

we see that for $k = 0, 1$,

$$\begin{aligned} \left\| a(\cdot)^{-1}\partial_t^{k+1}u(\cdot, s) \right\|^2_{L^2_{d\mu}} &= \int_\Omega a(x)^{-1}|\partial_t^{k+1}u(\cdot, s)|^2\, dx \\ &\le \|a(\cdot)^{-1}\Phi_\varepsilon(\cdot, t)^{-1}\|_{L^\infty(\Omega)} \int_\Omega |\partial_t^{k+1}u(\cdot, s)|^2 \Phi_\varepsilon\, dx \\ &\le \widetilde{C}(1+t)^{\frac{\alpha}{2-\alpha}} E_{\partial t}(t, \partial_t^k u) \end{aligned}$$

$$\leq \widetilde{C}M_{\varepsilon,k}(1+t)^{-\lambda_0-\frac{2-2\alpha}{2-\alpha}-2k}\|(u_0,u_1)\|^2_{H^{k+1}\times H^k}.$$

Therefore from Proposition 3.5 with $k=1$ and $k=0$ we have

$$\begin{aligned}
\mathcal{J}_1(t) &\leq \int_{t/2}^t \left\|a(\cdot)^{-1}u_{tt}(\cdot,s)\right\|_{L^2_{d\mu}}ds \\
&\leq \sqrt{\widetilde{C}M_1}\|(u_0,u_1)\|_{H^2\times H^1}\int_{t/2}^t (1+s)^{-\frac{\lambda_0}{2}-\frac{1-\alpha}{2-\alpha}-1}ds \\
&\leq \frac{2(2-\alpha)}{\lambda_0(2-\alpha)+1-\alpha}\sqrt{\widetilde{C}M_{\varepsilon,1}}(1+t)^{-\frac{\lambda_0}{2}-\frac{1-\alpha}{2-\alpha}}\|(u_0,u_1)\|_{H^2\times H^1}
\end{aligned}$$

and

$$\mathcal{J}_2(t) \leq \left\|a(\cdot)^{-1}u_t(\cdot,t/2)\right\|_{L^2_{d\mu}} \leq \sqrt{\widetilde{C}M_{\varepsilon,0}}(1+t)^{-\frac{\lambda_0}{2}-\frac{1-\alpha}{2-\alpha}}\|(u_0,u_1)\|_{H^1\times L^2}.$$

Moreover, by Lemma 2.2, we see by Cauchy–Schwarz inequality that for $t\geq 1$,

$$\begin{aligned}
\mathcal{J}_3(t) &\leq C\int_0^{t/2}(t-s)^{-\frac{N-\alpha}{2(2-\alpha)}-1}\left\|a(\cdot)^{-1}u_t(\cdot,s)\right\|_{L^1_{d\mu}}ds \\
&\leq C\left(\frac{t}{2}\right)^{-\frac{N-\alpha}{2(2-\alpha)}-1}\int_0^{t/2}\sqrt{\|\Phi_\varepsilon^{-1}(\cdot,s)\|_{L^1(\Omega)}E_{\partial t}(s;u)}\,ds.
\end{aligned}$$

Since

$$\begin{aligned}
\|\Phi^{-1}(\cdot,t)\|_{L^1(\Omega)} &\leq \int_{\mathbb{R}^N}\exp\left(-\frac{A_{1\varepsilon}}{h+2\varepsilon}\frac{|x|^{2-\alpha}}{1+t}\right)dx \\
&= (1+t)^{\frac{N}{2-\alpha}}\int_{\mathbb{R}^N}\exp\left(-\frac{A_{1\varepsilon}}{h+2\varepsilon}|y|^{2-\alpha}\right)dy,
\end{aligned}$$

we deduce

$$\begin{aligned}
\mathcal{J}_3(t) &\leq C'(1+t)^{-\frac{N-\alpha}{2(2-\alpha)}-1}\|(u_0,u_1)\|_{H^1\times L^2}\int_0^{t/2}(1+s)^{\frac{N-\alpha}{2(2-\alpha)}-\frac{\lambda_0}{2}-\frac{1-\alpha}{2-\alpha}}ds \\
&\leq C'\left(\frac{N-\alpha}{2(2-\alpha)}-\frac{\lambda_0}{2}+\frac{1}{2-\alpha}\right)(1+t)^{-\frac{N-\alpha}{2(2-\alpha)}-1}(1+t/2)^{\frac{N-\alpha}{2(2-\alpha)}-\frac{\lambda_0}{2}-\frac{1-\alpha}{2-\alpha}+1}\|(u_0,u_1)\|_{H^1\times L^2} \\
&\leq C''(1+t)^{-\frac{\lambda_0}{2}-\frac{1-\alpha}{2-\alpha}}\|(u_0,u_1)\|_{H^1\times L^2}.
\end{aligned}$$

Consequently, we obtain

$$\left\|u(\cdot,t)-e^{tL_*}[u_0+a(\cdot)^{-1}u_1]\right\|_{L^2_{d\mu}} \leq C'''(1+t)^{-\frac{\lambda_0}{2}-\frac{1-\alpha}{2-\alpha}}\|(u_0,u_1)\|_{H^2\times H^1}.$$

Next we show the assertion for $(u_0,u_1)$ satisfying $(u_0,u_1)\in(H^2\times H^1_0(\Omega))\times H^1_0(\Omega)$ (the compatibility condition of order 1) via an approximation argument. Fix $\phi\in C_c^\infty(\mathbb{R}^N,[0,1])$ such that $\phi\equiv 1$ on $\overline{B}(0,R_0)$ and $\phi\equiv 0$ on $\mathbb{R}^N\setminus B(0,R_0+1)$ and define for $n\in\mathbb{N}$,

$$\begin{pmatrix} u_{0n} \\ u_{1n}\end{pmatrix} = \begin{pmatrix} \phi\tilde{u}_{0n} \\ \phi\tilde{u}_{1n}\end{pmatrix}, \quad \begin{pmatrix} \tilde{u}_{0n} \\ \tilde{u}_{1n}\end{pmatrix} = \left(1+\frac{1}{n}\mathcal{A}\right)^{-1}\begin{pmatrix} u_0 \\ u_1\end{pmatrix},$$

where $\mathcal{A}$ is an $m$-accretive operator in $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$ associated with (1.1), that is,

$$\mathcal{A} = \begin{pmatrix} 0 & -1 \\ -\Delta & a(x) \end{pmatrix}$$

endowed with domain $D(\mathcal{A}) = (H^2 \cap H_0^1(\Omega)) \times H_0^1(\Omega)$. Then $(u_{0n}, u_{1n})$ satisfies $\mathrm{supp}(u_{0n}, u_{1n}) \subset \overline{B}(0, R_0 + 1)$ and the compatibility condition of order 2. Let $v_n$ be a solution of (1.1) with $(u_{0n}, u_{1n})$. Observe that

$$\begin{aligned} \|(u_{0n}, u_{1n})\|_{H^2\times H^1}^2 &\le C^2 \|\phi\|_{W^{2,\infty}}^2 \|(\tilde{u}_0, \tilde{u}_1)\|_{H^2\times H_1}^2 \\ &\le C'^2 \|\phi\|_{W^{2,\infty}}^2 (\|(\tilde{u}_0, \tilde{u}_1)\|_{\mathcal{H}}^2 + \|\mathcal{A}(\tilde{u}_0, \tilde{u}_1)\|_{\mathcal{H}}^2) \\ &\le C'^2 \|\phi\|_{W^{2,\infty}}^2 (\|(u_0, u_1)\|_{\mathcal{H}}^2 + \|\mathcal{A}(u_0, u_1)\|_{\mathcal{H}}^2) \\ &\le C''^2 \|\phi\|_{W^{2,\infty}}^2 \|(u_0, u_1)\|_{H^2\times H^1}^2 \end{aligned}$$

with suitable constants $C, C', C'' > 0$, and

$$\begin{pmatrix} u_{0n} \\ u_{1n} \end{pmatrix} \to \begin{pmatrix} \phi u_0 \\ \phi u_1 \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \quad \text{in } \mathcal{H}$$

as $n \to \infty$ and also $u_{0n} + a^{-1}u_{1n} \to u_0 + a^{-1}u_1$ in $L^2_{d\mu}$ as $n \to \infty$. Using the result of the previous step, we deduce

$$\left\| v_n(\cdot, t) - e^{tL_*}[u_{0n} + a(\cdot)^{-1}u_{1n}] \right\|_{L^2_{d\mu}} \le \tilde{C}(1+t)^{-\frac{\lambda_0}{2} - \frac{1-\alpha}{2-\alpha}} \|(u_0, u_1)\|_{H^2\times H^1}$$

with some constant $\tilde{C} > 0$. Letting $n \to \infty$, by continuity of the $C_0$-semigroup $e^{-t\mathcal{A}}$ in $\mathcal{H}$ we also obtain diffusion phenomena for initial data in $(H^2 \cap H_0^1(\Omega)) \cap H_0^1(\Omega)$. □

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**Conflict of Interest**

All authors declare no conflicts of interest in this paper.

**References**

1. M. Ikawa, *Mixed problems for hyperbolic equations of second order*, J. Math. Soc. Japan, **20** (1968), 580-608.
2. M. Ikawa, M. Ikawa, Hyperbolic partial differential equations and wave phenomena, American Mathematical Society, Providence, RI, 2000.
3. R. Ikehata, *Some remarks on the wave equation with potential type damping coefficients*, Int. J. Pure Appl. Math., **21** (2005), 19-24.

4. A. Matsumura, *On the asymptotic behavior of solutions of semi-linear wave equations*, Publ. Res. Inst. Math. Sci., **12** (1976), 169-189.

5. A. Matsumura, *Energy decay of solutions of dissipative wave equations*, Proc. Japan Acad., Ser. A, **53** (1977), 232-236.

6. K. Mochizuki, *Scattering theory for wave equations with dissipative terms*, Publ. Res. Inst. Math. Sci., **12** (1976), 383-390.

7. K. Nishihara, *$L^p$-$L^q$ estimates of solutions to the damped wave equation in 3-dimensional space and their application*, Math. Z., **244** (2003), 631-649.

8. P. Radu, G. Todorova, and B. Yordanov, *Higher order energy decay rates for damped wave equations with variable coefficients*, Discrete Contin. Dyn. Syst. Ser. S, **2** (2009), 609-629.

9. P. Radu, G. Todorova, and B. Yordanov, *Decay estimates for wave equations with variable coefficients*, Trans. Amer. Math. Soc., **362** (2010), 2279-2299.

10. M. Sobajima and Y. Wakasugi, *Diffusion phenomena for the wave equation with space-dependent damping in an exterior domain*, J. Differential Equations, **261** (2016), 5690-5718.

11. G. Todorova, and B. Yordanov, *Critical exponent for a nonlinear wave equation with damping*, J. Differential Equations, **174** (2001), 464-489.

12. G. Todorova, and B. Yordanov, *Weighted $L^2$-estimates for dissipative wave equations with variable coefficients*, J. Differential Equations, **246** (2009), 4497-4518.

13. Y. Wakasugi, *On diffusion phenomena for the linear wave equation with space-dependent damping*, J. Hyp. Diff. Eq., **11** (2014), 795-819.