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Fixed point and endpoint theorems for set-valued fuzzy contraction maps in fuzzy metric spaces

Fatemeh Kiany[1*] and Alireza Amini-Harandi[2,3]

* Correspondence: fatemehkianybs@yahoo.com
[1]Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran
Full list of author information is available at the end of the article

Abstract

In this paper, we first present a fixed point theorem for set-valued fuzzy contraction type maps in complete fuzzy metric spaces which extends and improves some well-know results in literature. Then by presenting an endpoint result we initiate endpoint theory for fuzzy contraction maps in fuzzy metric spaces.
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1. Introduction and preliminaries

Many authors have introduced the concept of fuzzy metric spaces in different ways [1-4]. Kramosil and Michalek [5] introduced the fuzzy metric space by generalizing the concept of the probabilistic metric space to fuzzy situation. George and Veeramani [6,7] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [5] and obtained a Hausdorff topology for this kind of fuzzy metric spaces. Recently, the fixed point theory in fuzzy metric spaces has been studied by many authors [8-18]. In [11], the following definition is given.

Definition 1.1. A sequence (t_n) of positive real numbers is said to be an s-increasing sequence if there exists $m_0 \in \mathbb{N}$ such that $t_m + 1 \leq t_{m+1}$, for all $m \geq m_0$.

Gregori and Sapena [11] proved the following fixed point theorem.

Theorem 1.2. *Let $(X, M, *)$ be a complete fuzzy metric space such that for every s-increasing sequence (t_n) and every $x, y \in X$*

$$\lim_{n\to\infty} *_{i=n}^{\infty} M(x, y, t_n) = 1.$$

Suppose $f : X \to X$ is a map such that for each $x, y \in X$ and $t > 0$, we have

$$M(fx, fy, kt) \geq M(x, y, t),$$

where $0 < k < 1$. Then, f has a unique fixed point.

In this article, we first give a fixed point theorem for set-valued contraction maps which improve and generalize the above-mentioned result of Gregori and Sapena. Then, in Section 2, we initiate endpoint theory in fuzzy metric spaces by presenting an endpoint result for set-valued maps.

To set up our results in the next section we recall some definitions and facts.

Definition 1.3 (3). A binary operation * : [0, 1] × [0, 1] → [0, 1] is called a continuous t-norm if ([0,1], *) is an abelian topological monoid with unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, \in [0, 1]$. Examples of t-norm are $a * b = ab$ and $a * b = \min\{a, b\}$.

Definition 1.4 (6). The 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary non-empty set, * is a continuous t-norm, and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions, for *each* $x, y, z \in X$ and $t, s > 0$,

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only *if* $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$,
(5) $M(x, y, t) : (0, \infty) \to [0,1]$ is continuous.

Example 1.5. [6] Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = \min\{a, b\}$) and for all $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric M induced by the metric d *the standard fuzzy metric.*

Definition 1.6. Let $(X, M, *)$ be a fuzzy metric space.

(1) A sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.
(2) A sequence $\{x_n\}$ is called a Cauchy sequence if

$$\lim_{m,n\to\infty} M(x_m, x_n, t) = 1,$$

for all $t > 0$.
(3) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.
(4) A subset $A \subseteq X$ is said to be closed if for each convergent sequence $\{x_n\}$ with $x_n \in A$ and $x_n \to x$, we have $x \in A$.
(5) A subset $A \subseteq X$is said to be compact if each sequence in A has a convergent subsequence.

Throughout the article, let $\mathcal{K}(X)$ denote the class of all compact subsets of X.

Lemma 1.7. [10]*For all* $x, y \in X$, $M(x, y, .)$ *is non-decreasing.*

Definition 1.8. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t),$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X^2 \times (0, \infty)$ which converges to a point $(x, y, t) \in X^2 \times (0, \infty)$; i.e.,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

Lemma 1.9. [10]*M is a continuous function on $X^2 \times (0, \infty)$.*

2. Fixed point theory

The following lemma is essential in proving our main result.

Lemma 2.1. *Let $(X, M, *)$ be a fuzzy metric space such that for every $x, y \in X$, $t > 0$ and $h > 1$*

$$\lim_{n\to\infty} *_{i=n}^{\infty} M(x, y, th^i) = 1. \tag{2.1}$$

Suppose $\{x_n\}$ is a sequence in X such that for all $n \in \mathbb{N}$,

$$M(x_n, x_{n+1}, \alpha t) \geq M(x_{n-1}, x_n, t),$$

where $0 < \alpha < 1$. Then $\{x_n\}$ is a Cauchy sequence.

Proof. For each $n \in \mathbb{N}$ and $t > 0$, we have

$$M(x_n, x_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{1}{\alpha}t\right) \geq M\left(x_{n-2}, x_{n-1}, \frac{1}{\alpha^2}t\right) \geq \cdots \geq M\left(x_0, x_1, \frac{1}{\alpha^{n-1}}t\right).$$

Thus for each $n \in \mathbb{N}$, we get

$$M(x_n, x_{n+1}, t) \geq M\left(x_0, x_1, \frac{1}{\alpha^{n-1}}t\right).$$

Pick the constants $h > 1$ and $l \in \mathbb{N}$ such that

$$h\alpha < 1 \text{ and } \sum_{i=l}^{\infty} \frac{1}{h^i} = \frac{\frac{1}{h^l}}{1 - \frac{1}{h}} < 1.$$

Hence, for $m \geq n$, we get

$$\begin{aligned}
M(x_n, x_m, t) &\geq M\left(x_n, x_m, \left(\frac{1}{h^l} + \frac{1}{h^{l+1}} + \cdots + \frac{1}{h^{l+m}}\right)t\right) \\
&\geq M\left(x_n, x_{n+1}, \frac{1}{h^l}t\right) * M\left(x_{n+1}, x_{n+2}, \frac{1}{h^{l+1}}t\right) * \cdots * M\left(x_{m-1}, x_m, \frac{1}{h^{l+m}}t\right) \\
&\geq M\left(x_0, x_1, \frac{1}{\alpha^{n-1}h^l}t\right) * M\left(x_0, x_1, \frac{1}{\alpha^n h^{l+1}}t\right) * \cdots * M\left(x_0, x_1, \frac{1}{\alpha^{m-2}h^{l+m-n-2}}t\right) \\
&\geq M\left(x_0, x_1, \frac{1}{(\alpha h)^{n-1}}t\right) * M\left(x_0, x_1, \frac{1}{(\alpha h)^n}t\right) * \cdots * M\left(x_0, x_1, \frac{1}{(\alpha h)^{m-2}}t\right) \\
&\geq *_{i=n}^{\infty} M\left(x_0, x_1, \frac{1}{(\alpha h)^{i-1}}t\right)
\end{aligned}$$

Then, from the above, we have

$$\lim_{m,n\to\infty} M(x_n, x_m, t) \geq \lim_{n\to\infty} *_{i=n}^{\infty} M\left(x_0, x_1, \frac{1}{(\alpha h)^{i-1}}t\right) = 1,$$

for each $t > 0$. Therefore, we get

$$\lim_{m,n\to\infty} M(x_n, x_m, t) = 1,$$

for each $t > 0$ and so $\{x_n\}$ is a Cauchy sequence.

In 2004, Rodríguez-López and Romaguera [19] introduced Hausdorff fuzzy metric on the set of the non-empty compact subsets of a given fuzzy metric space.

Definition 2.2. ([19]) Let $(X, M, *)$ be a fuzzy metric space. For each $A, B \in \mathcal{K}(X)$ and $t > 0$, set

$$H_M(A, B, t) = \min\{\inf_{x\in A}\sup_{y\in B} M(x, y, t), \inf_{y\in B}\sup_{x\in A} M(x, y, t)\}.$$

Lemma 2.3. [19]*Let $(X, M, *)$ be a fuzzy metric space. Then, the 3-tuple $(\mathcal{K}(X), H_M, *)$ is a fuzzy metric space.*

Now we are ready to prove our first main result.

Theorem 2.4. *Let $(X, M, *)$ be a complete fuzzy metric. Suppose $F : X \to X$ is a set-valued map with non-empty compact values such that for each $x, y \in X$ and $t > 0$, we have*

$$H_M(Fx, Fy, \alpha(d(x, y, t))t) \geq M(x, y, t), \tag{2.2}$$

where $\alpha : [0, \infty) \to [0,1)$ satisfying

$$\limsup_{r\to t^+} \alpha(r) < 1, \quad \forall \quad t \in [0, \infty),$$

and $d(x, y, t) = \dfrac{t}{M(x, y, t)} - t$. Furthermore, assume that $(X, M, *)$ satisfies (2.1) for some $x_0 \in X$ and $x_1 \in Fx_0$. Then F has a fixed point.

Proof. Let $t > 0$ be fixed. Notice first that if A and B are non-empty compact subsets of X and $x \in A$ then by [19, Lemma 1], there exists a $y \in B$ such that

$$H_M(A, B, t) \leq \sup_{b\in B} M(x, b, t) = M(x, B, t) = M(x, y, t).$$

Thus given $\alpha \leq H_M(A, B, t)$ there exists a point $y \in B$ such that

$$M(x, y, t) \geq \alpha.$$

Let $x_0 \in X$ and $x_1 \in Fx_0$. If $Fx_0 = Fx_1$ then $x_1 \in Fx_1$ and x_1 is a fixed point of F and we are finished. So, we may assume that $Fx_0 \neq Fx_1$. From (2.2), we get

$$H_M(Fx_0, Fx_1, \alpha(d(x_0, x_1, t))t) \geq M(x_0, x_1, t).$$

Since $x_1 \in Fx_0$ and F is compact valued then by Rodríguez-López and Romaguera [19, Lemma 1] there exists a $x_2 \in Fx_1$ satisfying

$$\begin{aligned} M(x_1, x_2, t) \geq M(x_1, x_2, \alpha(d(x_0, x_1, t))t) &= \sup_{y\in Fx_1} M(x_1, y, \alpha(d(x_0, x_1, t))t) \\ &\geq H_M(Fx_0, Fx_1, \alpha(d(x_0, x_1, t))t) \\ &\geq M(x_0, x_1, t). \end{aligned}$$

Continuing this process, we can choose a sequence $\{x_n\}_{n\geq 0}$ in X such that $x_{n+1} \in Fx_n$ satisfying

$$\begin{aligned} M(x_{n+1}, x_{n+2}, t) \geq M(x_{n+1}, x_{n+2}, \alpha(d(x_n, x_{n+1}, t))t) &= \sup_{y \in Fx_{n+1}} M(x_{n+1}, y, \alpha(d(x_n, x_{n+1}, t))t) \\ &\geq H_M(Fx_n, Fx_{n+1}, \alpha(d(x_n, x_{n+1}, t))t) \\ &\geq M(x_n, x_{n+1}, t). \end{aligned}$$

Then, the sequence $\{M(x_{n+1}, x_{n+2}, t)\}_n$ is non-decreasing.

Thus $\{d(x_{n+1}, x_{n+2}, t)\}_n$ is a non-negative non-increasing sequence and so is convergent, say to, $l \geq 0$. Since by the assumption

$$\limsup_{n\to\infty} \alpha(d(x_{n+1}, x_{n+2}, t)) \leq \limsup_{r\to t^+} \alpha(r) < 1,$$

then there exists $k < 1$ and $N \in \mathbb{N}$ such that

$$\alpha(d(x_{n+1}, x_{n+2}, t)) < k, \quad \forall \quad n > N. \tag{2.4}$$

Since $M(x, y,.)$ is non-decreasing then (2.3) together with (2.4) yield

$$M(x_{n+1}, x_{n+2}, kt) \geq M(x_{n+1}, x_{n+2}, \alpha(d(x_n, x_{n+1}, t))t) \geq M(x_n, x_{n+1}, t).$$

Then from the above, we get

$$M(x_{n+1}, x_{n+2}, kt) \geq M(x_n, x_{n+1}, t).$$

Hence by Lemma 2.1, we get $\{x_n\}$, which is a Cauchy sequence. Since $(X, M, *)$ is a complete fuzzy metric space, then there exists $\bar{x} \in X$ such that $\lim_{n\to\infty} x_n = \bar{x}$, that means $\lim_{n\to\infty} M(x_n, \bar{x}, t) = 1$, for each $t > 0$. Thus, $\lim_{n\to\infty} d(x_n, \bar{x}, t) = 0$, for each $t > 0$. Since

$$\limsup_{n\to\infty} \alpha(d(x_n, \bar{x}, t)) \leq \limsup_{r\to 0^+} \alpha(r) < 1,$$

then there exists $k < l < 1$ such that

$$\limsup_{n\to\infty} \alpha(d(x_n, \bar{x}, t)) < l.$$

Now we claim that $\bar{x} \in F\bar{x}$. To prove the claim notice first that since $H_M(Fx_n, F\bar{x}, lt) \geq H_M(Fx_n, F\bar{x}, kt) \geq H_M(Fx_n, F\bar{x}, \alpha(d(x_n, \bar{x}, t))t) \geq M(x_n, \bar{x}, t)$, and $\lim_{n\to\infty} M(x_n, \bar{x}, t) = 1$ then for each $t > 0$, we get

$$\lim_{n\to\infty} H_M(Fx_n, F\bar{x}, t) = 1. \tag{2.5}$$

Since $x_{n+1} \in Fx_n$ then from (2.5), we have

$$\lim_{n\to\infty} \sup_{y\in F\bar{x}} M(x_{n+1}, y, t) = 1.$$

Thus there exists a sequence $y_n \in F\bar{x}$ such that

$$\lim_{n\to\infty} M(x_n, y_n, t) = 1,$$

for each $t > 0$. For each $n \in \mathbb{N}$, we have

$$M(y_n, \bar{x}, s + t) \geq M(y_n, x_n, s) * M(x_n, \bar{x}, t).$$

Hence, from the above, we get

$$\lim_{n\to\infty} M(y_n, \bar{x}, t) = 1,$$

which means $\lim_{n\to\infty} y_n = \bar{x}$. Since $F\bar{x}$ is closed (note that $F\bar{x}$ is compact), $y_n \to \bar{x}$ and $y_n \in F\bar{x}$ then, we get $\bar{x} \in F\bar{x}$.

Corollary 2.5. *Let* $(X, M, *)$ *be a complete fuzzy metric. Suppose* $F : X \to X$ *is a set-valued map with non-empty compact values such that for each* $x, y \in X$ *and* $t > 0$, *we have*

$$H_M(Fx, Fy, kt) \geq M(x, y, t),$$

where $0 < k < 1$. *Furthermore, assume that* $(X, M, *)$ *satisfies* (2.1) *for some* $x_0 \in X$ *and* $x_1 \in Fx_0$. *Then F has a fixed point.*

From Corollary 2.5, we get the following improvement of the above mentioned result of Gregori and Sapena [11] (note that for each $t > 0$ and $h > 1$, the sequence $t_n = th^n$ is s-increasing).

Theorem 2.6. *Let* $(X, M, *)$ *be a complete fuzzy metric space. Suppose* $f : X \to X$ *is a map such that for each* $x, y \in X$ *and* $t > 0$, *we have*

$$M(fx, fy, kt) \geq M(x, y, t),$$

where $0 < k < 1$. *Furthermore, assume that* $(X, M, *)$ *satisfies* (2.1) *for some* $x_0 \in X$, *each* $t > 0$ *and* $h > 1$. *Then f has a fixed point.*

Let (X, d) be a metric space and A and B are non-empty closed bounded subsets of X. Now set

$$H(A, B) = \max\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{y\in B}\inf_{x\in A} d(x, y)\}.$$

Then H is called the Hausdorff metric. Now, we are ready to derive the following version of Mizoguchi-Takahashi fixed point theorem [20].

Corollary 2.7. *Let* (X, d) *be a complete metric space. Suppose* $F : M \to M$ *is a set-valued map with non-empty compact values such that for some* $k < 1$

$$H(Fx, Fy) \leq \alpha(d(x, y))d(x, y),$$

where $\alpha : [0, \infty) \to [0,1)$ *satisfying*

$$\limsup_{r\to t^+} \alpha(r) < 1, \quad \forall t \in [0, \infty).$$

Then F has a fixed point.

Proof. Let $(X, M, *)$ be standard fuzzy metric space induced by the metric d with $a * b = ab$. Now we show that the conditions of Theorem 2.4 are satisfied. Since (X, d) is a complete metric space then $(X, M, *)$ is complete. It is easy to see that $(X, M, *)$ satisfies (2.1). For each non-empty closed bounded subsets of X, we have

$$\begin{aligned}
H_M(A, B, t) &= \min\left\{\inf_{x\in A}\sup_{y\in B} M(x, y, t), \inf_{y\in B}\sup_{x\in A} M(x, y, t)\right\} \\
&= \min\left\{\inf_{x\in A}\sup_{y\in B} \frac{t}{t + d(x, y)}, \inf_{y\in B}\sup_{x\in A} \frac{t}{t + d(x, y)}\right\} \\
&= \min\left\{\frac{t}{t + \sup_{x\in A}\inf_{y\in B} d(x, y)}, \frac{t}{t + \sup_{y\in B}\inf_{x\in A} d(x, y)}\right\} \\
&= \frac{t}{t + \max\left\{\sup_{x\in A}\inf_{y\in B} d(x, y), \sup_{y\in B}\inf_{x\in A} d(x, y)\right\}} \\
&= \frac{t}{t + H(A, B)}.
\end{aligned}$$

By the above and our assumption, we have

$$\begin{aligned} H_M(Fx, Fy, \alpha(d(x,y,t))t) &= \frac{\alpha(d(x,y))t}{\alpha(d(x,y))t + H(Fx, Fy)} \\ &\geq \frac{\alpha(d(x,y))t}{\alpha(d(x,y))(t + d(x,y))} \\ &= \frac{t}{t + d(x,y)} \\ &= M(x,y,t), \end{aligned}$$

for each $t > 0$ and each $x, y \in X$. Therefore, the conclusion follows from Theorem 2.4.

3. Endpoint theory

Let X be a non-empty set and let $F : X \to 2^X$ be a set-valued map. An element $x \in X$ is said to be an endpoint (invariant or stationary point) of F, if $Fx = \{x\}$. The investigation of the existence and uniqueness of endpoints of set-valued contraction maps in metric spaces have received much attention in recent years [21-26].

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space and let $F : X \to X$ be a multi-valued mapping. We say that F is continuous if for any convergent sequence $x_n \to x_0$ we have $H_M(Fx_n, Fx_0, t) \to 1$ as $n \to \infty$, for each $t > 0$.

As far as we know the following is the first endpoint result for set-valued contraction type maps in fuzzy metric spaces.

Theorem 3.2. *Let $(X, M, *)$ be a complete fuzzy metric space and let $F : X \to \mathcal{K}(X)$be a continuous set-valued mapping. Suppose that for each $x \in X$ there exists $y \in Fx$ satisfying*

$$H_M(y, Fy, kt) \geq M(x,y,t), \quad \forall \quad t > 0, \tag{3.1}$$

where $k \in [0,1)$. Then, F has an endpoint.

Proof. For each $x \in X$, define the function $f : X \to [0, \infty)$ by $f(x, t) = H_M(x, Fx, t) = \inf_{y \in Fx} M(x, y, t)$, $x \in X$. Suppose that $\{x_n\}$ converges to x; then for any $y \in Fx$ and $z \in Fx_n$, we have

$$\begin{aligned} M(x,y,t) &\geq M\left(x, x_n, \frac{t}{3}\right) * M\left(x_n, z, \frac{t}{3}\right) * M\left(z, y, \frac{t}{3}\right) \\ &\geq M\left(x, x_n, \frac{t}{3}\right) * H_M\left(x_n, Fx_n, \frac{t}{3}\right) * H_M\left(z, Fx, \frac{t}{3}\right) \\ &\geq M\left(x, x_n, \frac{t}{3}\right) * f\left(x_n, \frac{t}{3}\right) * H_M\left(Fx_n, Fx, \frac{t}{3}\right). \end{aligned}$$

Since $y \in Fx$ is arbitrary then from the above, we get

$$f(x,t) = H_M(x, Fx, t) \geq M\left(x, x_n, \frac{t}{3}\right) * f\left(x_n, \frac{t}{3}\right) * H_M\left(Fx_n, Fx, \frac{t}{3}\right).$$

It follows from the continuity of F that

$$f(x,t) \geq \limsup_{n\to\infty} \left(M\left(x, x_n, \frac{t}{3}\right) * f(x_n) * H_M\left(Fx_n, Fx, \frac{t}{3}\right)\right) = \limsup_{n\to\infty} f\left(x_n, \frac{t}{3}\right).$$

Hence,

$$f(x,t) \geq \limsup_{n\to\infty} f\left(x_n, \frac{t}{3}\right),$$

whenever $x_n \to x$. Let $x_0 \in X$. Then by (3.1) there exists a $x_1 \in Fx_0$ such that

$$H_M(x_1, Fx_1, kt) \geq M(x_0, x_1, t).$$

Continuing this process, we can choose a sequence $\{x_n\}_{n\geq 0}$ in X such that $x_{n+1} \in Fx_n$ satisfying

$$H_M(x_{n+1}, Fx_{n+1}, kt) \geq M(x_n, x_{n+1}, t). \tag{3.2}$$

From the definition of $H_M(x_n, Tx_n)$, we have

$$M(x_n, x_{n+1}, t) \geq H_M(x_n, Fx_n, t). \tag{3.3}$$

From (3.2) and (3.3), we get

$$\begin{aligned} H_M(x_{n+1}, Fx_{n+1}, kt) &\geq M(x_n, x_{n+1}, t) \\ &\geq H_M(x_n, Fx_n, t) \\ &\geq H_M(x_n, Fx_n, kt) \\ &\geq M\left(x_{n-1}, x_n, \frac{1}{k}t\right), \end{aligned} \tag{3.4}$$

which implies that $\{H_M(x_n, Fx_n, kt)\}_n$ is a non-negative non-decreasing sequence of real numbers and so is convergent. To find the limit of $\{H(x_n, Fx_n, kt)\}_n$ notice that

$$\begin{aligned} H_M(x_{n+1}, Fx_{n+1}, kt) &\geq H_M(x_n, Fx_n, t) \\ &\geq H_M\left(x_{n-1}, Fx_{n-1}, \frac{1}{k}t\right) \geq \cdots \geq H_M\left(x_0, Fx_0, \frac{1}{k^n}t\right). \end{aligned} \tag{3.5}$$

Since Fx_0 is compact then there exists a $y_0 \in Fx_0$ such that

$$H_M\left(x_0, Fx_0, \frac{1}{k^n}t\right) = M\left(x_0, y_0, \frac{1}{k^n}t\right). \tag{3.6}$$

(3.5) together with (3.6) imply that for each $n \in \mathbb{N}$

$$H_M(x_{n+1}, Fx_{n+1}, kt) \geq M\left(x_0, y_0, \frac{1}{k^n}t\right).$$

From (2.1) we have $\lim_{n\to\infty} M\left(x_0, y_0, \frac{1}{k^n}t\right) = 1$ and so

$$\lim_{n\to\infty} H_M(x_n, Fx_n, t) = 1, \quad \forall \quad t > 0.$$

From (3.2), we get

$$M(x_n, x_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{1}{k}t\right),$$

from which and Lemma (2.1), we get $\{x_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is a complete fuzzy metric space then there exists a $\bar{x} \in X$ such that $\lim_{n\to\infty} x_n = \bar{x}$. By assumption the function $f(x) = H_M(x, Fx, t)$ is upper semicontinuous, then

$$H_M(\bar{x}, F\bar{x}, t) \geq \lim_{n\to\infty} H_M(x_n, Fx_n, t) = 1.$$

Thus

$$H_M(\bar{x}, F\bar{x}, t) = 1,$$

and so $F\bar{x} = \{\bar{x}\}$.

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Author details
[1]Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran [2]Department of Mathematics, University of Shahrekord, Shahrekord 88186-34141, Iran [3]School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran

Authors' contributions
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