Dong and Fang *Boundary Value Problems* (2016) 2016:106
DOI 10.1186/s13661-016-0612-9

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Differential equations of divergence form in separable Musielak-Orlicz-Sobolev spaces

Ge Dong[1,2†] and Xiaochun Fang[1*†]

*Correspondence: xfang@tongji.edu.cn
[1]Department of Mathematics, Tongji University, Siping Road, Shanghai, 200092, China
Full list of author information is available at the end of the article
†Equal contributors

Abstract

In this paper, we study the existence of weak solutions for differential equations of divergence form

$$-\operatorname{div}(a_1(x,Du)) + a_0(x,u) = f(x,u,Du),$$

in Ω coupled with a Dirichlet or Neumann boundary condition in separable Musielak-Orlicz-Sobolev spaces where a_1 satisfies the growth condition, the coercive condition, and the monotone condition, and a_0 satisfies the growth condition without any coercive condition or monotone condition. The right-hand side $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying a growth condition dependent on the solution u and its gradient Du. We prove the existence of weak solutions by using a linear functional analysis method. Some sufficient conditions guarantee the existence enclosure of weak solutions between sub- and supersolutions. Our method does not require any reflexivity of the Musielak-Orlicz-Sobolev spaces.

Keywords: separable Musielak-Orlicz-Sobolev spaces; differential equation; sub-supersolution

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Le [1] established a sub-supersolution method for variational inequalities with Leray-Lions operators in Sobolev spaces with variable exponents. Following [1], Fan [2] established a sub-supersolution method for the differential equations of divergence form

$$-\operatorname{div}\bigl(a_1(x,Du)\bigr) + a_0(x,u) = f(x,u), \tag{1.1}$$

in Ω coupled with Neumann or Dirichlet boundary condition in reflexive Musielak-Orlicz-Sobolev spaces $W_0^1L_\Phi(\Omega)$. Here a_1 and a_0 are supposed to satisfy growth conditions, coercive conditions, and monotone conditions, that is,

$$\bigl|a_1(x,\xi)\bigr| \le b_1\varphi\bigl(x,|\xi|\bigr) + g(x), \tag{1.2}$$

$$a_1(x,\xi)\xi \ge b_2\Phi\bigl(x,|\xi|\bigr) - h(x), \tag{1.3}$$

$$\bigl[a_1(x,\xi) - a_1(x,\eta)\bigr](\xi - \eta) \ge 0, \tag{1.4}$$

and

$$\left|a_0(x,t)\right| \le b_1\varphi\bigl(x,|t|\bigr) + g(x), \tag{1.5}$$

$$a_0(x,t)t \ge b_2\Phi\bigl(x,|t|\bigr) - h(x), \tag{1.6}$$

$$\bigl[a_0(x,s) - a_0(x,t)\bigr](s-t) \ge 0, \tag{1.7}$$

for $x \in \Omega$, $s,t \in \mathbb{R}$ and $\xi,\eta \in \mathbb{R}^N$, where $b_1, b_2 > 0$, $g \in E_{\overline{\Phi}}(\Omega)$, $g \ge 0$, $h \in L^1(\Omega)$, and $h \ge 0$. The right-hand side $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Liu *et al.* [3] proved the existence of weak solutions for (1.1) with $a_0 = 0$ in reflexive Musielak-Orlicz-Sobolev spaces.

However, there exist some nonreflexive Musielak-Orlicz-Sobolev spaces. For example, let $\Phi(x,t) = (1 + \frac{t}{p(x)})\ln(1 + \frac{t}{p(x)}) - \frac{t}{p(x)}$, for $x \in \Omega$ and $t > 0$, where $p : \Omega \to \mathbb{R}$ is a measurable function such that $1 < p_- := \inf_{x\in\Omega} p(x) \le p(x) \le p_+ := \sup_{x\in\Omega} p(x) < +\infty$. Then the Musielak-Orlicz-Sobolev space $W^1L_\Phi(\Omega)$ is separable and nonreflexive.

The purpose of this paper is to weaken the restriction of reflexivity of the Musielak-Orlicz spaces in [2] and study the existence of solutions for the following nonlinear problem:

$$-\operatorname{div}\bigl(a_1(x,Du)\bigr) + a_0(x,u) = f(x,u,Du), \tag{1.8}$$

in Ω coupled with Dirichlet or Neumann boundary condition, where a_1 satisfies the growth condition, the coercive condition, and the monotone condition, and a_0 satisfies the growth condition without any coercive condition or monotone condition. The right-hand side $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying a growth condition dependent on the solution u and its gradient Du.

One needs the following coercive condition of Φ in [2]:

$$\Phi(x,\alpha u) \ge \alpha G(\alpha)\Phi(x,u), \quad \text{for } x \in \Omega, t \in \mathbb{R} \text{ and } \alpha > 0, \tag{1.9}$$

where $G : (0,+\infty) \to \mathbb{R}$ is a function such that $G(\alpha) \to +\infty$ as $\alpha \to +\infty$. We will point out that the condition (1.9) can be omitted.

This paper is organized as follows: Section 2 contains some preliminaries and some technical lemmas which will be needed. We establish some basic properties for Musielak-Orlicz functions and some necessary and sufficient conditions for Musielak-Orlicz functions satisfying the Δ_2 condition. In Section 3, we establish a linear functional analysis method for differential equations of divergence form to prove the existence of weak solutions for (1.8) with Dirichlet boundary or Neumann boundary condition in separable Musielak-Orlicz-Sobolev spaces. We give the enclosure of weak solutions between sub- and supersolutions by using a sub-supersolution method. Our method does not require any monotonicity or coercivity of a_0. We point out that the coercive condition (1.9) of Φ can be omitted because of the reflexivity of the Musielak-Orlicz-Sobolev spaces in [2].

We refer to some results of sub-supersolution methods for variational inequalities and the existence of solutions for differential equations studied in variable exponent Sobolev or Orlicz-Sobolev spaces (see, *e.g.*, [4–11]). For some results we also refer to [12–14].

In this paper, we always assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary and denote by $L^0(\Omega)$ the set of all real measurable functions defined on Ω.

2 Preliminaries

Now we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces; for more details see [2, 15, 16], and [17].

A real function Φ defined on $\Omega \times \mathbb{R}_+$, where $\mathbb{R}_+ = [0, +\infty)$, will be said a generalized N-function (*i.e.* a Musielak-Orlicz function), denoted by $\Phi \in N(\Omega)$, if it satisfies the following conditions:

(i) $\Phi(x, u)$ is an N-function of the variable $u \geq 0$ for every $x \in \Omega$, *i.e.* is a convex, nondecreasing, continuous function of u such that $\Phi(x, 0) = 0$, $\Phi(x, u) > 0$ for $u > 0$, and we have the conditions

$$\lim_{u\to 0^+} \sup_{x\in\Omega} \frac{\Phi(x,u)}{u} = 0, \qquad \lim_{u\to +\infty} \inf_{x\in\Omega} \frac{\Phi(x,u)}{u} = +\infty.$$

(ii) $\Phi(x, u)$ is a measurable function of x for all $u \geq 0$.

Equivalently, Φ admits the representation

$$\Phi(x,u) = \int_0^u \varphi(x,\tau)\,d\tau, \tag{2.1}$$

where $\varphi(x, u)$ is the right-hand derivative of $\Phi(x, \cdot)$ at u, for a fixed $x \in \Omega$ and all $u \geq 0$. Then for every $x \in \Omega$, $\varphi(x, \tau)$ is a right-continuous and nondecreasing function of $\tau \geq 0$, $\varphi(x, 0) = 0$, $\varphi(x, \tau) > 0$ for $\tau > 0$, and $\lim_{u\to+\infty} \inf_{x\in\Omega} \varphi(x, \tau) = +\infty$.

Let $\Phi \in N(\Omega)$, then $\Phi(x, u) \leq u\varphi(x, u) \leq \Phi(x, 2u)$, for $x \in \Omega$, $u \geq 0$.

The complementary function $\overline{\Phi}$ to a Musielak-Orlicz function Φ is defined as follows:

$$\overline{\Phi}(x,v) = \sup_{u\geq 0}\bigl\{uv - \Phi(x,u)\bigr\}, \quad \text{for all } v \geq 0, x \in \Omega.$$

Then $\overline{\Phi}$ is a Musielak-Orlicz function and Φ is also the complementary function to $\overline{\Phi}$. Equivalently, $\overline{\Phi}$ admits the representation

$$\overline{\Phi}(x,v) = \int_0^v \phi(x,\sigma)\,d\sigma, \tag{2.2}$$

where ϕ is given by

$$\phi(x,\sigma) = \sup\bigl\{\tau : \varphi(x,\tau) \leq \sigma\bigr\}, \quad \text{for all } x \in \Omega. \tag{2.3}$$

Similar to the proof in [18], we can deduce that

$$\phi\bigl(x,\varphi(x,u)\bigr) \geq u, \qquad \varphi\bigl(x,\phi(x,v)\bigr) \geq v, \quad \text{for } u \geq 0, v \geq 0 \text{ and } x \in \Omega, \tag{2.4}$$

and

$$\begin{aligned} &\phi\bigl(x,\varphi(x,u)-\varepsilon\bigr) \leq u, \quad \text{for } u \geq 0, 0 < \varepsilon \leq \varphi(x,u) \text{ and } x \in \Omega, \\ &\varphi\bigl(x,\phi(x,v)-\varepsilon\bigr) \leq v, \quad \text{for } v \geq 0, 0 < \varepsilon \leq \phi(x,v) \text{ and } x \in \Omega. \end{aligned}$$

For $\Phi \in N(\Omega)$, the following inequality is called the Young inequality:

$$uv \leq \Phi(x,u) + \overline{\Phi}(x,v), \quad \text{for all } u, v \geq 0, x \in \Omega, \tag{2.5}$$

and the equality holds if and only if $u = \phi(x,v)$ or $v = \varphi(x,u)$, *i.e.*

$$u\varphi(x,u) = \Phi(x,u) + \overline{\Phi}\big(x,\varphi(x,u)\big), \qquad \phi(x,v)v = \Phi\big(x,\phi(x,v)\big) + \overline{\Phi}(x,v). \tag{2.6}$$

Let $\Phi \in N(\Omega)$. Φ is said to satisfy the Δ_2 condition ($\Phi \in \Delta_2$, for short), if there exist a positive constant $K > 1$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\Phi(x,2u) \leq K\Phi(x,u) + h(x), \quad \text{for all } u \geq 0 \text{ and a.e. } x \in \Omega. \tag{2.7}$$

Clearly, by the proof of Proposition 1.3(6) in [2], if $\Phi \in \Delta_2$, then there exist $K > 1$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$\overline{\Phi}\big(x,\varphi(x,u)\big) \leq (K-1)\Phi(x,u) + h(x), \quad \text{for all } u \geq 0 \text{ and a.e. } x \in \Omega. \tag{2.8}$$

For each $x \in \Omega$, the inverse function of $\Phi(x,\cdot)$ is denoted by $\Phi^{-1}(x,\cdot)$, *i.e.*

$$\Phi^{-1}\big(x,\Phi(x,u)\big) = \Phi\big(x,\Phi^{-1}(x,u)\big) = u, \quad \text{for } u \geq 0.$$

Let $\Psi, \Upsilon \in N(\Omega)$. $\Psi \preceq \Upsilon$ means that Ψ is weaker than Υ, *i.e.*, there exist positive constants K_1, K_2 and a nonnegative function $h_1 \in L^1(\Omega)$ such that

$$\Psi(x,u) \leq K_1\Upsilon(x,K_2u) + h_1(x), \quad \text{for all } u \geq 0 \text{ and a.e. } x \in \Omega. \tag{2.9}$$

Φ is called locally integrable, if $\int_\Omega \Phi(x,u)\,dx < \infty$ for every $u > 0$.

The following assumptions will be used.

(Φ_1) $\inf_{x\in\Omega} \Phi(x,1) = c_1 > 0$.

(Φ_2) For every $t_0 > 0$ there exists $c = c(t_0) > 0$ such that

$$\inf_{x\in\Omega} \frac{\Phi(x,t)}{t} \geq c \tag{2.10}$$

and

$$\inf_{x\in\Omega} \frac{\overline{\Phi}(x,t)}{t} \geq c, \tag{2.11}$$

for all $t \geq t_0$.

Obviously, (2.10) implies (Φ_1).

Let $\Phi \in N(\Omega)$. The Musielak-Orlicz space (*i.e.* the generalized Orlicz space) $L_\Phi(\Omega)$ is defined by

$$L_\Phi(\Omega) = \left\{u \in L^0(\Omega) : \int_\Omega \Phi\left(x, \frac{|u(x)|}{\lambda}\right) dx < \infty, \text{ for some } \lambda > 0\right\},$$

with the (Luxemburg) norm

$$\|u\|_\Phi = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}.$$

Moreover, the set

$$K_\Phi(\Omega) = \left\{ u \in L^0(\Omega) : \int_\Omega \Phi\bigl(x, \bigl|u(x)\bigr|\bigr)\, dx < \infty \right\},$$

will be called the Musielak-Orlicz class (*i.e.* the generalized Orlicz class). A function $u \in L^0(\Omega)$ will be called a finite element of $L_\Phi(\Omega)$, if $\lambda u \in K_\Phi(\Omega)$ for every $\lambda > 0$. The space of all finite elements of $L^0(\Omega)$ will be denoted by $E_\Phi(\Omega)$. Then $K_\Phi(\Omega)$ is a convex subset of $L_\Phi(\Omega)$, $L_\Phi(\Omega)$ is the smallest vector subspace of $L^0(\Omega)$ containing $K_\Phi(\Omega)$, and $E_\Phi(\Omega)$ is the largest vector subspace of $L^0(\Omega)$ contained in $K_\Phi(\Omega)$.

If Φ is locally integrable, then $E_\Phi(\Omega)$ is a separable space, and $E_\Phi(\Omega) = K_\Phi(\Omega) = L_\Phi(\Omega)$ if and only if $\Phi \in \Delta_2$.

If Φ is locally integrable and satisfy (2.10), then $(E_\Phi(\Omega))^* = L_{\overline{\Phi}}(\Omega)$. Moreover, if $\overline{\Phi}$ is locally integrable satisfying (2.11), and $\Phi, \overline{\Phi} \in \Delta_2$, then $L_\Phi(\Omega)$ is reflexive.

The Musielak-Orlicz-Sobolev space $W^1L_\Phi(\Omega)$ is defined by

$$W^1L_\Phi(\Omega) = \bigl\{ u \in L_\Phi(\Omega) : \forall |\alpha| \leq 1, D^\alpha u \in L_\Phi(\Omega) \bigr\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ with nonnegative integers α_i, $i = 1, \ldots, N$, $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_N|$ and $D^\alpha u$ denote the distributional derivatives.

Let

$$\varrho_\Phi(u) = \sum_{|\alpha| \leq 1} \int_\Omega \Phi\bigl(x, \bigl|D^\alpha u(x)\bigr|\bigr)\, dx \quad \text{and} \quad \|u\|_{\Phi,\Omega} = \inf\left\{ \lambda > 0 : \varrho_\Phi\left(\frac{u}{\lambda}\right) \leq 1 \right\},$$

for $u \in W^1L_\Phi(\Omega)$. $\varrho_\Phi(u)$ is a convex modular and $\|u\|_{\Phi,\Omega}$ is a norm on $W^1L_\Phi(\Omega)$, respectively. The pair $(W^1L_\Phi(\Omega), \|u\|_{\Phi,\Omega})$ is a Banach space if Φ is locally integrable and satisfies (Φ_1).

Taking $\Phi(x,u) = \Phi(u)$, $W^1L_\Phi(\Omega)$ is the Orlicz-Sobolev space. Taking $\Phi(x,|u|) = |u|^{p(x)}$, $W^1L_\Phi(\Omega)$ is the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$.

It is easy to see that

$$W^1L_\Phi(\Omega) = \bigl\{ u \in L_\Phi(\Omega) : |Du| \in L_\Phi(\Omega) \bigr\}.$$

Denote $\|Du\|_\Phi = \||Du|\|_\Phi$ and $\|u\|_{1,\Phi} = \|u\|_\Phi + \|Du\|_\Phi$. Then $\|u\|_{1,\Phi}$ and $\|u\|_{\Phi,\Omega}$ are two equivalent norms.

The space $W^1L_\Phi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq 1} L_\Phi(\Omega) = \prod L_\Phi$; this subspace is $\sigma(\prod L_\Phi, \prod E_{\overline{\Phi}})$ closed. Let $W_0^1L_\Phi(\Omega)$ be the $\sigma(\prod L_\Phi, \prod E_{\overline{\Phi}})$ closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1L_\Phi(\Omega)$.

Let $W^1E_\Phi(\Omega) = \{u \in E_\Phi(\Omega) : \forall |\alpha| \leq 1, D^\alpha u \in E_\Phi(\Omega)\}$, and $W_0^1E_\Phi(\Omega)$ is the (norm) closure of $\mathcal{D}(\Omega)$ in $W^1L_\Phi(\Omega)$.

The proof of the following lemma is similar to [19].

Lemma 2.1 *Let* meas Ω *be bounded,* $\Phi \in N(\Omega)$, *and* φ *is the right-hand derivative of* Φ. *Then*

$$\frac{\int_\Omega \varphi(x, |Du|)|Du|\, dx}{\int_\Omega |Du|\, dx} \to +\infty, \quad \textit{if} \int_\Omega |Du|\, dx \to +\infty. \tag{2.12}$$

Proof Let us assume that there is a sequence $\{u_n\}$ with $\int_\Omega |Du_n(x)|\,dx \to +\infty$ and $K_0 < \infty$ such that

$$\frac{\int_\Omega \varphi(x,|Du_n(x)|)|Du_n(x)|\,dx}{\int_\Omega |Du_n(x)|\,dx} \le K_0.$$

Since $\Phi \in N(\Omega)$, there exists $R > 0$ such that

$$\inf_{x\in\Omega} \varphi(x,R) \ge \inf_{x\in\Omega} \frac{\Phi(x,R)}{R} > 2K_0.$$

We define $\widetilde{\Omega}(R,n) := \{x \in \Omega \,|\, |Du_n(x)| \ge R\}$ and take for all n with $\int_\Omega |Du_n(x)|\,dx \ge 2R \operatorname{meas}\Omega$, then

$$\begin{aligned}
&\frac{\int_\Omega \varphi(x,|Du_n(x)|)|Du_n(x)|\,dx}{\int_\Omega |Du_n(x)|\,dx} \\
&\quad\ge \inf_{x\in\Omega} \varphi(x,R) \frac{\int_{\widetilde{\Omega}(R,n)} |Du_n(x)|\,dx}{\int_{\widetilde{\Omega}(R,n)} |Du_n(x)|\,dx + R \cdot \operatorname{meas}(\Omega)} \\
&\quad\ge \frac{1}{2} \inf_{x\in\Omega} \varphi(x,R) > K_0.
\end{aligned}$$

This is a contradiction, thus (2.12) holds. □

Lemma 2.2 (see [20], Remark 2.1) *Let V be a vector space of finite dimension and $A : V \to V'$ be a continuous mapping with*

$$\lim_{\|u\|_V \to +\infty} \frac{(A(u),u)}{\|u\|_V} = +\infty,$$

where V' is the dual space of V, then A is surjective.

Lemma 2.3 (see [21], Lemma 2.1) *If $u \in W^1L_\Phi(\Omega)$, then $u^+, u^- \in W^1L_\Phi(\Omega)$, and*

$$Du^+ = \begin{cases} Du, & \text{if } u > 0, \\ 0, & \text{if } u \le 0, \end{cases} \quad \text{and} \quad Du^- = \begin{cases} 0, & \text{if } u \ge 0, \\ -Du, & \text{if } u < 0. \end{cases}$$

Here $u^+ = \max\{u,0\}$, $u^- = -\min\{u,0\}$. This lemma holds in $W_0^1L_\Phi(\Omega)$ as well.

Lemma 2.4 (see [17]) *If a sequence $g_n \in L_{\overline{\Phi}}(\Omega)$ converges in measure to a measurable function g and if g_n remains bounded in $L_{\overline{\Phi}}(\Omega)$, then $g \in L_{\overline{\Phi}}(\Omega)$ and $g_n \to g$ for $\sigma(L_{\overline{\Phi}}(\Omega), E_\Phi(\Omega))$.*

The following propositions refer to Theorems 1.6-1.8 in [16], Theorem 4.2 in [22], and Theorem 2.1 in [18].

Proposition 2.1 *Let $\Phi \in N(\Omega)$ and*

$$\Phi_1(x,u) = a\Phi(x,bu) \quad (a,b > 0), \textit{ for all } u \ge 0, x \in \Omega. \tag{2.13}$$

Then $\Phi_1 \in N(\Omega)$ *and the complementary function* $\overline{\Phi_1}$ *to* Φ_1 *is given by*

$$\overline{\Phi_1}(x,v) = a\overline{\Phi}\left(x, \frac{v}{ab}\right), \quad \textit{for all } v \geq 0, x \in \Omega, \tag{2.14}$$

where $\overline{\Phi}$ *is the complementary function to* Φ.

Proof It is easy to see that $\Phi_1 \in N(\Omega)$. We only need to show (2.14). By (2.1) and (2.13), we can deduce that

$$\varphi_1(x,\tau) = ab\varphi(x, b\tau), \quad \text{for all } \tau \geq 0, x \in \Omega,$$

where φ and φ_1 are the right-hand derivatives of Φ and Φ_1, respectively.

From (2.3), $\phi_1(x,\sigma) = \frac{1}{b}\sup\{b\tau : \varphi(x,b\tau) \leq \frac{\sigma}{ab}\} = \frac{1}{b}\phi(x, \frac{\sigma}{ab})$, $\forall \sigma \geq 0$ and $x \in \Omega$.

For $\forall v \geq 0$, by (2.2), $\overline{\Phi}_1(x,v) = a\int_0^v \phi(x, \frac{\sigma}{ab})\,d\frac{\sigma}{ab}$, $\forall v \geq 0$ and $x \in \Omega$. Define $s = \frac{\sigma}{ab}$. Then $\overline{\Phi}_1(x,v) = a\int_0^{\frac{v}{ab}} \phi(x,s)\,ds = a\overline{\Phi}(x, \frac{v}{ab})$, $\forall v \geq 0$ and $x \in \Omega$. □

Proposition 2.2 *Let* $\Phi_1, \Phi_2 \in N(\Omega)$ *and*

$$\Phi_1(x,u) \leq \Phi_2(x,u) + h(x), \quad \textit{for some } h \in L^1(\Omega), \textit{ all } u \geq 0 \textit{ and } x \in \Omega. \tag{2.15}$$

Then

$$\overline{\Phi_2}(x,v) \leq \overline{\Phi_1}(x,v) + h(x), \quad \textit{for all } v \geq 0 \textit{ and } x \in \Omega,$$

where $\overline{\Phi_1}$ *and* $\overline{\Phi_2}$ *are the complementary functions to* Φ_1 *and* Φ_2, *respectively.*

Proof By (2.5) and (2.6), one has $\Phi_2(x,\phi_2(x,v)) + \overline{\Phi_2}(x,v) = \phi_2(x,v)\cdot v \leq \Phi_1(x,\phi_2(x,v)) + \overline{\Phi_1}(x,v)$, $\forall v \geq 0$ and $x \in \Omega$.

In view of (2.15), $\Phi_2(x,\phi_2(x,v)) + h(x) \geq \Phi_1(x,\phi_2(x,v))$, $\forall v \geq 0$ and $x \in \Omega$. Therefore, $\overline{\Phi_2}(x,v) \leq \overline{\Phi_1}(x,v) + h(x)$, $\forall v \geq 0$ and $x \in \Omega$. □

Proposition 2.3 *Let* $\Phi \in N(\Omega)$ *and its complementary function is* $\overline{\Phi}$. φ *and* ϕ *are given by* (2.1) *and* (2.2), *respectively. Then the following assertions are equivalent.*

(1) $\Phi \in \Delta_2$.

(2) $\forall l_1 > 1$, *there exist* $K' > 1$ *and a nonnegative function* $\tilde{h}_1 \in L^1(\Omega)$ *such that*

$$\Phi(x, l_1 u) \leq K'\Phi(x,u) + \tilde{h}_1(x), \quad \textit{for all } u \geq 0 \textit{ and a.e. } x \in \Omega.$$

(3) $\forall l_2 > 1$, *there exist* $\varepsilon \in (0,1)$ *and a nonnegative function* $\tilde{h}_2 \in L^1(\Omega)$ *such that*

$$\Phi\big(x,(1+\varepsilon)u\big) \leq l_2\Phi(x,u) + \tilde{h}_2(x), \quad \textit{for all } u \geq 0 \textit{ and a.e. } x \in \Omega.$$

(4) $\forall l_3 > 1$, *there exist* $\delta > 0$ *and a nonnegative function* $\tilde{h}_3 \in L^1(\Omega)$ *such that*

$$(l_3 + \delta)\overline{\Phi}(x,v) \leq \overline{\Phi}(x, l_3 v) + \tilde{h}_3(x), \quad \textit{for all } v \geq 0 \textit{ and a.e. } x \in \Omega.$$

(5) $\forall l_4 > 1$, *there exist* $l_0 > 1$ *and a nonnegative function* $\tilde{h}_4 \in L^1(\Omega)$ *such that*

$$\overline{\Phi}(x, v) \le \frac{1}{l_0 l_4} \overline{\Phi}(x, l_4 v) + \tilde{h}_4(x), \quad \text{for all } v \ge 0 \text{ and a.e. } x \in \Omega.$$

(6) *There exist* $l_5 > 1$ *and a nonnegative function* $\tilde{h}_5 \in L^1(\Omega)$ *such that*

$$\overline{\Phi}(x, v) \le \frac{1}{2l_5} \overline{\Phi}(x, l_5 v) + \tilde{h}_5(x), \quad \text{for all } v \ge 0 \text{ and a.e. } x \in \Omega.$$

(7) *There exist* $l_6 > 0$ *and a nonnegative function* $\tilde{h}_6 \in L^1(\Omega)$ *such that*

$$u\varphi(x, 2u) \le l_6 u\varphi(x, u) + \tilde{h}_6(x), \quad \text{for all } u \ge 0 \text{ and a.e. } x \in \Omega.$$

(8) $\forall m_1 > 1$, *there exist* $l_7 > 0$ *and a nonnegative function* $\tilde{h}_7 \in L^1(\Omega)$ *such that*

$$u\varphi(x, m_1 u) \le l_7 u\varphi(x, u) + \tilde{h}_7(x), \quad \text{for all } u \ge 0 \text{ and a.e. } x \in \Omega.$$

Proof (1)$\Rightarrow$(2). Since $\Phi \in \Delta_2$, by (2.7), there exist $K > 1$ and a nonnegative function $h \in L^1(\Omega)$ such that $\Phi(x, 2u) \le K\Phi(x, u) + h(x)$, $\forall u \ge 0$ and a.e. $x \in \Omega$. For every $l_1 > 1$, there exists $n \in \mathbb{N}$ such that $2^n \ge l_1$. Then

$$\begin{aligned}
\Phi(x, l_1 u) &\le \Phi\bigl(x, 2^n u\bigr) \le K\Phi\bigl(x, 2^{n-1}u\bigr) + h(x) \\
&\le K^2\Phi\bigl(x, 2^{n-2}u\bigr) + (K+1)h(x) \\
&\le \cdots \le K^n\Phi(x, u) + \bigl(K^{n-1} + \cdots + K + 1\bigr)h(x) \\
&= K^n\Phi(x, u) + \frac{K^n - 1}{K - 1} h(x),
\end{aligned}$$

$\forall u \ge 0$ and a.e. $x \in \Omega$. Taking $K' = K^n$ and $\tilde{h}_1 = \frac{K^n-1}{K-1} h(x)$, we can deduce the assertion (2).

(2)$\Rightarrow$(3). For every $l_2 > 1$, by the assertion (2), there exist $K' > l_2$ and a nonnegative function $\tilde{h}_1 \in L^1(\Omega)$ such that

$$\Phi(x, 2u) \le K'\Phi(x, u) + \tilde{h}_1(x), \quad \text{for all } u \ge 0 \text{ and a.e. } x \in \Omega.$$

Take $\varepsilon = \frac{l_2-1}{K'-1}$, then $\varepsilon \in (0, 1)$. Hence,

$$\begin{aligned}
\Phi\bigl(x, (1+\varepsilon)u\bigr) &= \Phi\bigl(x, (1-\varepsilon)u + 2\varepsilon u\bigr) \le (1-\varepsilon)\Phi(x, u) + \varepsilon\Phi(x, 2u) \\
&\le (1-\varepsilon)\Phi(x, u) + K'\varepsilon\Phi(x, u) + \varepsilon\tilde{h}_1(x) = l_2\Phi(x, u) + \varepsilon\tilde{h}_1(x),
\end{aligned}$$

for all $u \ge 0$ and a.e. $x \in \Omega$. Taking $\tilde{h}_2 = \varepsilon\tilde{h}_1$, we complete the assertion (3).

(3)$\Rightarrow$(4). By the assertion (3), $\forall l_3 > 1$, there exist $\varepsilon \in (0, 1)$ and a nonnegative function $\tilde{h}_2 \in L^1(\Omega)$ such that

$$\Phi\bigl(x, (1+\varepsilon)u\bigr) \le l_3\Phi(x, u) + \tilde{h}_2(x), \quad \text{for all } u \ge 0 \text{ and a.e. } x \in \Omega.$$

It implies that $\frac{1}{l_3}\Phi(x,(1+\varepsilon)u) \le \Phi(x,u) + \frac{1}{l_3}\tilde{h}_2(x)$. Denote $\Phi_1(x,u) = \frac{1}{l_3}\Phi(x,(1+\varepsilon)u)$. By Proposition 2.1, $\overline{\Phi_1}(x,v) = \frac{1}{l_3}\overline{\Phi}(x,\frac{l_3}{1+\varepsilon}v)$, $\forall v \ge 0$ and a.e. $x \in \Omega$. By Proposition 2.2, we get

$$\overline{\Phi_1}(x,v) \le \frac{1}{l_3}\overline{\Phi}\left(x,\frac{l_3}{1+\varepsilon}v\right) + \frac{1}{l_3}\tilde{h}_2(x) \le \frac{1}{l_3(1+\varepsilon)}\overline{\Phi}(x,l_3v) + \frac{1}{l_3}\tilde{h}_2(x),$$

$\forall v \ge 0$, and a.e. $x \in \Omega$. Thus, we have $l_3(1+\varepsilon)\overline{\Phi_1}(x,v) \le \overline{\Phi}(x,l_3v) + (1+\varepsilon)\tilde{h}_2(x)$. Taking $\delta = l_3\varepsilon$ and $\tilde{h}_3 = (1+\varepsilon)\tilde{h}_2$, we complete the assertion (4).

(4)$\Rightarrow$(5). By the assertion (4), $\forall l_4 > 1$, there exist $\delta > 0$ and a nonnegative function $\tilde{h}_3 \in L^1(\Omega)$ such that

$$(l_4+\delta)\overline{\Phi}(x,v) \le \overline{\Phi}(x,l_4v) + \tilde{h}_3(x), \quad \forall v \ge 0 \text{ and a.e. } x \in \Omega.$$

Hence, $\overline{\Phi}(x,v) \le \frac{1}{l_4(1+\frac{\delta}{l_4})}\overline{\Phi}(x,l_4v) + \frac{1}{l_4(1+\frac{\delta}{l_4})}\tilde{h}_3(x)$. Taking $l_0 = 1 + \frac{\delta}{l_4}$ and $\tilde{h}_4 = \frac{1}{l_4(1+\frac{\delta}{l_4})}\tilde{h}_3$, we complete the assertion (5).

(5)$\Rightarrow$(1). By the assertion (5), $\forall l_4 > 1$, there exist $l_0 > 1$ and a nonnegative function $\tilde{h}_4 \in L^1(\Omega)$ such that

$$\overline{\Phi}(x,v) \le \frac{1}{l_0l_4}\overline{\Phi}(x,l_4v) + \tilde{h}_4(x), \quad \forall v \ge 0 \text{ and a.e. } x \in \Omega.$$

By Proposition 2.1 and Proposition 2.2, we obtain $\Phi(x,l_0u) \le l_0l_4\Phi(x,u) + l_0l_4\tilde{h}_4(x)$, $\forall u \ge 0$ and a.e. $x \in \Omega$. Take $n_0 \in \mathbb{N}$ such that $l_0^{n_0} \ge 2$. Then $\Phi(x,2u) \le \Phi(x,l_0^{n_0}u) \le l_0^{n_0}l_4^{n_0}\Phi(x,u) + \frac{l_0^{n_0}l_4^{n_0}-1}{l_0l_4-1}\tilde{h}_4(x)$. Denote $l_0^{n_0}l_4^{n_0} = K$ and $\frac{l_0^{n_0}l_4^{n_0}-1}{l_0l_4-1}\tilde{h}_4 = h$. We deduce (2.7), *i.e.* $\Phi \in \Delta_2$.

(6)$\Rightarrow$(1). Define $\Psi_1(x,v) = \frac{1}{2l_5}\overline{\Phi}(x,l_5v)$. By Proposition 2.1, $\overline{\Psi_1}(x,u) = \frac{1}{2l_5}\Phi(x,2u)$, $\forall u \ge 0$ and a.e. $x \in \Omega$. By Proposition 2.2, $\Phi(x,2u) \le 2l_5\Phi(x,u) + 2l_5\tilde{h}_5(x)$, $\forall u \ge 0$ and a.e. $x \in \Omega$. Therefore, $\Phi \in \Delta_2$.

Similarly, (1) implies (6).

(1)$\Rightarrow$(7). By (2), there exist $K' > 0$ and $\tilde{h}_1 \in L^1(\Omega)$ such that

$$\Phi(x,4u) \le K'\Phi(x,u) + \tilde{h}_1(x), \quad \text{for all } u \ge 0 \text{ and a.e. } x \in \Omega.$$

On the other hand, we have $2u\varphi(x,2u) \le \Phi(x,4u)$ and $\Phi(x,u) \le u\varphi(x,u)$, for $x \in \Omega$, $u \ge 0$. Hence,

$$u\varphi(x,2u) \le \frac{K'}{2}u\varphi(x,u) + \frac{1}{2}\tilde{h}_1(x), \quad \text{for all } u \ge 0 \text{ and a.e. } x \in \Omega.$$

Consequently, the assertion (7) holds by taking $l_6 = \frac{K'}{2}$ and $\tilde{h}_6 = \frac{1}{2}\tilde{h}_1$.

(7)$\Rightarrow$(8). For every $m_1 > 1$, there is $n_0 \in \mathbb{N}^+$ such that $2^{n_0} \ge m_1$. Then $u\varphi(x,m_1u) \le u\varphi(x,2^{n_0}u) \le l_6^{n_0}u\varphi(x,u) + \frac{l_6^{n_0}-1}{l_6-1}\tilde{h}_6(x)$, $\forall u \ge 0$ and a.e. $x \in \Omega$. Taking $l_7 = l_6^{n_0}$ and $\tilde{h}_7 = \frac{l_6^{n_0}-1}{l_6-1}\tilde{h}_6$, we complete (8).

(8)$\Rightarrow$(1). For every $l_1 > 1$, we have $\Phi(x,l_1u) \le l_1u\varphi(x,l_1u)$. By (8), there exist $l_7 > 0$ and $\tilde{h}_7 \in L^1(\Omega)$ such that

$$u\varphi(x,l_1u) \le l_7u\varphi\left(x,\frac{u}{2}\right) + \tilde{h}_7(x), \quad \text{for all } u \ge 0 \text{ and a.e. } x \in \Omega.$$

It follows that $\Phi(x, l_1 u) \leq l_1 l_7 u\varphi(x, \frac{u}{2}) + l_1\tilde{h}_7(x) \leq 2l_1 l_7\Phi(x, u) + l_1\tilde{h}_7(x)$, for all $u \geq 0$ and a.e. $x \in \Omega$. Taking $K' = 2l_1 l_7$ and $\tilde{h}_1 = l_1\tilde{h}_7$, we deduce (2). Immediately, (1) holds. □

Example 2.1 Let $\Phi(x, |t|) = (1 + \frac{|t|}{p(x)})\ln(1 + \frac{|t|}{p(x)}) - \frac{|t|}{p(x)}$, for $x \in \Omega$ and $t \in \mathbb{R}$, where $p : \Omega \to \mathbb{R}$ is a measurable function such that $1 < p_- \leq p(x) \leq p_+ < +\infty$. Then $\varphi(x, |t|) = \frac{1}{p(x)}\ln(1 + \frac{|t|}{p(x)})$, $\phi(x, |s|) = p(x)(\exp(p(x)|s|) - 1)$ and $\overline{\Phi}(x, |s|) = \exp(p(x)|s|) - p(x)|s| - 1$. It follows that $\Phi \in N(\Omega)$ and $\Phi \in \Delta_2$. But $\overline{\Phi} \notin \Delta_2$. Moreover, both Φ and $\overline{\Phi}$ are locally integrable. Therefore, $L_\Phi(\Omega)$ is separable, but $L_\Phi(\Omega)$ is not reflexive.

Remark 2.1 Let $\Phi(x, |t|) = \exp(p(x)|t|) - 1$, for $x \in \Omega$ and $t \in \mathbb{R}$, where $p : \Omega \to \mathbb{R}$ is a measurable function such that $1 < p_- \leq p(x) \leq p_+ < +\infty$. It is worth noting that Φ does not satisfy the condition $\lim_{u\to 0^+}\sup_{x\in\Omega}\frac{\Phi(x,u)}{u} = 0$. Therefore, $\Phi \notin N(\Omega)$.

Clearly, by (2.9), Proposition 2.1 and Proposition 2.2, we can deduce the following proposition.

Proposition 2.4 *If* $\Phi \preceq \Psi$, *then* $\overline{\Psi} \preceq \overline{\Phi}$.

3 Existence theorems

Let $\Phi \in N(\Omega)$, and satisfy the condition

(Φ) $\Phi \in \Delta_2$, $\overline{\Phi}$ is a complementary function to Φ, both Φ and $\overline{\Phi}$ are locally integrable and satisfy (Φ_2).

We assume that there exists $\Psi \in N(\Omega)$ satisfying the condition

(Ψ) $\Psi \in \Delta_2$, $\overline{\Psi}$ is a complementary function to Ψ, both Ψ and $\overline{\Psi}$ are locally integrable and satisfy (Φ_2), $\Phi \preceq \Psi$, and the embedding $W^1L_\Phi(\Omega) \hookrightarrow L_\Psi(\Omega)$ is compact.

Note that, in this case, the spaces $L_\Phi(\Omega)$, $L_\Psi(\Omega)$, $W^1L_\Phi(\Omega)$, $W^1_0L_\Phi(\Omega)$ are separable Banach spaces.

For $u, v \in L^0(\Omega)$, we denote $u \wedge v = \min\{u, v\}$, $u \vee v = \max\{u, v\}$, $u^+ := u \vee 0$, $u^- := -u \wedge 0$, $u \leq v \Leftrightarrow u(x) \leq v(x)$ for a.e. $x \in \Omega$.

Let $a_1 : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function satisfying the following conditions:

(A_1) For a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^N$,

$$\bigl|a_1(x, \xi)\bigr| \leq b_1\overline{\Phi}^{-1}\bigl(x, \Phi\bigl(x, |\xi|\bigr)\bigr) + g_1(x), \tag{3.1}$$

$$a_1(x, \xi)\xi \geq b_2\Phi\bigl(x, |\xi|\bigr) - g_2(x), \tag{3.2}$$

$$\bigl[a_1(x, \xi) - a_1(x, \eta)\bigr](\xi - \eta) > 0, \quad \xi \neq \eta, \tag{3.3}$$

where $b_1, b_2 > 0$, $g_1 \in E_{\overline{\Phi}}(\Omega)$, $g_1 \geq 0$, $g_2 \in L^1(\Omega)$, and $g_2 \geq 0$.

Let $a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following conditions:

(A_0) For a.e. $x \in \Omega$ and all $t \in \mathbb{R}$,

$$\bigl|a_0(x, t)\bigr| \leq b_1\overline{\Phi}^{-1}\bigl(x, \Phi\bigl(x, |t|\bigr)\bigr) + g_1(x), \tag{3.4}$$

where $b_1 > 0$, $g_1 \in E_{\overline{\Phi}}(\Omega)$, and $g_1 \geq 0$.

Example 3.1

(1) Let $\Phi(x,|t|) = \frac{1}{p(x)}|t|^{p(x)}$, $a_1(x,\xi) = |\xi|^{p(x)-2}\xi$, for $x \in \Omega$ and $t \in \mathbb{R}$, where $p : \Omega \to \mathbb{R}$ is a measurable function such that $2 \le p_- \le p(x) \le p_+ < +\infty$. Then Φ satisfies (Φ) and we get the $p(x)$-Laplace operator $\operatorname{div}(|Du|^{p(x)-2}Du)$.

(2) Let $\Phi(x,|t|) = \frac{1}{p(x)}[(1+|t|^2)^{p(x)/2} - 1]$, $a_1(x,\xi) = (1+|\xi|^2)^{(p(x)-2)/2}\xi$, for $x \in \Omega$ and $t \in \mathbb{R}$, where $p : \Omega \to \mathbb{R}$ is a measurable function such that $2 \le p_- \le p(x) \le p_+ < +\infty$. Then Φ satisfies (Φ) and we obtain the generalized mean curvature operator $\operatorname{div}((1+|Du|^2)^{(p(x)-2)/2}Du)$. Moreover, by Proposition 2.3(6), $\overline{\Phi} \in \Delta_2$.

(3) Let $\Phi(x,|t|) = (1+\frac{|t|}{p(x)})\ln(1+\frac{|t|}{p(x)}) - \frac{|t|}{p(x)}$, for $x \in \Omega$ and $t \in \mathbb{R}$, where $p : \Omega \to \mathbb{R}$ is a measurable function such that $1 < p_- \le p(x) \le p_+ < +\infty$. Clearly, it can be verified that Φ satisfies (Φ). Put $a_1(x,\xi) = \varphi(x,|\xi|)\frac{\xi}{|\xi|}$, and $a_0(x,t) = \varphi(x,|t|)$, for $x \in \Omega$, $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$, where $\varphi(x,|t|) = \frac{1}{p(x)}\ln(1+\frac{|t|}{p(x)})$. Then a_1 and a_0 satisfy (A_1) and (A_0), respectively.

Remark 3.1 Clearly, the condition (1.2) (resp. (1.5)) implies (3.1) (resp. (3.4)).

Consider the following Dirichlet boundary value problem:

$$\begin{cases} -\operatorname{div}(a_1(x,Du)) + a_0(x,u) = f(x,u,Du), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function. Denote by F the Nemytskii operator associated to f, that is,

$$F(u)(x) = f\bigl(x,u(x),Du(x)\bigr), \quad \text{for } x \in \Omega.$$

A function u is called a (weak) solution of (3.5) if $u \in W_0^1L_\Phi(\Omega)$, $F(u) \in L_{\overline{\Psi}}(\Omega)$ and u satisfies the equation

$$\int_\Omega a_1(x,Du)Dv\,dx + \int_\Omega a_0(x,u)v\,dx = \int_\Omega f(x,u,Du)v\,dx, \quad \text{for all } v \in W_0^1L_\Phi(\Omega). \tag{3.6}$$

A function u is called a subsolution (resp. supersolution) of (3.5) if $u \in W_0^1L_\Phi(\Omega)$, $F(u) \in L_{\overline{\Psi}}(\Omega)$ and (3.6) holds with '=' replaced by '$\le$' (resp. '$\ge$') for every nonnegative functions v in $W_0^1L_\Phi(\Omega)$ (see [2]).

Theorem 3.1 *Suppose that* $\underline{u}_1,\dots,\underline{u}_k$ *and* $\overline{u}_1,\dots,\overline{u}_m$ *are subsolutions and supersolutions of* (3.5), *respectively, that satisfy*

$$\underline{u} := \underline{u}_1 \vee \underline{u}_2 \vee \cdots \vee \underline{u}_k \le \overline{u}_1 \wedge \overline{u}_2 \wedge \cdots \wedge \overline{u}_m := \overline{u}.$$

Let (Φ), (Ψ), (A_1), (A_0) *hold. Assume the nonlinear term* g *satisfies the following local growth condition*:

$$\bigl|f(x,t,\xi)\bigr| \le q(x) + b_3\overline{\Phi}^{-1}\bigl(x,\Phi\bigl(x,|t|\bigr)\bigr) + b_4\overline{\Psi}^{-1}\bigl(x,\Phi\bigl(x,|\xi|\bigr)\bigr), \tag{3.7}$$

for a.e. $x \in \Omega$ *and* $\forall t \in [\underline{u}(x),\overline{u}(x)]$, *with* $q \in E_{\overline{\Psi}}(\Omega)$, $b_3, b_4 > 0$. *Then there exists a solution* u *of* (3.5) *such that* $\underline{u} \le u \le \overline{u}$.

Proof Denote $V = W_0^1 L_\Phi(\Omega)$. For $x \in \Omega$, we put

$$Tu(x) = \begin{cases} \overline{u}(x), & \text{if } u(x) > \overline{u}(x), \\ u(x), & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x), \\ \underline{u}(x), & \text{if } u(x) < \underline{u}(x), \end{cases} \quad \text{for } u \in V.$$

Then $Tu = u \vee \underline{u} + u \wedge \overline{u} - u$. By Remark 3.1 in [2], $T : V \to V$ is continuous. It is easy to see that T is bounded. From Proposition 2.4, we obtain $F(Tu) \in L_{\overline{\Psi}}(\Omega)$, $\forall u \in V$.

We define the cutoff function $l : \Omega \times \mathbb{R} \to \mathbb{R}$ given by

$$l(x,s) = \begin{cases} \overline{\Phi}^{-1}(x, \Phi(x, s - \overline{u}(x))), & \text{if } s > \overline{u}(x), \\ 0, & \text{if } \underline{u}(x) \le s \le \overline{u}(x), \\ -\overline{\Phi}^{-1}(x, \Phi(x, \underline{u}(x) - s)), & \text{if } s < \underline{u}(x), \end{cases}$$

for $x \in \Omega$, $s \in \mathbb{R}$. Then l satisfies the following condition:

$$\bigl|l(x,s)\bigr| \le \overline{\Phi}^{-1}\bigl(x, \Phi\bigl(x, 2|s|\bigr)\bigr) + \overline{\Phi}^{-1}\bigl(x, \Phi\bigl(x, 2\bigl|\overline{u}(x)\bigr|\bigr)\bigr) + \overline{\Phi}^{-1}\bigl(x, \Phi\bigl(x, 2\bigl|\underline{u}(x)\bigr|\bigr)\bigr), \tag{3.8}$$

for $x \in \Omega$ and all $s \in \mathbb{R}$.

For all $u \in V$, since $\Phi \in \Delta_2$, there exists $K_1 > 1$ such that

$$\begin{aligned}
&\int_\Omega l(x,u)u\,dx \\
&\quad = \int_{\{u>\overline{u}\}} \overline{\Phi}^{-1}\bigl(x, \Phi(x, u - \overline{u})\bigr)(u - \overline{u})\,dx \\
&\qquad + \int_{\{u>\overline{u}\}} \overline{\Phi}^{-1}\bigl(x, \Phi(x, u - \overline{u})\bigr)\overline{u}\,dx \\
&\qquad + \int_{\{u<\underline{u}\}} \overline{\Phi}^{-1}\bigl(x, \Phi(x, \underline{u} - u)\bigr)(\underline{u} - u)\,dx \\
&\qquad - \int_{\{u<\underline{u}\}} \overline{\Phi}^{-1}\bigl(x, \Phi(x, \underline{u} - u)\bigr)\underline{u}\,dx \\
&\quad \ge \int_{\{u>\overline{u}\}} \Phi(x, u - \overline{u})\,dx - \int_{\{u>\overline{u}\}} \biggl[\frac{1}{2}\Phi(x, u - \overline{u}) + \Phi\bigl(x, 2|\overline{u}|\bigr)\biggr]\,dx \\
&\qquad + \int_{\{u<\underline{u}\}} \Phi(x, \underline{u} - u)\,dx - \int_{\{u<\underline{u}\}} \biggl[\frac{1}{2}\Phi(x, \underline{u} - u) + \Phi\bigl(x, 2|\underline{u}|\bigr)\biggr]\,dx \\
&\quad = \frac{1}{2}\int_{\{u>\overline{u}\}} \Phi(x, u - \overline{u})\,dx - \int_{\{u>\overline{u}\}} \Phi\bigl(x, 2|\overline{u}|\bigr)\,dx \\
&\qquad + \frac{1}{2}\int_{\{u<\underline{u}\}} \Phi(x, \underline{u} - u)\,dx - \int_{\{u<\underline{u}\}} \Phi\bigl(x, 2|\underline{u}|\bigr)\,dx \\
&\quad \ge \frac{1}{2}\int_{\{u>\overline{u}\}} \biggl[2\Phi\biggl(x, \frac{|u|}{2}\biggr) - \Phi\bigl(x, |\overline{u}|\bigr)\biggr]\,dx - \int_\Omega \Phi\bigl(x, 2|\overline{u}|\bigr)\,dx \\
&\qquad + \frac{1}{2}\int_{\{u<\underline{u}\}} \biggl[2\Phi\biggl(x, \frac{|u|}{2}\biggr) - \Phi\bigl(x, |\underline{u}|\bigr)\biggr]\,dx \\
&\qquad - \int_\Omega \Phi\bigl(x, 2|\underline{u}|\bigr)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\geq \int_{\{u>\overline{u}\}\cup\{u<\underline{u}\}} \Phi\left(x, \frac{|u|}{2}\right) dx - C \\
&\quad + \int_{\{\overline{u}\leq u\leq \underline{u}\}} \left[\Phi\left(x, \frac{|u|}{2}\right) - \Phi\left(x, \frac{|\overline{u}| \vee |\underline{u}|}{2}\right)\right] dx \\
&= \int_\Omega \Phi\left(x, \frac{|u|}{2}\right) dx - C \\
&\geq \frac{1}{K_1} \int_\Omega \Phi\bigl(x, |u|\bigr)\, dx - C,
\end{aligned}
\tag{3.9}
$$

for some constant $C > 0$ independent of u, where $\{u < \underline{u}\} = \{x \in \Omega : u(x) < \underline{u}(x)\}$, $\{u > \overline{u}\} = \{x \in \Omega : u(x) > \overline{u}(x)\}$, and $\{\underline{u} \leq u \leq \overline{u}\} = \{x \in \Omega : \underline{u}(x) \leq u(x) \leq \overline{u}(x)\}$.

Let us consider the auxiliary equation of finding $u \in V$ such that

$$
\begin{aligned}
&\int_\Omega a_1(x, Du)Dv\, dx + \int_\Omega a_0(x, Tu)v\, dx + \lambda \int_\Omega l(x, u)v\, dx \\
&\quad = \int_\Omega F(Tu)v\, dx, \quad \forall v \in V,
\end{aligned}
\tag{3.10}
$$

where $\lambda > 0$ is a parameter to be specified later.

Define $\Gamma_T : V \to V^*$,

$$
(\Gamma_T u, v) := \int_\Omega a_1(x, Du)Dv\, dx + \int_\Omega a_0(x, Tu)v\, dx + \lambda \int_\Omega l(x, u)v\, dx - \int_\Omega F(Tu)v\, dx,
$$

$\forall v \in V$. Then Γ_T is well defined.

Since $\Phi \in \Delta_2$, there exists a sequence $\{w_n\} \subset V$ such that $\{w_n\}$ is dense in V. Let $V_m = \operatorname{span}\{w_1, \ldots, w_m\}$ and consider $\Gamma_T|_{V_m}$. For every $u \in V_m$, $\|Du\|_\Phi$ and $\int_\Omega |Du|\, dx$ are two norms of V_m equivalent to the usual norm of finite dimensional vector spaces.

Similar to the proof of Proposition 3.1 in [20], we can deduce that the mapping $u \to \Gamma_T|_{V_m} u : V_m \to V_m^*$ is continuous.

In view of (3.7), one has

$$
\begin{aligned}
&\left|\int_\Omega F(Tu)u\, dx\right| \\
&\quad \leq C^* \|q\|_{\overline{\Psi}} \|u\|_{1,\Phi} + 2b_3 \int_\Omega \Phi\bigl(x, |u|\bigr)\, dx + b_3 \int_\Omega \Phi\bigl(x, |\overline{u}|\bigr)\, dx + b_3 \int_\Omega \Phi\bigl(x, |\underline{u}|\bigr)\, dx \\
&\qquad + b_4 \varepsilon_1 \int_\Omega \Phi\bigl(x, |Du|\bigr)\, dx + b_4 \int_\Omega \varepsilon_1 \Psi\left(x, \frac{1}{\varepsilon_1} |\overline{u}| \vee |\underline{u}|\right) dx + b_4 \int_\Omega \Psi\bigl(x, |\overline{u}|\bigr)\, dx \\
&\qquad + b_4 \int_\Omega \Psi\bigl(x, |\underline{u}|\bigr)\, dx + b_4 \int_\Omega \Phi\bigl(x, |D\overline{u}|\bigr)\, dx + b_4 \int_\Omega \Phi\bigl(x, |D\underline{u}|\bigr)\, dx,
\end{aligned}
\tag{3.11}
$$

for all $u \in V$, where $\varepsilon_1 = \frac{b_2}{2b_4}$ and the constant $C^* > 0$.

Thanks to (3.4) and (2.8), there exist $K_2 > 1$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$
\begin{aligned}
&\left|\int_\Omega a_0(x, Tu)u\, dx\right| \\
&\quad \leq b_1(K_2 - 1) \int_\Omega \bigl[\Phi\bigl(x, |u|\bigr) + \Phi\bigl(x, |\underline{u}|\bigr) + \Phi\bigl(x, |\overline{u}|\bigr)\bigr]\, dx + b_1 \int_\Omega h(x)\, dx
\end{aligned}
$$

$$
\begin{aligned}
&+ (b_1+1)\int_\Omega \Phi\bigl(x,|u|\bigr)\,dx + \int_\Omega \overline{\Phi}\bigl(x,\bigl|g_1(x)\bigr|\bigr)\,dx \\
&= (b_1K_2+1)\int_\Omega \Phi\bigl(x,|u|\bigr)\,dx + C,
\end{aligned}
\tag{3.12}
$$

for all $u \in V$, where the constant $C > 0$ is independent of u.

Let $\lambda > K_1(b_1K_2 + 1 + 2b_3)$. Combining (3.2), (3.9), (3.11), and (3.12), we obtain

$$
\begin{aligned}
(\Gamma_T u, u) &\geq \frac{b_2}{2}\int_\Omega \Phi\bigl(x,|Du|\bigr)\,dx + \left(\frac{\lambda}{K_1} - b_1K_2 - 1 - 2b_3\right)\int_\Omega \Phi\bigl(x,|u|\bigr)\,dx \\
&\quad - C - C^*\|q\|_{\overline{\Psi}}\|u\|_{1,\Phi} \\
&\geq \frac{b_2}{2}\int_\Omega \Phi\bigl(x,|Du|\bigr)\,dx - C - C^*\|q\|_{\overline{\Psi}}\|u\|_{1,\Phi},
\end{aligned}
\tag{3.13}
$$

for all $u \in V$ and some $C > 0$ independent of u. By Proposition 1.9 in [2], there exists $C_1 > 0$ such that $\|u\|_\Phi \leq C_1\|Du\|_\Phi$. In view of (3.13), for all $u \in V_m$, we have

$$
\begin{aligned}
\frac{(\Gamma_T|_{V_m}u,u)}{\|u\|_{1,\Phi}} &\geq \frac{b_2\int_\Omega \Phi(x,|Du|)\,dx}{2(1+C_1)\|Du\|_\Phi} - \frac{C}{\|u\|_{1,\Phi}} - C^*\|q\|_{\overline{\Psi}} \\
&\geq \frac{b_2\int_\Omega \Phi(x,|Du|)\,dx}{2C_2(1+C_1)\int_\Omega |Du|\,dx} - \frac{C}{\|u\|_{1,\Phi}} - C^*\|q\|_{\overline{\Psi}},
\end{aligned}
$$

for some constant $C_2 > 0$. By Lemma 2.1, we get

$$
\frac{(\Gamma_T|_{V_m}u,u)}{\|u\|_{1,\Phi}} \to +\infty, \quad \text{as } \|u\|_{1,\Phi} \to +\infty.
\tag{3.14}
$$

By Lemma 2.2, there exists a Galerkin solution $u_m \in V_m$ for every $m \in \mathbb{N}$ such that $(\Gamma_T u_m, v) = 0$, $v \in V_m$. Using the density of $\{w_m\}$, we deduce that

$$
(\Gamma_T u_m, v) = 0, \quad \forall v \in V.
\tag{3.15}
$$

For $u \in V$, define $\rho(u) = \int_\Omega(\Phi(x,|Du|) + \Phi(x,|u|))\,dx$ and $\|u\|_\rho = \inf\{\lambda > 0 : \rho(\frac{u}{\lambda}) \leq 1\}$. Then $\|u\|_\rho$ is a norm of V equivalent to $\|u\|_{1,\Phi}$ (see [2]).

Taking $\alpha_0 = \min\{\frac{b_2}{2}, \frac{\lambda}{K_1} - b_1K_2 - 1 - 2b_3\}$, we have

$$
\begin{aligned}
(\Gamma_T u, u) &\geq \alpha_0\left[\int_\Omega \Phi\bigl(x,|Du|\bigr)\,dx + \int_\Omega \Phi\bigl(x,|u|\bigr)\,dx\right] - C - C^*\|q\|_{\overline{\Psi}}\|u\|_{1,\Phi} \\
&\geq \alpha_0\bigl(\|u\|_\rho - \varepsilon\bigr)\left[\int_\Omega \Phi\left(x,\frac{|Du|}{\|u\|_\rho - \varepsilon}\right)dx + \int_\Omega \Phi\left(x,\frac{|u|}{\|u\|_\rho - \varepsilon}\right)dx\right] \\
&\quad - C - C^*\|q\|_{\overline{\Psi}}\|u\|_{1,\Phi} \\
&\geq \alpha_0\bigl(\|u\|_\rho - \varepsilon\bigr) - C - C^*\|q\|_{\overline{\Psi}}\|u\|_{1,\Phi},
\end{aligned}
$$

for all $u \in V$, as $\|u\|_{1,\Phi}$ is large enough. Therefore, by (3.15), we get a sequence $\{u_m\}$ that is bounded in V. Hence, there exist $u_0 \in V$ and a subsequence $\{u_k\}$ of $\{u_m\}$, such that

$$u_k \rightharpoonup u_0 \quad \text{weakly in } V \text{ for } \sigma\Big(\prod L_\Phi, \prod E_{\overline{\Phi}}\Big), \tag{3.16}$$

$$u_k \to u_0 \quad \text{strongly in } L_\Psi(\Omega), \tag{3.17}$$

$$u_k \to u_0 \quad \text{a.e. in } \Omega, \tag{3.18}$$

as $k \to \infty$.

By (3.4) and (3.8), $\{a_0(x, Tu_k)\}$ and $\{l(x, u_k)\}$ are bounded in $L_{\overline{\Phi}}(\Omega)$. By Lemma 2.4,

$$a_0(x, Tu_k) \rightharpoonup a_0(x, Tu_0) \quad \text{weakly in } L_{\overline{\Phi}}(\Omega) \text{ for } \sigma(L_{\overline{\Phi}}, E_\Phi)$$

and

$$l(x, u_k) \rightharpoonup l(x, u_0) \quad \text{weakly in } L_{\overline{\Phi}}(\Omega) \text{ for } \sigma(L_{\overline{\Phi}}, E_\Phi),$$

as $k \to \infty$.

On the other hand, by the Lebesgue theorem, we deduce that

$$\int_\Omega a_0(x, Tu_k)(u_k - u_0)\,dx \to 0, \qquad \int_\Omega l(x, u_k)(u_k - u_0)\,dx \to 0, \quad \text{as } k \to \infty.$$

Thanks to (3.7), $\{F(Tu_k)\}$ is bounded in $L_{\overline{\Psi}}(\Omega)$. Hence,

$$\int_\Omega F(Tu_k)(u_k - u_0)\,dx \to 0, \quad \text{as } k \to \infty.$$

Thus we obtain

$$\int_\Omega a_1(x, Du_k)(Du_k - Du_0)\,dx \to 0, \quad \text{as } k \to \infty. \tag{3.19}$$

Similar to the proof of Proposition 3.1 in [20], we can construct a subsequence still denoted by $\{u_k\}$ such that

$$Du_k \to Du_0 \quad \text{a.e. in } \Omega, \text{ as } k \to \infty. \tag{3.20}$$

Hence

$$a_1(x, Du_k) \to a_1(x, Du_0) \quad \text{a.e. in } \Omega, \text{ as } k \to \infty. \tag{3.21}$$

In view of (3.1), $\{a_1(x, Du_k)\}$ is bounded in $(L_{\overline{\Phi}}(\Omega))^N$, then by Lemma 2.4, we have

$$a_1(x, Du_k) \to a_1(x, Du_0) \quad \text{weakly in } \big(L_{\overline{\Phi}}(\Omega)\big)^N \text{ for } \sigma\Big(\big(L_{\overline{\Phi}}(\Omega)\big)^N, \big(E_\Phi(\Omega)\big)^N\Big), \tag{3.22}$$

as $k \to \infty$. Similarly,

$$F(Tu_k) \rightharpoonup F(Tu_0) \quad \text{weakly in } L_{\overline{\Psi}}(\Omega) \text{ for } \sigma(L_{\overline{\Psi}}, E_\Psi), \text{as } k \to \infty.$$

Hence, $(\Gamma_T u_k, v) = (\Gamma_T u_0, v)$, $\forall v \in V$. By (3.15), $(\Gamma_T u_0, v) = 0$, $\forall v \in V$, *i.e.*, u_0 is a solution of (3.10).

For every $m \in \mathbb{N}$, taking $v = (u_m - \overline{u})^+ \in V$ in (3.15) as a test function, we get

$$\begin{aligned}
&\int_\Omega \bigl[a_1(x, Du_m) - a_1(x, D\overline{u})\bigr] D(u_m - \overline{u})^+ \, dx \\
&\quad + \int_\Omega \bigl[a_0(x, Tu_m) - a_0(x, \overline{u})\bigr](u_m - \overline{u})^+ \, dx + \lambda \int_\Omega l(x, u_m)(u_m - \overline{u})^+ \, dx \\
&\quad \le \int_\Omega \bigl[F(Tu_m) - F(\overline{u})\bigr](u_m - \overline{u})^+ \, dx. \qquad (3.23)
\end{aligned}$$

By (3.3), we have

$$\begin{aligned}
&\int_\Omega \bigl[a_1(x, Du_m) - a_1(x, D\overline{u})\bigr] D(u_m - \overline{u})^+ \, dx \\
&\quad = \int_{\{u_m > \overline{u}\}} \bigl[a_1(x, Du_m) - a_1(x, D\overline{u})\bigr] D(u_m - \overline{u}) \, dx \ge 0.
\end{aligned}$$

Since

$$\int_\Omega \bigl[a_0(x, Tu_m) - a_0(x, \overline{u})\bigr](u_m - \overline{u})^+ \, dx = 0$$

and

$$\int_\Omega \bigl[F(Tu_m) - F(\overline{u})\bigr](u_m - \overline{u})^+ \, dx = 0,$$

we get

$$0 \ge \int_\Omega l(x, u_m)(u_m - \overline{u})^+ \, dx \ge \int_{\{u_m > \overline{u}\}} \Phi(x, u_m - \overline{u}) \, dx \ge 0.$$

It follows that $u_m \le \overline{u}$. Using arguments similar to those above we can prove that $u_m \ge \underline{u}$.

Thanks to (3.18), one has $\underline{u} \le u_0 \le \overline{u}$. From the definitions of $l(\cdot, u_0(\cdot))$ and Tu_0, we have

$$l\bigl(x, u_0(x)\bigr) = 0, \qquad a_0\bigl(x, Tu_0(x)\bigr) = a_0\bigl(x, u_0(x)\bigr)$$

and

$$f\bigl(x, Tu_0(x), DTu_0(x)\bigr) = f\bigl(x, u_0(x), Du_0(x)\bigr),$$

for a.e. $x \in \Omega$. We note that then (3.10) reduces to (3.6), which completes the proof. □

Remark 3.2 Our proof does not need the conditions $\overline{\Phi} \in \Delta_2$ and (Φ_3) in [2].

Remark 3.3 Our method needs the strict monotonicity (3.3) of a_1, but does not require monotonicity (1.7) or coercivity (1.6) of a_0. However, if $\overline{\Phi} \in \Delta_2$, then we can deduce (3.22) by following the lines of Theorem 4.1 in [23] when (3.3) is replaced by (1.4).

Remark 3.4 Assume that (1.7) holds and the assumptions of Theorem 3.1 hold. If $f(x,u,Du) = f(x) \in L_{\overline{\Psi}}(\Omega)$, then it is easy to see that (3.5) has a unique solution.

Remark 3.5 Now we consider the following Neumann boundary value problem:

$$\begin{cases} -\operatorname{div}(a_1(x,Du)) + a_0(x,u) = f(x,u,Du), & \text{in } \Omega, \\ a_1(x,Du)\cdot\gamma = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.24}$$

where γ is the outward unit normal to $\partial\Omega$.

We also assume that there is a function $G : [k,+\infty) \to \mathbb{R}$ for some $k > 0$ such that $G(s) \to +\infty$ as $s \to +\infty$ and

$$\Phi(x,su) \geq G(s)s\Phi(x,u) - sh(x), \quad \text{for all } s > 0, u \geq 0, \text{ a.e. } x \in \Omega, \tag{3.25}$$

and some $h \in L^1(\Omega)$, $h \geq 0$.

Assume that (3.25) holds and the assumptions of Theorem 3.1 hold. Replacing V by $W^1L_\Phi(\Omega)$ in the proof of Theorem 3.1, and (3.13)-(3.14) by the following lines, we can deduce a similar theorem to Theorem 3.1 for the Neumann boundary value problem (3.24).

$$\begin{aligned} (\Gamma_T u,u) &\geq \frac{b_2}{2}\int_\Omega \Phi\bigl(x,|Du|\bigr)\,dx + \left(\frac{\lambda}{K_1} - b_1K_2 - 1 - 2b_3\right)\int_\Omega \Phi\bigl(x,|u|\bigr)\,dx \\ &\quad - C - C^*\|q\|_{\overline{\Psi}}\|u\|_{1,\Phi} \\ &\geq \alpha_0\left[\int_\Omega \Phi\bigl(x,|Du|\bigr)\,dx + \int_\Omega \Phi\bigl(x,|u|\bigr)\,dx\right] - C - C^*\|q\|_{\overline{\Psi}}\|u\|_{1,\Phi}, \end{aligned} \tag{3.26}$$

for all $u \in V$ and some $C > 0$ independent of u, where $\alpha_0 = \min\{\frac{b_2}{2}, \frac{\lambda}{K_1} - b_1K_2 - 1 - 2b_3\}$.

Combining (3.25) and (3.26), we can deduce that, for any $\varepsilon > 0$,

$$\begin{aligned} &(\Gamma_T u,u) \\ &\quad\geq \alpha_0\left[\int_\Omega \Phi\left(x, \bigl(\|u\|_\rho - \varepsilon\bigr)\frac{|Du|}{\|u\|_\rho - \varepsilon}\right)dx + \int_\Omega \Phi\left(x, \bigl(\|u\|_\rho - \varepsilon\bigr)\frac{|u|}{\|u\|_\rho - \varepsilon}\right)dx\right] \\ &\qquad - C - C^*\|q\|_{\overline{\Psi}}\|u\|_{1,\Phi} \\ &\quad\geq \alpha_0\bigl(\|u\|_\rho - \varepsilon\bigr)G\bigl(\bigl(\|u\|_\rho - \varepsilon\bigr)\bigr)\left[\int_\Omega \Phi\left(x, \frac{|Du|}{\|u\|_\rho - \varepsilon}\right)dx + \int_\Omega \Phi\left(x, \frac{|u|}{\|u\|_\rho - \varepsilon}\right)dx\right] \\ &\qquad - \alpha_0\bigl(\|u\|_\rho - \varepsilon\bigr)\int_\Omega \bigl|h(x)\bigr|\,dx - C - C^*\|q\|_{\overline{\Psi}}\|u\|_{1,\Phi} \\ &\quad\geq \alpha_0\bigl(\|u\|_\rho - \varepsilon\bigr)G\bigl(\bigl(\|u\|_\rho - \varepsilon\bigr)\bigr) - \alpha_0\bigl(\|u\|_\rho - \varepsilon\bigr)\int_\Omega \bigl|h(x)\bigr|\,dx - C \\ &\qquad - C^*\|q\|_{\overline{\Psi}}\|u\|_{1,\Phi}, \end{aligned}$$

$\forall u \in V$, as $\|u\|_{1,\Phi}$ is large enough. Since ε is arbitrary, we get

$$(\Gamma_T u,u) \geq \alpha_0\|u\|_\rho G\bigl(\|u\|_\rho\bigr) - \alpha_0\|u\|_\rho\int_\Omega \bigl|h(x)\bigr|\,dx - C - C^*\|q\|_{\overline{\Psi}}\|u\|_{1,\Phi},$$

$\forall u \in V$, as $\|u\|_{1,\Phi}$ is large enough. Therefore, we obtain

$$\frac{(\Gamma_T|_{V_m}u,u)}{\|u\|_{1,\Phi}} \to +\infty, \quad \text{as } \|u\|_{1,\Phi} \to +\infty.$$

Proposition 3.1 *If $\overline{\Phi} \in \Delta_2$, then there are functions $h \in L^1(\Omega)$, $h \geq 0$, and $G : [k,+\infty) \to \mathbb{R}$ for some $k > 2$ such that $G(s) \to +\infty$ as $s \to +\infty$ and* (3.25) *holds.*

Proof The proof of (3.25) is similar to the proof of Lemma 3.14 of [24].

Since $\overline{\Phi} \in \Delta_2$, there exist a positive constant $k > 1$ and a nonnegative function $h \in L^1(\Omega)$ such that $\overline{\Phi}(x,2v) \leq k\overline{\Phi}(x,v) + h(x)$, for all $v \geq 0$ and a.e. $x \in \Omega$. Necessarily, $k > 2$. Defining a function $F : [1,+\infty) \to [k,+\infty)$ by

$$F(r) = r\bigl((1-\lambda)k^n + \lambda k^{n+1}\bigr) \quad \text{if } r \in \bigl[2^n, 2^{n+1}\bigr] \text{ and } r = (1-\lambda)2^n + \lambda 2^{n+1},$$

we obtain

$$\begin{aligned}\overline{\Phi}(x,rv) &\leq \bigl[(1-\lambda)k^n + \lambda k^{n+1}\bigr]\overline{\Phi}(x,v) + \left[(1-\lambda)\frac{k^n-1}{k-1} + \lambda\frac{k^{n+1}-1}{k-1}\right]h(x) \\ &\leq \bigl[(1-\lambda)k^n + \lambda k^{n+1}\bigr]\overline{\Phi}(x,v) + \bigl[(1-\lambda)k^n + \lambda k^{n+1}\bigr]h(x) \\ &\leq F(r)\overline{\Phi}(x,v) + \frac{F(r)}{r}h(x).\end{aligned}$$

Hence $\frac{1}{F(r)}\overline{\Phi}(x,rv) \leq \overline{\Phi}(x,v) + \frac{1}{r}h(x)$. Taking $\Psi_1(x,v) = \frac{1}{F(r)}\overline{\Phi}(x,rv)$, by Proposition 2.1 and Proposition 2.2, we have $\Phi(x,u) \leq \frac{1}{F(r)}\Phi(x,\frac{F(r)}{r}u) + \frac{1}{r}h(x)$, for all $u \geq 0$ and a.e. $x \in \Omega$. It follows that $F(r)\Phi(x,u) \leq \Phi(x,\frac{F(r)}{r}u) + \frac{F(r)}{r}h(x)$, for all $u \geq 0$ and a.e. $x \in \Omega$. Since $\frac{F(r)}{r}$ strictly increases from k to $+\infty$ as $r \in [1,+\infty)$, its reciprocal function $G(s)$ is well defined and strictly increases from 1 to $+\infty$ as $s \in [k,+\infty)$, and we have $sG(s)\Phi(x,u) \leq \Phi(x,su) + sh(x)$, *i.e.*

$$\Phi(x,su) \geq sG(s)\Phi(x,u) - sh(x), \quad \text{for } s \geq k, u \geq 0 \text{ and a.e. } x \in \Omega. \qquad \square$$

Remark 3.6 Clearly, (1.9) can be replaced by (3.25) in the proof of Theorem 2.1 in [2]. Therefore, by Proposition 3.1, the condition (1.9) can be omitted since $\overline{\Phi} \in \Delta_2$ in [2].

Denote $\mathcal{S} = \{u \in W_0^1L_\Phi(\Omega) : u \text{ is a solution of (3.5) and } \underline{u} \leq u \leq \overline{u}\}$. Under the assumptions of Theorem 3.1, the solution set $\mathcal{S}$ is nonempty and we can deduce the following corollary.

Corollary 3.1 *Under the assumptions of Theorem* 3.1, *the following assertions about $\mathcal{S}$ are true*.

(a) *The set $\mathcal{S}$ is compact in $W_0^1L_\Phi(\Omega)$.*
(b) *$\mathcal{S}$ is a direct set in both directions, that is, if $u_1, u_2 \in \mathcal{S}$ then there exist $u, v \in \mathcal{S}$ such that $u \geq u_1 \vee u_2$ and $v \leq u_1 \wedge u_2$.*
(c) *$\mathcal{S}$ has least and greatest elements with respect to the ordering '$\leq$', that is, there are $u_*, u^* \in \mathcal{S}$ such that $u_* \leq u \leq u^*$, for all $u \in \mathcal{S}$.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Tongji University, Siping Road, Shanghai, 200092, China. [2]Department of Basic Teaching, Shanghai Jian Qiao University, Huchenghuan Road 1111, Shanghai, 201306, China.

Acknowledgements
The authors are highly grateful for the referees' careful reading and comments on this paper. The first author was supported by 'Chen Guang' Project (supported by Shanghai Municipal Education Commission and Shanghai Education Development Foundation) (10CGB25). The second author was supported by the National Natural Science Foundation of China (11371279).

Received: 14 December 2015 Accepted: 12 May 2016 Published online: 31 May 2016

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