Kutbi *Fixed Point Theory and Applications* 2013, **2013**:135
http://www.fixedpointtheoryandapplications.com/content/2013/1/135

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Common fixed point and invariant approximation results

Marwan A Kutbi*

*Correspondence: mkutbi@yahoo.com
Department of Mathematics, King Abdul Aziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia

Abstract

Some common fixed point results for Banach operator pairs in strongly *M*-starshaped metric spaces are obtained. As application, invariant approximation theorems are derived.

MSC: 47H10; 54H25

Keywords: common fixed point; Banach operator pair; strongly *M*-starshaped metric space; invariant approximation

1 Introduction and preliminaries

We first review needed definitions. Let X be a metric space with metric d, $M \subset X$ and $J = [0, 1]$. The space X is called

(1) M-starshaped [1] if there exists a continuous mapping $W : X \times M \times J \to X$ satisfying

$$d\bigl(x, W(y, q, \lambda)\bigr) \le \lambda d(x, y) + (1 - \lambda)d(x, q)$$

for all $x, y \in X$, $q \in M$ and all $\lambda \in J$;

(2) strongly M-starshaped [2, 3] if it is M-starshaped and satisfies the property (I), that is,

$$d\bigl(W(x, q, \lambda), W(y, q, \lambda)\bigr) \le \lambda d(x, y)$$

for all $x, y \in X$, $q \in M$ and all $\lambda \in J$;

(3) (strongly) convex if it is (strongly) X-starshaped;

(4) starshaped if it is $\{q\}$-starshaped for some $q \in X$.

A strongly convex metric space is also said to be a metric space of hyperbolic type (see Ciric [4]). Obviously, every normed space X is a strongly convex metric space with W defined by $W(x, q, \lambda) = \lambda x + (1 - \lambda)q$ for all $x, q \in X$ and all $\lambda \in J$. More generally, if X is a linear space with a translation invariant metric satisfying $d(\lambda x + (1 - \lambda)y, 0) \le \lambda d(x, 0) + (1 - \lambda)d(y, 0)$, then X is a strongly convex metric space. A subset D of an M-starshaped metric space X is called q-starshaped if there exists $q \in D \cap M$ such that $W(x, q, \lambda) \in D$ for all $x \in D$ and all $\lambda \in J$. For details, we refer the reader to Al-Thagafi [2], Guay *et al.* [5] and Takahashi [1].

Let $I, T : X \to X$ be two mappings and $D \subset X$. Then T is called

(5) I-nonexpansive on D if $d(Tx, Ty) \le d(Ix, Iy)$ for all $x, y \in D$;

(6) I-contraction on D if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \le kd(Ix, Iy)$ for all $x, y \in D$.

A point $x \in D$ is a coincidence point (common fixed point) of I and T if $Ix = Tx$ ($x = Ix = Tx$). The set of coincidence points of I and T is denoted by $C(I,T)$. The mappings I and T are called

(7) commuting on D if $ITx = TIx$ for all $x \in D$;
(8) weakly compatible if they commute at their coincidence points, *i.e.*, if $ITx = TIx$ whenever $Ix = Tx$.

The ordered pair (I,T) of two self-maps of a metric space X is called a Banach operator pair if the set $\mathrm{Fix}(T)$ is I-invariant, namely $I(\mathrm{Fix}(T)) \subseteq \mathrm{Fix}(T)$. Obviously, a commuting pair (I,T) is a Banach operator pair but not conversely in general, see [6–8].

Let $S \subset X$ and $\widehat{x} \in X$. Then $P_S(\widehat{x}) = \{x \in S : d(x,\widehat{x}) = d(\widehat{x},S)\}$ is called the set of best S-approximants to $\widehat{x}$, where $d(\widehat{x},S) = \inf\{d(\widehat{x},y) : y \in S\}$ and $C_S^I(\widehat{x}) = \{x \in S : Ix \in P_S(\widehat{x})\}$.

In 1963, Meinardus [9] employed the Schauder fixed point theorem to prove a result regarding invariant approximation. In 1979, Singh [10] proved the following extension of the result of Meinardus.

Theorem 1.1 *Let T be a nonexpansive operator on a normed space X, let M be a nonempty subset of X, $T(M) \subset M$ and $u \in F(T)$. If $P_M(u)$ is nonempty compact and starshaped, then $P_M(u) \cap F(T) \neq \emptyset$.*

Hicks and Humphries [11] found that Singh's results remain true if $T(M) \subset M$ is replaced by $T(\partial M) \subset M$. In 1988, Sahab *et al.* [12] established the following result which contains the result of Hicks and Humphries and Theorem 1.1.

Theorem 1.2 *Let I and T be self-maps of a normed space X with $u \in F(I) \cap F(T)$, $M \subset X$ with $T(\partial M) \subset M$, and $q \in F(I)$. If $D = P_M(u)$ is compact and q-starshaped, $I(D) = D$, I is continuous and linear on D, I and T are commuting on D and T is I-nonexpansive on $D \cup \{u\}$, then $P_M(u) \cap F(T) \cap F(I) \neq \emptyset$.*

Invariant approximation results for commuting maps due to Al-Thagafi [13] extended and generalized Theorems 1.1-1.2 and the works of [11, 14, 15]. Al-Thagafi results were further extended by [7, 8, 16–26] to R-subweakly commuting, pointwise R-subweakly commuting and a Banach operator pair.

The aim of this paper is to establish certain common fixed point theorem for a Banach operator pair in the setup of strongly M-starshaped metric spaces. As application, invariant approximation results for this class of maps are derived. Our results extend and unify the work of Al-Thagafi [2, 13], Dotson [27], Habiniak [14], Hicks and Humphries [11], Hussain and Berinde [28], Hussain *et al.* [22], Naz [3], Latif [29], Sahab *et al.* [12] and Singh [10, 15].

The following result will be needed.

Lemma 1.3 [2] *Let D be a subset of an M-starshaped metric space (X,d) and $\widehat{x} \in X$. Then $P_D(\widehat{x}) \subset \partial D \cap D$.*

2 Main results

The following result will be needed (see Lemma 2.10 [7] and Lemma 2.2 [8]).

Lemma 2.1 *Let S be a nonempty subset of a metric space (X,d), and let T,f be self-maps of S. If $F(f)$ is nonempty, $clT(F(f)) \subseteq F(f)$, $cl(T(M))$ is complete, and T and f satisfy for all $x,y \in S$ and $0 \le h < 1$,*

$$d(Tx,Ty) \le h \max\{d(fx,fy), d(Tx,fx), d(Ty,fy), d(Tx,fy), d(Ty,fx)\}, \tag{2.1}$$

then $S \cap F(T) \cap F(f)$ is a singleton.

Theorem 2.2 *Let S be a nonempty subset of a strongly M-starshaped metric space X and let T,f be self-maps of S. Suppose that $F(f)$ is q-starshaped, $clT(F(f)) \subseteq F(f)$, $cl(T(S))$ is compact, T is continuous on S and*

$$\begin{aligned}\|Tx - Ty\| \le \max\{&\|fx - fy\|, \operatorname{dist}(fx,[q,Tx]), \operatorname{dist}(fy,[q,Ty]),\\ &\operatorname{dist}(fy,[q,Tx]), \operatorname{dist}(fx,[q,Ty])\},\end{aligned} \tag{2.2}$$

for all $x,y \in S$, then $S \cap F(T) \cap F(f) \neq \emptyset$.

Proof Define $T_n : F(f) \to F(f)$ by $T_n x = W(Tx,q,k_n)$ for all $x \in F(f)$ and a fixed sequence of real numbers k_n ($0 < k_n < 1$) converging to 1. Since $F(f)$ is q-starshaped and $clT(F(f)) \subseteq F(f)$, therefore $clT_n(F(f)) \subseteq F(f)$ for each $n \ge 1$. Also, by (2.2),

$$\begin{aligned}d(T_n x, T_n y) &= d\bigl(W(Tx,q,k_n), W(Ty,q,k_n)\bigr)\\ &= k_n d(Tx,Ty)\\ &\le k_n \max\{d(fx,fy), \operatorname{dist}(fx,[q,Tx]), \operatorname{dist}(fy,[q,Ty]),\\ &\quad \operatorname{dist}(fx,[q,Ty]), \operatorname{dist}(fy,[q,Tx])\}\\ &\le k_n \max\{d(fx,fy), d(fx,T_n x), d(fy,T_n y), d(fy,T_n x), d(fx,T_n y)\}\end{aligned}$$

for each $x,y \in F(f)$ and $0 < k_n < 1$. If $cl(T(S))$ is compact for each $n \ge 1$, then $cl(T_n(S))$ is compact and hence complete. By Lemma 2.1, for each $n \ge 1$, there exists $x_n \in F(f)$ such that $x_n = fx_n = T_n x_n$. The compactness of $cl(T(M))$ implies that there exists a subsequence $\{Tx_m\}$ of $\{Tx_n\}$ such that $Tx_m \to z \in cl(T(M))$ as $m \to \infty$. Since $\{Tx_m\}$ is a sequence in $T(F(f))$ and $clT(F(f)) \subseteq F(f)$, therefore $z \in F(f)$. Further, $x_m = T_m x_m = W(Tx_m,q,k_m) \to z$. By the continuity of T, we obtain $Tz = z = fz$. Thus, $S \cap F(T) \cap F(f) \neq \emptyset$. □

Corollary 2.3 *Let S be a nonempty subset of a strongly M-starshaped metric space X and let T,f be self-maps of S. Suppose that $F(f)$ is q-starshaped, $clT(F(f)) \subseteq F(f)$, $cl(T(S))$ is compact, T is continuous on S and T is f-nonexpansive on S, then $S \cap F(T) \cap F(f) \neq \emptyset$.*

Corollary 2.4 *Let S be a nonempty subset of a strongly M-starshaped metric space X and let T,f be self-maps of S. Suppose that $F(f)$ is closed and q-starshaped, (T,f) is a Banach operator pair, $cl(T(S))$ is compact, T is continuous on S and T satisfies* (2.2) *or T is f-nonexpansive on S, then $S \cap F(T) \cap F(f) \neq \emptyset$.*

Corollary 2.5 ([13], Theorem 2.1) *Let M be a nonempty closed and q-starshaped subset of a normed space X and let T and f be self-maps of M such that $T(M) \subseteq f(M)$. Suppose that*

T commutes with f and $q \in F(f)$. If $cl(T(M))$ is compact, f is continuous and linear and T is f-nonexpansive on M, then $M \cap F(T) \cap F(f) \neq \emptyset$.

Corollary 2.6 (([30], Theorem 3.3)) *Let M be a nonempty subset of a normed space X and let T and f be self-maps of M. Suppose that $F(f)$ is q-starshaped, $clT(F(f)) \subseteq F(f)$, $cl(T(M))$ is compact, T is continuous on M and* (2.2) *holds for all $x, y \in M$. Then $M \cap F(T) \cap F(f) \neq \emptyset$.*

Corollary 2.7 ([7], Theorem 2.11) *Let M be a nonempty subset of a normed space X and let T, f be self-maps of M. Suppose that $F(f)$ is q-starshaped and closed $cl(T(M))$ is compact, T is continuous on M, (T,f) is a Banach operator pair and satisfies* (2.2) *for all $x, y \in M$. Then $M \cap F(T) \cap F(f) \neq \emptyset$.*

Corollary 2.8 *Let X be a strongly M-starshaped metric space, let $f, T : X \to X$ be two mappings, S be a subset of X such that $T(\partial S \cap S) \subset S$ and $\widehat{x} \in F(T) \cap F(f)$. Suppose that $P_S(\widehat{x})$ is nonempty closed and q-starshaped with $q \in F(f) \cap M$ and $cl(T(P_S(\widehat{x})))$ is compact and $f(P_S(\widehat{x})) = P_S(\widehat{x})$. If T is continuous, $clT(F(f)) \subseteq F(f)$ and satisfies, for all $x \in P_S(\widehat{x}) \cup \{\widehat{x}\}$,*

$$d(Tx, Ty) \leq \begin{cases} d(fx, fu) & \text{if } y = u, \\ \max\{d(fx, fy), \operatorname{dist}(fx, [q, Tx]), \operatorname{dist}(fy, [q, Ty]), & \\ \quad \operatorname{dist}(fx, [q, Ty]), \operatorname{dist}(fy, [q, Tx])\} & \text{if } y \in P_S(\widehat{x}), \end{cases} \tag{2.3}$$

then $P_S(\widehat{x}) \cap F(T) \cap F(f) \neq \emptyset$.

Proof Let $x \in P_S(\widehat{x})$. Then by Lemma 1.3, $x \in \partial S \cap S$ and so $Tx \in S$ since $T(\partial S \cap S) \subset S$. As T satisfies (2.3) on $P_S(\widehat{x}) \cup \{\widehat{x}\}$ and $I(P_S(\widehat{x})) = P_S(\widehat{x})$, we have

$$d(Tx, \widehat{x}) = d(Tx, T\widehat{x}) \leq d(Ix, I\widehat{x}) = d(Ix, \widehat{x}) = d(\widehat{x}, S).$$

This implies that $Tx \in P_S(\widehat{x})$. Thus $T(P_S(\widehat{x})) \subset P_S(\widehat{x}) = f(P_S(\widehat{x}))$. Now Theorem 2.2 implies that $P_S(\widehat{x}) \cap F(T) \cap F(f) \neq \emptyset$. □

Theorem 2.9 *Let X be a strongly M-starshaped metric space, let $f, T : X \to X$ be two mappings, S be a subset of X such that $T(\partial S \cap S) \subset S$ and $\widehat{x} \in F(T) \cap F(f)$. Suppose that $P_S(\widehat{x})$ is nonempty closed and q-starshaped with $q \in F(f) \cap M$ and $cl(T(P_S(\widehat{x})))$ is compact and $f(P_S(\widehat{x})) = P_S(\widehat{x})$. If T is continuous, $clT(F(f)) \subseteq F(f)$ and T is f-nonexpansive on $P_S(\widehat{x}) \cup \{\widehat{x}\}$, then $P_S(\widehat{x}) \cap F(T) \cap F(f) \neq \emptyset$.*

Remark 2.10 A subset S of a strongly M-starshaped metric space X is said to have the property (N) w.r.t. T [22, 28] if

(i) $T : S \to S$,

(ii) $W(Tx, q, k_n) \in S$ for some $q \in S \cap M$ and a fixed sequence of real numbers k_n $(0 < k_n < 1)$ converging to 1 and for each $x \in S$.

All results of the paper (Theorem 2.2-Theorem 2.9) remain valid provided f is assumed to be surjective and q-starshapedness of the set $F(f)$ is replaced by the property (N) respectively. Consequently, recent results due to Hussain and Berinde [28] and Hussain *et al.* [22] are improved and extended.

Remark 2.11 Recently, in [31], the author obtained certain fixed point theorems in convex metric spaces. Using Theorems 3.2 and 3.4 [31] and the technique in [7], we can prove more common fixed point and approximation results for Banach pairs satisfying generalized nonexpansive conditions in a strongly M-starshaped metric space X.

Remark 2.12 All results of the paper can be proved for multivalued Banach operator pairs defined and studied in [32].

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author has read and approved the final manuscript.

Acknowledgements
This research was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The author acknowledges with thanks DSR, KAU for financial support.

Received: 25 November 2012 Accepted: 7 May 2013 Published: 27 May 2013

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doi:10.1186/1687-1812-2013-135
Cite this article as: Kutbi: Common fixed point and invariant approximation results. *Fixed Point Theory and Applications* 2013 **2013**:135.