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Modified generalized projective synchronization of fractional-order chaotic Lü systems

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Abstract

This paper addresses new modified generalized projective synchronization (MGPS) of fractional-order chaotic systems based on the stability theory of fractional-order systems, where the drive and response systems could be asymptotically synchronized up to a desired transformation matrix, not a diagonal matrix. MGPS between the hyperchaotic Lorenz system and the Lü system of the base order 0.95 is implemented as an example. Numerical simulations show the effectiveness and feasibility of the method.

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Keywords: fractional-order system; chaotic system; Lyapunov exponent; modified generalized projective synchronization

1 Introduction

It is well known that fractional calculus is a classical mathematical notion, with a history as long as calculus itself. But its applications to physics and engineering are a subject of only recent interest [1, 2]. It was found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives, for instance, viscoelastic systems [3], electromagnetic waves [4], dielectric polarization [5], quantitative finance [6], quantum evolution of complex systems [7], and so forth.

More recently, many investigations were devoted to the chaotic behavior, chaotic control, and synchronization of fractional-order dynamical systems. For example, it has been shown that Chua's circuit with an order as low as 2.7 can produce a chaotic attractor [8]. In [9], it was shown that nonautonomous Duffing systems with an order less than 2.0 can still behave in a chaotic manner. In [10], chaotic behavior of the fractional-order 'jerk' model, in which chaotic attractors can be obtained with a system of the order as low as 2.1, was studied. Bifurcations and chaos in the fractional-order simplified Lorenz system [11], chaotic behavior and its control in the fractional-order Chen system [12] were reported. In [13], chaotic and hyperchaotic behaviors in fractional-order Rössler equations were studied. Chaotic dynamics and synchronization of fractional-order Arneodo systems [14], the Lü system [15], and a unified system [16], synchronization of fractional-order hyperchaotic modified systems [17] were also reported.

Recently, projective synchronization (PS) has been especially extensively studied because it can be used to obtain faster communication with its proportional feature and the unpredictability of the scaling factor can additionally enhance the security of communication [18–20]. In [21, 22], Wu *et al.* presented the generalized projective synchronization (GPS) method for fractional-order Chen hyperchaotic systems, which associates with the projective synchronization and the generalized one, where the drive and response systems could be synchronized up to scaling factors θ_i. In the practical applications, a more general form synchronization, called modified generalized projective synchronization (MGPS), where the drive and response systems can be synchronized up to a transformation matrix that is not diagonal, will increase the complexity of the synchronization and further increase the diversity and the security of communications. Moreover, to the best of our knowledge, most of the existing papers only consider constant scaling factors which is a diagonal matrix, and the MGPS, which has rarely been explored, will contain GPS with constant scaling factors and extend previous works. Therefore, MGPS of fractional-order chaotic systems becomes a new meaningful problem.

Motivated by the above discussion, this paper introduces a fractional-order chaotic Lü system and aims to investigate this new MGPS of fractional-order chaotic systems with different structure, where the drive and response systems could be asymptotically synchronized up to a desired transformation matrix, not a diagonal matrix. Based on the stability theory of fractional-order systems, the controllers are designed to make the drive and response systems synchronize up to the desired transformation matrix.

This paper is organized as follows. In Section 2, a brief review of the fractional derivative and numerical algorithm for the fractional-order system is given. Dynamics of a novel fractional-order chaotic system is numerically studied and demonstrated by computer simulation. In Section 3, a general method of MGPS for coupled fractional-order chaotic systems is presented based on the stability theory of fractional-order systems. MGPS between the fractional-order hyperchaotic Lorenz system and the Lü system is derived, and numerical simulations show the effectiveness and feasibility of the proposed synchronization scheme. Finally, the conclusions are given in Section 4.

2 A novel fractional-order chaotic system

2.1 Fractional derivative and its approximation method

There are many definitions of fractional differential operators [1]. The definition of the Riemann-Liouville derivative is given as follows:

$$D^{\alpha}f(t) = \frac{d^m}{dt^m}J^{m-\alpha}f(t), \quad \alpha > 0, \tag{2.1}$$

where $m = \lceil\alpha\rceil$, *i.e.*, m is the first integer which is not less than α, J^{β} is the β-order Riemann-Liouville integral operator as described by

$$J^{\beta}z(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{z(\tau)}{(t-\tau)^{1-\beta}}\,d\tau, \quad 0 < \beta \leq 1, \tag{2.2}$$

where $\Gamma(\bullet)$ denotes the gamma function $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,dt$.

Here and throughout, the following Caputo definition is applied:

$$D_*^{\alpha}f(t) = \frac{d^m}{dt^m}J^{m-\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(m-q)}\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)\,d\tau, & m-1 < q < m, \\ \frac{d^m}{dt^m}f(t), & q = m, \end{cases} \tag{2.3}$$

where $m = \lceil\alpha\rceil$.

Here we choose the Caputo version and use an improved predictor-corrector algorithm, *i.e.*, the Adams-Bashforth-Moulton predictor-correctors scheme for fractional differential equations [23–27], where the numerical approximation is a time-domain approach that is more accurate, and the computational cost is greatly reduced.

The improved fractional predictor-corrector algorithm is based on the analytical property of the following differential equation:

$$\begin{cases} D_*^\alpha x = f(t,x), & 0 \le t \le T, \\ x^{(k)}(0) = x_{(0)}^{(k)}, & k = 0,1,2,\ldots,m-1 \ (m = \lceil \alpha \rceil) \end{cases} \tag{2.4}$$

which is equivalent to the Volterra integral equation

$$x(t) = \sum_{k=0}^{m-1} x_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau,x)}{(t-\tau)^{1-\alpha}} \, d\tau. \tag{2.5}$$

Now, set $h = T/N$, $t_n = nh$ ($n = 0,1,2,\ldots,N \in Z^+$). Equation (2.5) can be written as

$$x_h(t_{n+1}) = \sum_{k=0}^{m-1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} f\bigl(t_{n+1}, x_h^\theta(t_{n+1})\bigr) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum a_{j,n+1} f\bigl(t_j, x_h(t_j)\bigr), \tag{2.6}$$

where the predicted value $x_h^\theta(t_{n+1})$ is determined by

$$x_h^\theta(t_{n+1}) = \sum_{k=0}^{m-1} x_0^{(k)} \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} f\bigl(t_{n+1}, x_h^\theta(t_{n+1})\bigr) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum b_{j,n+1} f\bigl(t_j, x_h(t_j)\bigr) \tag{2.7}$$

and

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha+1}, & j = 0, \\ (n-j+2)^{\alpha+1} + n - j^{\alpha+2} - 2(n-j+1)^{\alpha+1}, & 1 \le j \le n, \end{cases} \tag{2.8}$$

$$b_{j,n+1} = \frac{h^\alpha}{\alpha} \bigl((n-j+1)^\alpha - (n-j)^\alpha\bigr). \tag{2.9}$$

The estimation error in this method is

$$e = \max\bigl(\bigl|x(t_j) - x_h(t_j)\bigr|\bigr) = O\bigl(h^\theta\bigr) \quad (j = 0,1,\ldots,N), \theta = \min(2, 1+\alpha).$$

2.2 Dynamic analysis of a novel fractional-order system

2.2.1 Novel fractional-order system model

In 2006, Wang *et al.* introduced a new modified hyperchaotic Lü system [28], further investigated its dynamical behaviors, and physically implemented the system which is described by

$$\begin{cases} \dot{x} = a(y - x + yz), \\ \dot{y} = -xz + by + w, \\ \dot{z} = xy - cz, \\ \dot{w} = -dx, \end{cases} \tag{2.10}$$

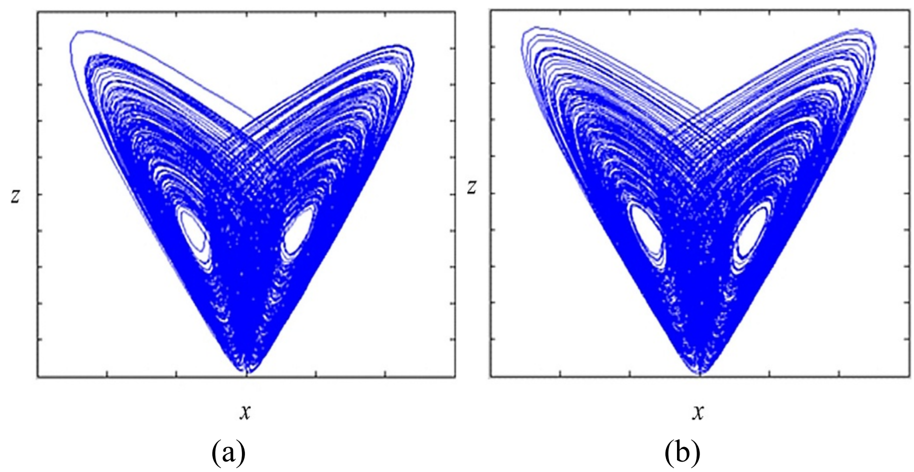

Figure 1 Phase portraits in the *x*-*z* plane of integer-order hyperchaotic real system (2.10) for *a* = 70, *b* = 15, *c* = 12, *d* = 5: (a) *x* = 1, *y* = 2, *z* = 3, *w* = 4; (b) *x* = 5, *y* = 3, *z* = 5, *w* = 1.

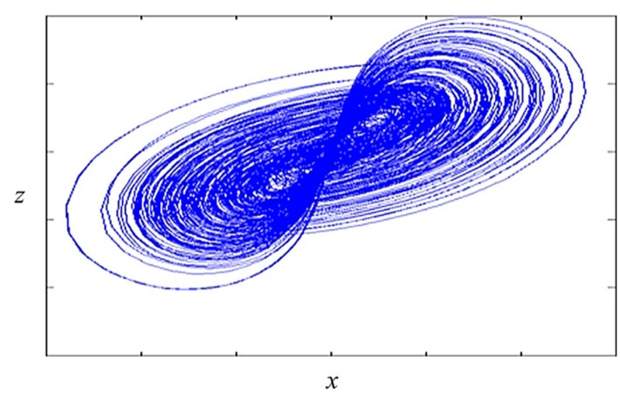

Figure 2 Phase portraits in the *x*-*z* plane of hyperchaotic complex system (2.11) for *a* = 70, *b* = 15, *c* = 12, *d* = 5 and *x* = 1, *y* = 2, *z* = 3, *w* = 4.

where $(x, y, z, w) \in R^4$, and a, b, c, d are real constant parameters. The hyperchaotic attractors of system (2.10) for $a = 70$, $b = 15$, $c = 12$, $d = 5$ are plotted in Figure 1.

In 2011, Mahmoud *et al.* studied modified hyperchaotic complex Lü systems [29] given by

$$\begin{cases} \dot{x} = a(y - x + yz), \\ \dot{y} = -xz + by + w, \\ \dot{z} = \frac{1}{2}(\bar{x}y + x\bar{y}) - cz, \\ \dot{w} = -\frac{d}{2}(\bar{x} + x), \end{cases} \tag{2.11}$$

where $(x, y) \in C^2$ are complex variables, $(z, w) \in R^2$ are real variables, and a, b, c and d are real (or complex) positive constant parameters. The hyperchaotic attractors of (2.11) using the same choice of the parameters and initial conditions as in the real system (2.10) are plotted in Figure 2. More detailed complex dynamics of the modified hyperchaotic complex Lü system can be seen in [29].

In the present paper, based on the above descriptions, we modify the derivative operator in Eq. (2.10) to be with respect to the fractional order α ($0 < \alpha \leq 1$). Thus the fractional version of the hyperchaotic Lü system is given by

$$\begin{cases} D_*^\alpha x = a(y - x + yz), \\ D_*^\alpha y = -xz + by + w, \\ D_*^\alpha z = xy - cz, \\ D_*^\alpha w = -dx, \end{cases} \tag{2.12}$$

where a, b, c and d are real positive constant parameters. When $\alpha = 1$, system (2.12) reduces to the classical integer-order modified hyperchaotic Lü system (2.10).

2.2.2 *Dynamic analysis of the fractional-order Lü system*

According to the numerical algorithm for fractional differential systems in Section 2.1, system (2.12) for initial condition (x_0, y_0, z_0, w_0) can be written as

$$\begin{cases} x_{n+1} = x_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}[a(y_{n+1}^p - x_{n+1}^p + y_{n+1}^p z_{n+1}^p) + \sum_{j=0}^n a_{j,n+1}(y_j - x_j + y_j z_j)], \\ y_{n+1} = y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}[-x_{n+1}^p z_{n+1}^p + by_{n+1}^p + w_{n+1}^p + \sum_{j=0}^n a_{j,n+1}(-x_j z_j + by_j + w_j)], \\ z_{n+1} = z_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}[x_{n+1}^p y_{n+1}^p - cz_{n+1}^p + \sum_{j=0}^n a_{j,n+1}(x_j y_j - cz_j)], \\ w_{n+1} = w_0 + \frac{h^\alpha}{\Gamma(\alpha+2)}[-dx_{n+1}^p + \sum_{j=0}^n a_{j,n+1}(-dx_j)], \end{cases} \tag{2.13}$$

where

$$\begin{cases} x_{n+1}^p = x_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1}(y_j - x_j + y_j z_j), \\ y_{n+1}^p = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1}(-x_j z_j + by_j + w_j), \\ z_{n+1}^p = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1}(x_j y_j - cz_j), \\ w_{n+1}^p = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1}(-dx_j), \end{cases} \tag{2.14}$$

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha+1}, & j = 0, \\ (n-j+2)^{\alpha+1} + n - j^{\alpha+2} - 2(n-j+1)^{\alpha+1}, & 1 \leq j \leq n, \end{cases} \tag{2.15}$$

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}\big((n-j+1)^\alpha - (n-j)^\alpha\big). \tag{2.16}$$

In the following simulations, the system parameters are always chosen as $a = 70$, $b = 15$, $c = 12$, $d = 5$. The simulation results demonstrate that chaos indeed exists in the fractional-order system (2.12) with order less than 4. For example, when $\alpha = 0.95$, a chaotic attractor is found, and the phase portraits of the system are displayed in Figure 3. Equation (2.12) has no solution when $\alpha < 0.8320$. A chaotic attractor is found for $\alpha = 0.8321$, and its phase portrait is displayed in Figure 4. Thus the lower limit of fractional order for this system to be chaotic is between $\alpha = 0.8320$ and $\alpha = 0.8321$, and the lowest order we found for this system to yield chaotic is 3.3284.

The largest Lyapunov exponent of the simulation time series [30, 31] is positive only when $\alpha \geq 0.8321$, for example, $\alpha = 0.8321$, $\lambda_1 = 19.78383$. Obviously, the fractional-order system (2.12) is chaotic.

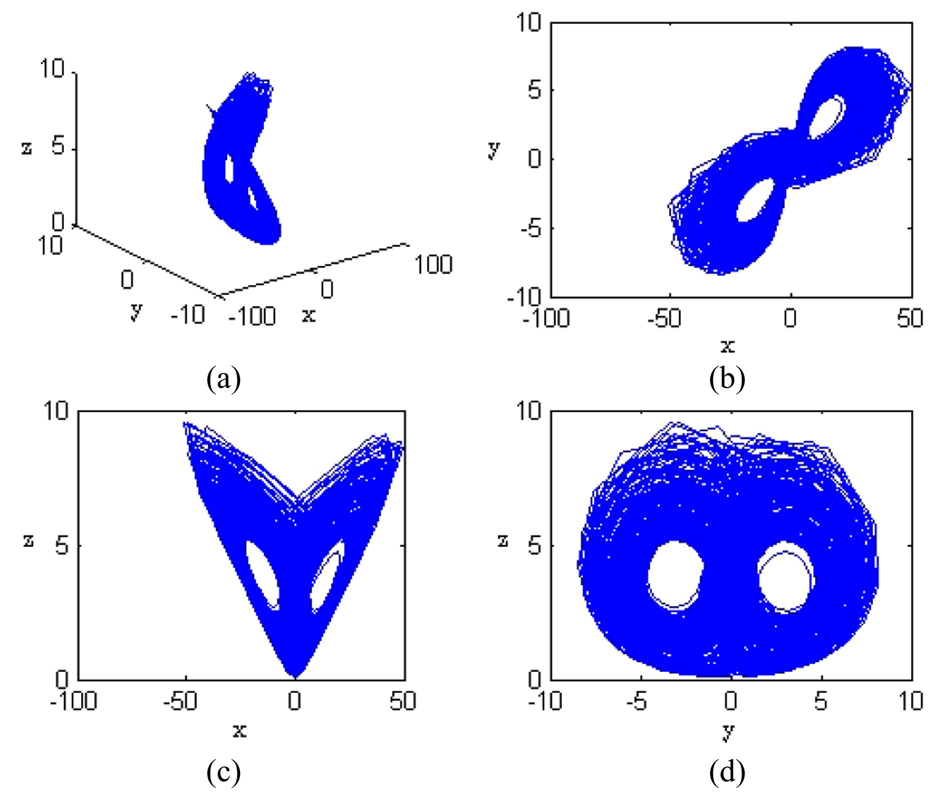

Figure 3 Phase portraits of fractional-order chaotic system (2.12) for $\alpha = 0.95$.

3 MGPS between the different fractional-order systems

3.1 A general method for MGPS of fractional-order systems

In this section we apply stability analysis to fractional-order systems. Fractional-order differential equations are at least as stable as their integer-order counterparts because systems with memory are typically more stable than those without memory [23, 26].

Consider the fractional-order chaotic drive and response systems as

$$D_*^\alpha X = f(X) \tag{3.1}$$

and

$$D_*^\alpha Y = g(Y) + U(X,Y), \tag{3.2}$$

where the state vectors $X, Y \in R^n$, $f, g : R^n \to R^n$ are continuous vector functions and $U(X,Y) : R^{2n} \to R^n$ is a controller to be determined later.

Decompose the fractional-order systems (3.1) and (3.2) as

$$D_*^\alpha X = AX + F(X) \tag{3.3}$$

and

$$D_*^\alpha Y = BY + G(Y) + U(X,Y), \tag{3.4}$$

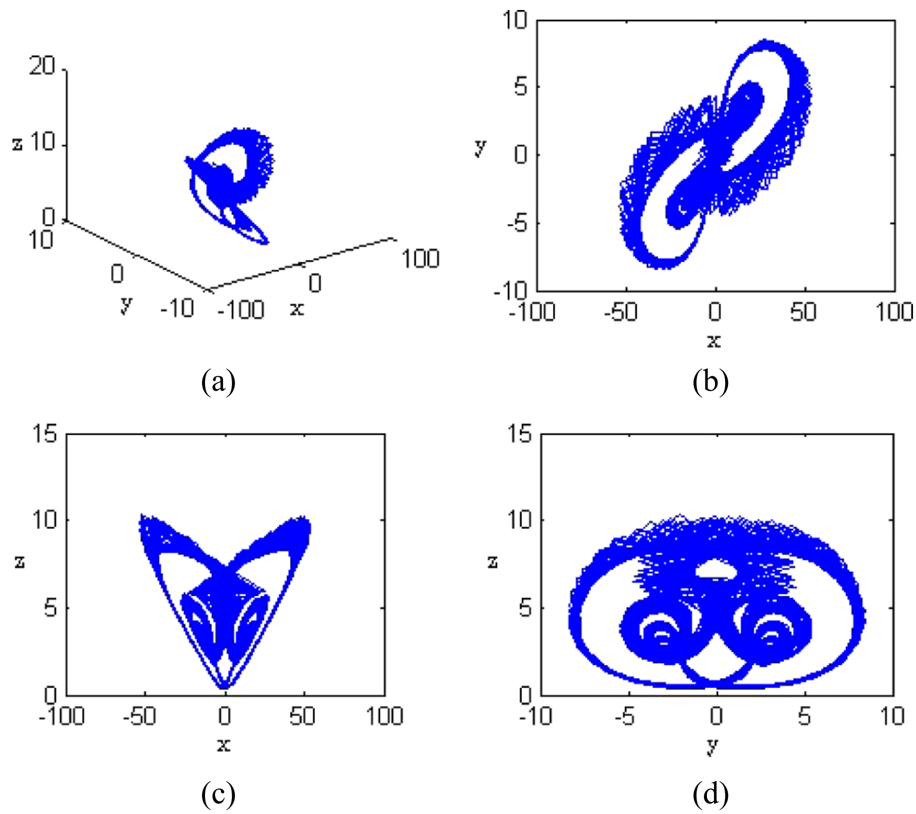

Figure 4 Phase portraits of fractional-order chaotic system (2.12) for $\alpha = 0.8321$. (a) three-dimensional phase portrait; **(b)** *x-y* phase plane; **(c)** *x-z* phase plane; **(d)** *y-z* phase plane.

where $A, B : R^{n\times n} \to R^{n\times n}$ is the Jacobian matrix of the system at the origin and $F, G : R^n \to R^n$ are the nonlinear parts.

Remark 1 Many fractional-order chaotic systems belong to the class individualized by (3.3). Examples include the fractional-order hyperchaotic modified Rössler system [13, 21], the Chen system [19], and the Lorenz systems [20, 22].

Definition 1 The drive system (3.3) and response system (3.4) are said to achieve MGPS if there exists a controller U such that $\lim_{t\to\infty} \|Y - CX\| = 0$, where $C : R^{n\times n} \to R^{n\times n}$ is a transformation matrix of the drive system (3.3).

Remark 2 In particular, if the transformation matrix is diagonal and all diagonal elements are the same, the MGPS is simplified to the GPS.

Hence, the error system is defined as

$$e = Y - CX, \tag{3.5}$$

which means that system (3.4) synchronizes with the projection of system (3.3). If $\lim_{t\to\infty} \|e\| = 0$, systems (3.3) and (3.4) substituted into (3.5) give an error system that can be expressed as

$$D_*^\alpha e = D_*^\alpha Y - CD_*^\alpha X = Be + BCX + G(Y) - CAX - CF(X) + U(X,Y). \tag{3.6}$$

Then the discussions of MGPS between the two coupled systems (3.3) and (3.4) can be translated into the analysis of the asymptotic stability of the zero solution of the error system (3.6). Next, a suitable controller is provided to ensure the asymptotic stability of the zero solution of the error system (3.6) based on the stability theorem of linear fractional-order systems.

For a given autonomous fractional-order linear system

$$D_*^\alpha z = Mz, \tag{3.7}$$

with $z(0) = z_0$, where $0 < \alpha < 1$ and $z \in R^n$, M is a constant matrix.

Lemma 1 [32] *System* (3.1) *is*

(i) *asymptotically stable if and only if*

$$\left|\arg\bigl(\lambda_i(M)\bigr)\right| > \alpha\pi/2 \quad (i = 1, 2, \dots, n), \tag{3.8}$$

where $\arg(\lambda_i(M))$ *denotes the argument of the eigenvalue* λ_i *of* M. *In this case, each component of the states decays toward* 0 *like* $t^{-\alpha}$,

(ii) *stable if and only if* $|\arg(\lambda_i(M))| \geq \alpha\pi/2$ $(i = 1, 2, \dots, n)$, *and those critical eigenvalues* λ_i *that satisfy* $|\arg(\lambda_i(M))| = \alpha\pi/2$ $(i = 1, 2, \dots, n)$ *have geometric multiplicity one.*

Now, due to Lemma 1, the following results can be obtained.

Theorem 1 *Given a fractional-order drive system* (3.3) *and a response system* (3.4), *there exists a suitable controller*

$$U = CAX + CF(X) - BCX - G(Y) + Ke, \tag{3.9}$$

where $K \in R^{n\times n}$ *is a gain matrix. MGPS between systems* (3.3) *and* (3.4) *can be achieved if and only if all the eigenvalues of* $B + K$ *satisfy* $|\arg(\lambda_i(B + K))| > \alpha\pi/2$ $(i = 1, 2, \dots, n)$.

Proof Substituting controller (3.9) into system (3.6), the error system (3.6) can be rewritten as

$$D_*^\alpha e = (B + K)e. \tag{3.10}$$

Due to Lemma 1, we arrive at the conclusion that system (3.10) is asymptotically stable if and only if all the eigenvalues $\lambda_i(B + K)$ satisfy $|\arg(\lambda_i(B + K))| > \alpha\pi/2$ $(i = 1, 2, \dots, n)$. That is, $\lim_{t\to\infty} \|e\| = 0$, or systems (3.3) and (3.4) realize MGPS. This completes the proof. □

3.2 MGPS between the fractional-order hyperchaotic Lorenz system and the Lü system

In this section the MGPS behavior between two different fractional-order systems, the hyperchaotic Lorenz system and the modified Lü system, is made. It is assumed that the fractional-order Lü system drives the hyperchaotic Lorenz system. Therefore, we define the Lü system as a master system and the hyperchaotic Lorenz system as a slave system as follows. The master system described through Eq. (2.12) is

$$\begin{cases} D_*^\alpha x_1 = a(x_2 - x_1 + x_2x_3), \\ D_*^\alpha x_2 = bx_2 + x_4 - x_1x_3, \\ D_*^\alpha x_3 = -cx_3 + x_1x_2, \\ D_*^\alpha x_4 = -dx_1. \end{cases} \tag{3.11}$$

The slave system described in [22] is

$$\begin{cases} D_*^\alpha y_1 = k(y_2 - y_1) + u_1, \\ D_*^\alpha y_2 = my_1 + y_2 - y_1y_3 - y_4 + u_2, \\ D_*^\alpha y_3 = y_1y_2 - ny_3 + u_3, \\ D_*^\alpha y_4 = py_2y_3 + u_4, \end{cases} \tag{3.12}$$

where k, m, n, p are parameters and $x, y, z, w \in R$ are variables, four functions $u_i(t)$ ($i = 1, 2, 3, 4$) are the controller to be determined later so that the drive and response systems can be synchronized in the sense of MGPS.

Comparing systems (3.11) and (3.12) with systems (3.3) and (3.4), one has

$$A = \begin{bmatrix} -a & a & 0 & 0 \\ 0 & b & 0 & 1 \\ 0 & 0 & -c & 0 \\ -d & 0 & 0 & 0 \end{bmatrix}, \qquad F(X) = \begin{pmatrix} ax_2x_3 \\ -x_1x_3 \\ x_1x_2 \\ 0 \end{pmatrix},$$

$$B = \begin{bmatrix} -k & k & 0 & 0 \\ m & 1 & 0 & -1 \\ 0 & 0 & n & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad G(Y) = \begin{pmatrix} 0 \\ -y_1y_3 \\ y_1y_2 \\ py_2y_3 \end{pmatrix}, \qquad U(X,Y) = \begin{pmatrix} u_1(X,Y) \\ u_2(X,Y) \\ u_3(X,Y) \\ u_4(X,Y) \end{pmatrix}.$$

Here, choose a transformation matrix

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}. \tag{3.13}$$

The error system $e = Y - CX$ between systems (3.11) and (3.12) is

$$\begin{cases} e_1 = y_1 - (x_1 + x_2), \\ e_2 = y_2 - (x_2 + x_3), \\ e_3 = y_3 - (x_3 + x_4), \\ e_4 = y_4 - (x_1 + x_4). \end{cases} \tag{3.14}$$

According to Theorem 1, with a suitable controller $U = CAx + CF(x) - BCx - G(y) + Ke$, $K \in R^{n\times n}$, the error dynamical system can be obtained in the form

$$D_*^\alpha e = (B + K)e. \tag{3.15}$$

Then Theorem 1 assures that there exists a feedback gain matrix K so that systems (3.11) and (3.12) realize the synchronization.

There is not a unique choice for such a matrix K. For example, when $\alpha = 0.95$, $(k, m, n, p) = (10, 28, 8/3, 0.1)$, and $(a, b, c, d) = (70, 15, 12, 5)$, we can set $B + K = \mathrm{diag}(-1, -2, -1, -3)$ with the matrix

$$K = \begin{bmatrix} 9 & -10 & 0 & 0 \\ -28 & -3 & 0 & 1 \\ 0 & 0 & -11/3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \tag{3.16}$$

so that all the eigenvalues $\lambda_i = (-1, -2, -1, -3)$ of $B + K$ satisfy $|\arg(\lambda_i(B + K))| > 0.95\pi/2$ $(i = 1, 2, \ldots, n)$.

In the numerical simulations, the initial values of the drive and response systems are arbitrarily chosen as $(x_1(0), x_2(0), x_3(0), x_4(0)) = (1, 2, 3, 4)$, and $(y_1(0), y_2(0), y_3(0), y_4(0)) = (0.3, 0.6, 0.3, 1)$. The time evolutions of the states are plotted in Figure 5(a)-(d). Figure 5(e) displays the evolution of the MGPS error $e = (e_1, e_2, e_3, e_4)^T$ which tends to zero as $t \to \infty$, which implies that the error system (3.14) between the drive and response systems (3.11)-(3.12) is globally and asymptotically stable.

What should be mentioned is that the fractional order and control gains, *i.e.*, the elements of K, play an important role in the synchronization rate.

4 Conclusion

In this paper, MGPS of a novel fractional-order chaotic system is presented. First, we numerically study the chaotic behavior of a novel fractional-order four-dimensional Lü system according to the generalized Adams-Bashforth-Moulton predictor-corrector algorithm. Then, based on the stability theory of fractional-order systems, the MGPS method for a class of fractional-order chaotic systems is presented, and a nonlinear controller is given to control the slave system to become a projection of the master system. In MGPS, the drive and response systems could be asymptotically synchronized up to a desired transformation matrix, but not a diagonal matrix.

This synchronization method can be easily extended to other chaotic systems. The classical 'P-C' complete synchronization, anti-synchronization, and generalized projective synchronization can be considered as special cases of our scheme. For verifying the effectiveness and feasibility of the presented synchronization scheme, some numerical simulations were performed.

It is worth mentioning that there are still many interesting problems about chaos synchronization of different fractional-order systems that warrant investigation in future, such as lag MGPS and adaptive MGPS.

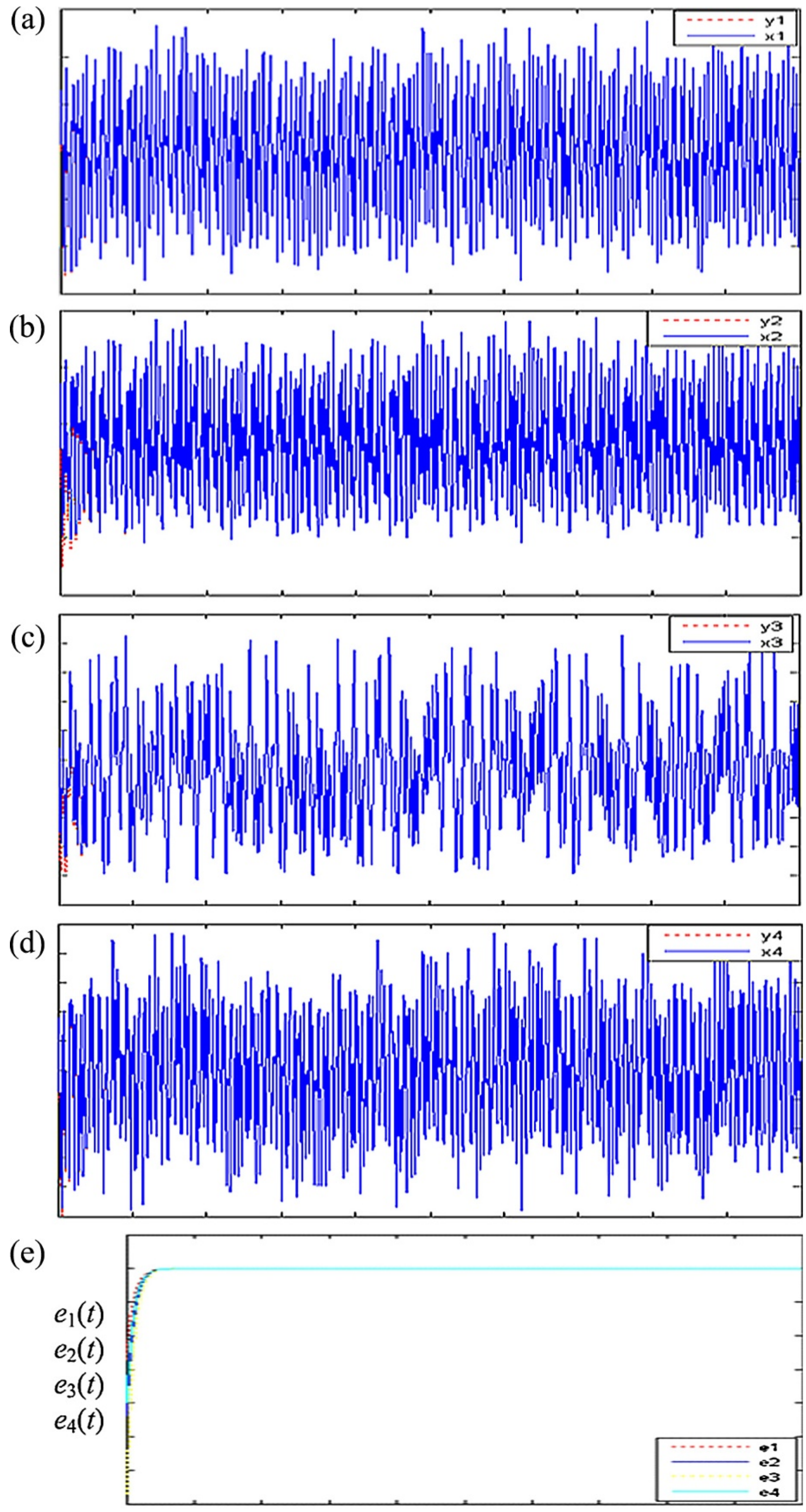

Figure 5 MGPS of the novel fractional-order hyperchaotic Lorenz system (3.12) and Lü system (3.11) for $\alpha = 0$. **(a)** Time response of $x_1(t)$ and $y_1(t)$; **(b)** Time response of $x_2(t)$ and $y_2(t)$; **(c)** Time response of $x_3(t)$ and $y_3(t)$; **(d)** Time response of $x_4(t)$ and $y_4(t)$; **(e)** Time response of synchronization errors.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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References
1. Podlubny, I: Fractional Differential Equations. Academic Press, New York (1999)
2. Hilfer, R: Applications of Fractional Calculus in Physics. World Scientific, River Edge (2001)
3. Bagley, RL, Calico, RA: Fractional-order state equations for the control of viscoelastically damped structures. J. Guid. Control Dyn. **14**(2), 304-311 (1991)
4. Heaviside, O: Electromagnetic Theory. Chelsea, New York (1971)
5. Sun, HH, Abdelwahad, AA, Onaral, B: Linear approximation of transfer function with a pole of fractional-order. IEEE Trans. Autom. Control **29**(5), 441-444 (1984). doi:10.1109/TAC.1984.1103551
6. Laskin, N: Fractional market dynamics. Physica A **287**(3-4), 482-492 (2000)
7. Kunsezov, D, Bulagc, A, Dang, GD: Quantum Levy processes and fractional kinetics. Phys. Rev. Lett. **82**(6), 1136-1139 (1999)
8. Hartley, TT, Lorenzo, CF, Qammer, HK: Chaos in a fractional-order Chua's system. IEEE Trans. Circuits Syst. I **42**(8), 485-490 (1995). doi:10.1109/81.404062
9. Arena, P, Caponetto, R, Fortuna, L, Porto, D: Chaos in a fractional-order Duffing system. In: Proceedings of European Conference on Circuit Theory and Design, vol. 3, pp. 1259-1262 (1997)
10. Ahmad, WM, Sprott, JC: Chaos in fractional-order autonomous nonlinear systems. Chaos Solitons Fractals **16**(2), 339-351 (2003). doi:10.1016/S0960-0779(02)00438-1
11. Sun, KH, Wang, X, Sprott, JC: Bifurcations and chaos in fractional-order simplified Lorenz system. Int. J. Bifurc. Chaos **20**(4), 1209-1219 (2010). doi:10.1142/S0218127410026411
12. Li, CG, Chen, GR: Chaos in the fractional-order Chen system and its control. Chaos Solitons Fractals **22**, 549-554 (2004). doi:10.1016/j.chaos.2004.02.035
13. Li, CG, Chen, GR: Chaos and hyperchaos in the fractional-order Rössler equations. Physica A **341**, 55-61 (2004). doi:10.1016/j.physa.2004.04.113
14. Lu, JG: Chaotic dynamics and synchronization of fractional-order Arneodo's systems. Chaos Solitons Fractals **26**(4), 1125-1133 (2005). doi:10.1016/j.chaos.2005.02.023
15. Lu, JG: Chaotic dynamics of the fractional-order Lü system and its synchronization. Phys. Lett. A **354**(4), 305-311 (2006). doi:10.1016/j.physleta.2006.01.068
16. Wu, XJ, Li, J, Chen, GR: Chaos in the fractional-order unified system and its synchronization. J. Franklin Inst. **345**(4), 392-401 (2008). doi:10.1016/S0960-0779(02)00438-1
17. Yu, YG, Li, HX: The synchronization of fractional-order Rössler hyperchaotic systems. Physica A **387**(5-6), 1393-1403 (2008). doi:10.1016/j.physa.2007.10.052
18. Wang, XY, He, YJ: Projective synchronization of fractional order chaotic system based on linear separation. Phys. Lett. A **372**(4), 435-441 (2008)
19. Mahmoud, GM, Mahmoud, EE, Arafa, AA: On projective synchronization of hyperchaotic complex nonlinear systems based on passive theory for secure communications. Phys. Scr. **87**(5), 055002 (2013). doi:10.1088/0031-8949/87/05/055002
20. Mahmoud, GM, Bountis, T: The dynamics of systems of complex nonlinear oscillators: a review. Int. J. Bifurc. Chaos **14**(11), 3821-3846 (2004). doi:10.1142/S0218127404011624
21. Wu, XJ, Lu, Y: Generalized projective synchronization of the fractional-order Chen hyperchaotic system. Nonlinear Dyn. **57**(1-2), 25-35 (2009). doi:10.1007/s11071-008-9416-5
22. Wu, XJ, Wang, H, Lu, HT: Modified generalized projective synchronization of a new fractional-order hyperchaotic system and its application to secure communication. Nonlinear Anal., Real World Appl. **13**(3), 1441-1450 (2012). doi:10.1016/j.nonrwa.2011.11.008
23. Diethelm, K, Ford, NJ, Freed, AD: A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dyn. **29**(1-4), 3-22 (2002). doi:10.1023/A:1016592219341
24. Diethelm, K: The Analysis of Fractional Differential Equations. An Application-Oriented Exposition Using Differential Operators of Caputo Type. LNM, vol. 2004. Springer, Berlin (2010). doi:10.1007/978-3-642-14574-2
25. Deng, WH: Numerical algorithm for the time fractional Fokker-Planck equation. J. Comput. Phys. **227**(2), 1510-1522 (2007). doi:10.1016/j.jcp.2007.09.015
26. Tavazoei, MS, Haeri, M: A necessary condition for double scroll attractor existence in fractional-order systems. Phys. Lett. A **367**(1-2), 102-113 (2007). doi:10.1016/j.physleta.2007.05.081
27. Dadras, S, Momeni, HR, Qi, GY, Wang, ZL: Four-wing hyperchaotic attractor generated from a new 4D system with one equilibrium and its fractional-order form. Nonlinear Dyn. **67**(2), 1161-1173 (2012)
28. Wang, GY, Zhang, X, Zheng, Y, Li, YX: A new modified hyperchaotic Lü system. Physica A **371**(2), 1913-1926 (2006). doi:10.1016/j.physa.2006.03.048
29. Mahmoud, GM, Ahmed, ME, Sabor, N: On autonomous and nonautonomous modified hyperchaotic complex Lü systems. Int. J. Bifurc. Chaos **21**(7), 1913-1926 (2011). doi:10.1142/S0218127411029525

30. Wolf, A, Swift, JB, Swinney, HL, Vstano, JA: Determining Lyapunov exponent from a time series. Physica D **16**(3), 285-317 (1985). doi:10.1016/0167-2789(85)90011-9
31. Rosenstein, MT, Collins, JJ, De Luca, CJ: A practical method for calculating largest Lyapunov exponents from small data sets. Physica D **65**(1-2), 117-134 (1993). doi:10.1016/0167-2789(93)90009-P
32. Matignon, D: Stability results for fractional differential equations with applications to control processing. In: IMACS, IEEE-SMC, Lille, France (1996)

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