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Existence results for nonlinear fractional differential equations involving different Riemann-Liouville fractional derivatives

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Abstract

By applying an iterative technique, a necessary and sufficient condition is obtained for the existence of the unique solution of nonlinear fractional differential equations involving two Riemann-Liouville derivatives of different fractional orders. Finally, an example is also given to illustrate the availability of our main results.

Keywords: different fractional-order; nonlinear fractional differential equations; Riemann-Liouville derivative; monotone iterative technique

1 Introduction

Recently, the study of fractional differential equations has acquired popularity, see books [1–5] for more information. In this paper, we consider the following nonlinear fractional differential equations:

$$\begin{cases} D^\alpha u(t) = f(t, D^\alpha u(t), D^\beta u(t), u(t)), \\ D^\beta u(0) = 0, \quad u(0) = 0, \end{cases} \tag{1.1}$$

where $t \in J = [0, T]$ $(0 < T < \infty)$, $f \in C(J \times \mathbb{R}^3, \mathbb{R})$, D is the standard Riemann-Liouville fractional derivative, $1 < \alpha \le 2$, $0 < \beta \le 1$ and $0 < \alpha - \beta \le 1$. It is worthwhile to indicate that the nonlinear term f involves the unknown function's Riemann-Liouville fractional derivatives with different orders.

The method of upper and lower solutions coupled with the monotone iterative technique is an interesting and powerful mechanism. The importance and advantage of the method needs no special emphasis [6, 7]. There have appeared some papers dealing with the existence of the solution of nonlinear Riemann-Liouville-type fractional differential equations [8–18] or nonlinear Caputo-type fractional differential equations [19–22] by using the method. For example, by employing the method of lower and upper solutions combined with the monotone iterative technique, Lakshmikanthan and Vatsala [13], McRae [14] and Zhang [17] successfully investigated the initial value problems of Riemann-Liouville fractional differential equation $D^\alpha u(t) = f(t, u(t))$, where $0 < \alpha \le 1$.

However, in the existing literature [8–18], only one case when $\alpha \in (0, 1]$ is considered. The research, involving Riemann-Liouville fractional derivative of order $1 < \alpha \le 2$, proceeds slowly and there appear some new difficulties in employing the monotone iterative method. To overcome these difficulties, we apply a substitution $D^\alpha u(t) = y(t)$. Note that the technique has been discussed for fractional problems in papers [10, 11]. To the best of our knowledge, it is the first paper, in which the monotone iterative method is applied to nonlinear Riemann-Liouville-type fractional differential equations, involving two different fractional derivatives D^α and D^β.

We organize the rest of this paper as follows. In Section 2, by using the monotone iterative technique and the method of upper and lower solutions, the minimal and maximal solutions of an equivalent problem of (1.1) are investigated and two explicit monotone iterative sequences, converging to the corresponding minimal and maximal solution, are given. In addition, the uniqueness of the solution for fractional differential equations (1.1) is discussed. In Section 3, an example is given to illustrate our results.

2 Existence results

Lemma 2.1 *For a given function $y \in C(J,\mathbb{R})$, the following problem*

$$\begin{cases} D^\alpha u(t) = y(t), \\ D^\beta u(0) = u(0) = 0, \end{cases} \tag{2.1}$$

has a unique solution $u(t) = I^\alpha y(t)$, where I is the fractional integral and $I^\alpha y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds$, $1 < \alpha \le 2$, $0 < \beta \le 1$ and $0 < \alpha - \beta \le 1$.

Proof One can reduce equation $D^\alpha u(t) = y(t)$ to an equivalent integral equation

$$u(t) = I^\alpha y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} \tag{2.2}$$

for some $c_1, c_2 \in \mathbb{R}$.

By $u(0) = 0$, it follows $c_2 = 0$. Consequently, the general solution of (2.2) is

$$u(t) = I^\alpha y(t) + c_1 t^{\alpha-1}. \tag{2.3}$$

Thus, we have

$$\begin{aligned} D^\beta u(t) &= I^{\alpha-\beta} y(t) + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} \\ &= \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} y(s)\,ds + c_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}. \end{aligned} \tag{2.4}$$

By the condition $D^\beta u(0) = 0$, it follows that $c_1 = 0$. Therefore, we have $u(t) = I^\alpha y(t)$.

Conversely, by a direct computation, we can get $D^\alpha u(t) = y(t)$ and $D^\beta u(t) = I^{\alpha-\beta} y(t)$. It is easy to verify $u(t) = I^\alpha y(t)$ satisfies (2.1).

This completes the proof. □

Combined with Lemma 2.1, we see that (1.1) can be translated into the following system

$$y(t) = f\bigl(t, y(t), I^{\alpha-\beta} y(t), I^\alpha y(t)\bigr), \tag{2.5}$$

where $y(t) = D^\alpha u(t)$, $\forall t \in J$ and I^α, $I^{\alpha-\beta}$ are the standard fractional integrals.

Now, we list for convenience the following condition:

(H_1) There exist $y_0, z_0 \in C(J, \mathbb{R})$ satisfying $y_0 \leq z_0$ such that

$$\begin{cases} y_0(t) \leq f(t, y_0(t), I^{\alpha-\beta} y_0(t), I^{\alpha} y_0(t)), \\ z_0(t) \geq f(t, z_0(t), I^{\alpha-\beta} z_0(t), I^{\alpha} z_0(t)). \end{cases}$$

(H_2) There exists a function $M \in C(J, (-1, +\infty))$ such that

$$f\big(t, u(t), I^{\alpha-\beta} u(t), I^{\alpha} u(t)\big) - f\big(t, v(t), I^{\alpha-\beta} v(t), I^{\alpha} v(t)\big) \geq -M(t)(u-v)(t),$$

where $y_0 \leq v \leq u \leq z_0$, $\forall t \in J$.

(H_3) There exist functions $N, K, L \in C(J, [0, +\infty))$ such that

$$\begin{aligned} &f\big(t, u(t), I^{\alpha-\beta} u(t), I^{\alpha} u(t)\big) - f\big(t, v(t), I^{\alpha-\beta} v(t), I^{\alpha} v(t)\big) \\ &\quad \leq N(t)(u-v)(t) + K(t) I^{\alpha-\beta}(u-v)(t) + L(t) I^{\alpha}(u-v)(t), \end{aligned}$$

where $y_0 \leq v \leq u \leq z_0$, $\forall t \in J$.

Theorem 2.1 *Assume that* (H_1) *and* (H_2) *hold. Then problem* (2.5) *has the minimal and maximal solution* y^*, z^* *in the ordered interval* $[y_0, z_0]$. *Moreover, there exist explicit monotone iterative sequences* $\{y_n\}, \{z_n\} \subset [y_0, z_0]$ *such that* $\lim_{n\to\infty} y_n(t) = y^*(t)$ *and* $\lim_{n\to\infty} z_n(t) = z^*(t)$, *where* $y_n(t)$, $z_n(t)$ *are defined as*

$$\begin{aligned} y_n(t) &= \frac{1}{1+M(t)}\big[f\big(t, y_{n-1}(t), I^{\alpha-\beta} y_{n-1}(t), I^{\alpha} y_{n-1}(t)\big) + M(t) y_{n-1}(t)\big], \\ &\quad \forall t \in J, n = 1, 2, \ldots, \\ z_n(t) &= \frac{1}{1+M(t)}\big[f\big(t, z_{n-1}(t), I^{\alpha-\beta} z_{n-1}(t), I^{\alpha} z_{n-1}(t)\big) + M(t) z_{n-1}(t)\big], \\ &\quad \forall t \in J, n = 1, 2, \ldots, \end{aligned} \tag{2.6}$$

and

$$y_0 \leq y_1 \leq \cdots \leq y_n \leq \cdots \leq y^* \leq z^* \leq \cdots \leq z_n \leq \cdots \leq z_1 \leq z_0. \tag{2.7}$$

Proof Define an operator $Q : [y_0, z_0] \to C(J, \mathbb{R})$ by $x = Q\eta$, where x is the unique solution of the corresponding linear problem corresponding to $\eta \in [y_0, z_0]$ and

$$Q\eta = \frac{1}{1+M(t)}\big[f\big(t, \eta(t), I^{\alpha-\beta}\eta(t), I^{\alpha}\eta(t)\big) + M(t)\eta(t)\big]. \tag{2.8}$$

Then, the operator Q has the following properties:

$$\begin{aligned} &\text{(a)} \quad y_0 \leq Qy_0, \qquad Qz_0 \leq z_0; \\ &\text{(b)} \quad Qh_1 \leq Qh_2, \quad \forall h_1, h_2 \in [y_0, z_0], h_1 \leq h_2. \end{aligned} \tag{2.9}$$

Firstly, we show that (a) holds. Let $y_1 = Qy_0$, $p = y_1 - y_0$. By (H_1) and the definition of Q, we know that

$$\begin{aligned} p(t) &= \frac{1}{1+M(t)}\bigl[f\bigl(t, y_0(t), I^{\alpha-\beta}y_0(t), I^{\alpha}y_0(t)\bigr) + M(t)y_0(t)\bigr] - y_0(t) \\ &\geq \frac{1}{1+M(t)}\bigl[y_0(t) + M(t)y_0(t)\bigr] - y_0(t) \\ &= 0. \end{aligned}$$

Thus, we can obtain $p(t) \geq 0$, $\forall t \in J$. That is, $y_0 \leq Qy_0$. Similarly, we can prove that $Qz_0 \leq z_0$. Then, (a) holds.

Secondly, let $q = Qh_2 - Qh_1$, by (2.8) and (H_2), we have

$$\begin{aligned} q(t) &= \frac{1}{1+M(t)}\bigl[f\bigl(t, h_2(t), I^{\alpha-\beta}h_2(t), I^{\alpha}h_2(t)\bigr) + M(t)h_2(t)\bigr] \\ &\quad - \frac{1}{1+M(t)}\bigl[f\bigl(t, h_1(t), I^{\alpha-\beta}h_1(t), I^{\alpha}h_1(t)\bigr) + M(t)h_1(t)\bigr] \\ &\geq \frac{1}{1+M(t)}\bigl[-M(t)(h_2 - h_1)(t) + M(t)(h_2 - h_1)(t)\bigr] \\ &= 0. \end{aligned}$$

Hence, we have $q(t) \geq 0$, $\forall t \in J$. That is, $Qh_2 \geq Qh_1$. Then, (b) holds.

Now, put

$$y_n = Qy_{n-1}, \qquad z_n = Qz_{n-1}, \quad n = 1, 2, \ldots. \tag{2.10}$$

By (2.9), we can get

$$y_0 \leq y_1 \leq \cdots \leq y_n \leq \cdots \leq z_n \leq \cdots \leq z_1 \leq z_0.$$

Obviously, y_n, z_n satisfy

$$\begin{aligned} y_n(t) &= f\bigl(t, y_{n-1}(t), I^{\alpha-\beta}y_{n-1}(t), I^{\alpha}y_{n-1}(t)\bigr) - M(t)(u_n - y_{n-1})(t), \\ z_n(t) &= f\bigl(t, z_{n-1}(t), I^{\alpha-\beta}z_{n-1}(t), I^{\alpha}z_{n-1}(t)\bigr) - M(t)(z_n - z_{n-1})(t). \end{aligned} \tag{2.11}$$

Employing the same arguments used in Ref. [17], we see that $\{y_n\}$, $\{z_n\}$ converge to their limit functions y^*, z^*, respectively. That is, $\lim_{n\to\infty} y_n(t) = y^*(t)$ and $\lim_{n\to\infty} z_n(t) = z^*(t)$. Moreover, $y^*(t)$, $z^*(t)$ are solutions of (2.5) in $[y_0, z_0]$. (2.7) is true.

Finally, we prove that $y^*(t)$, $z^*(t)$ are the minimal and the maximal solution of (2.5) in $[y_0, z_0]$. Let $w \in [y_0, z_0]$ be any solution of (2.5), then $Qw = w$. By $y_0 \leq w \leq z_0$, (2.9) and (2.10), we can obtain

$$y_n \leq w \leq z_n, \quad n = 1, 2, \ldots. \tag{2.12}$$

Thus, taking limit in (2.12) as $n \to +\infty$, we have $y^* \leq w \leq z^*$. That is, y^*, z^* are the minimal and maximal solution of (2.5) in the ordered interval $[y_0, z_0]$, respectively.

This completes the proof. □

Theorem 2.2 *Let $N(t) \geq -M(t)$. Assume conditions* (H_1)-(H_3) *hold. If*

$$\lambda(t) = N(t) + \frac{K(t)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{L(t)t^{\alpha}}{\Gamma(\alpha+1)} < 1,$$

then problem (2.5) *has a unique solution* $x(t) \in [y_0, z_0]$.

Proof By Theorem 2.1, we have proved that y^*, z^* are the minimal and maximal solution of (2.5) and

$$y_0(t) \leq y^*(t) \leq z^*(t) \leq z_0(t), \quad \forall t \in J.$$

Now, we are going to show that problem (2.5) has a unique solution x, *i.e.*, $y^*(t) = z^*(t) = x(t)$.

Let $p(t) = z^*(t) - y^*(t)$, by (H_3), we have

$$\begin{aligned}
0 \leq p(t) &\leq f\big(t, z^*(t), I^{\alpha-\beta}z^*(t), I^{\alpha}z^*(t)\big) - f\big(t, y^*(t), I^{\alpha-\beta}y^*(t), I^{\alpha}y^*(t)\big) \\
&\leq N(t)\big(z^* - y^*\big)(t) + K(t)I^{\alpha-\beta}\big(z^* - y^*\big)(t) + L(t)I^{\alpha}\big(z^* - y^*\big)(t) \\
&= N(t)p(t) + K(t)\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}p(s)\,ds + L(t)\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}p(s)\,ds \\
&\leq \left[N(t) + \frac{K(t)t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{L(t)t^{\alpha}}{\Gamma(\alpha+1)}\right]\max_{t\in J} p(t) \\
&\triangleq \lambda(t)\max_{t\in J} p(t),
\end{aligned}$$

which implies that $\max_{t\in J} p(t) \leq 0$. Since $p(t) \geq 0$, then it holds $p(t) = 0$. That is, $y^*(t) = z^*(t)$. Therefore, problem (2.5) has a unique solution $x \in [y_0, z_0]$. □

Let $x(t)$ be the unique solution of (2.5). Noting that $x \in [y_0, z_0]$ and $u(t) = I^{\alpha}x(t)$, we can easily obtain the following theorem.

Theorem 2.3 *Let all conditions of Theorem* 2.2 *hold. Then problem* (1.1) *has a unique solution* $u \in [I^{\alpha}y_0, I^{\alpha}z_0]$, $\forall t \in J$.

3 Example

Consider the following problem:

$$\begin{cases} D^{\frac{3}{2}}u(t) = \frac{t}{10}[1 - D^{\frac{3}{2}}u(t)]^2 + \frac{t^2}{5}D^{\frac{3}{2}}u(t) + \frac{t^2}{15}[1 - D^{\frac{1}{2}}u(t)]^3 + \frac{t^3}{20}u^2(t), \\ D^{\frac{1}{2}}u(0) = 0, \quad u(0) = 0, \end{cases} \tag{3.1}$$

where $t \in [0, 1]$.

Let $D^{\frac{3}{2}}u(t) = y(t)$, then $D^{\frac{1}{2}}u(t) = I^1 y(t)$, $u(t) = I^{\frac{3}{2}}y(t)$. So, (3.1) can be translated into the following problem

$$y(t) = \frac{t}{10}\big[1 - y(t)\big]^2 + \frac{t^2}{5}y(t) + \frac{t^2}{15}\big[1 - I^1 y(t)\big]^3 + \frac{t^3}{20}\big(I^{\frac{3}{2}}y(t)\big)^2, \tag{3.2}$$

Noting that $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, then

$$f\bigl(t, y, I^{\alpha-\beta}y, I^{\alpha}y\bigr) = \frac{t}{10}[1-y]^2 + \frac{t^2}{5}y + \frac{t^2}{15}\bigl[1 - I^1 y\bigr]^3 + \frac{t^3}{20}\bigl(I^{\frac{3}{2}}y\bigr)^2.$$

Take $y_0(t) = 0$, $z_0(t) = 1$, we have

$$\begin{cases} y_0(t) = 0 \le \frac{t}{10} + \frac{t^2}{15} = f(t, y_0(t), I^{\alpha-\beta}y_0(t), I^{\alpha}y_0(t)), \\ z_0(t) = 1 \ge \frac{t^2}{5}y + \frac{t^2}{15}(1-t)^3 + \frac{4t^6}{45\pi} = f(t, z_0(t), I^{\alpha-\beta}z_0(t), I^{\alpha}z_0(t)). \end{cases}$$

Hence, condition (H_1) holds.

For $y_0 \le y \le z \le z_0$, we have

$$\begin{aligned} &f\bigl(t, z, I^{\alpha-\beta}z, I^{\alpha}z\bigr) - f\bigl(t, y, I^{\alpha-\beta}y, I^{\alpha}y\bigr) \\ &\quad = \frac{t}{10}\bigl[(1-z)^2 - (1-y)^2\bigr] + \frac{t^2}{5}(z-y) \\ &\qquad + \frac{t^2}{15}\bigl[\bigl(1 - I^1 z\bigr)^3 - \bigl(1 - I^1 y\bigr)^3\bigr] + \frac{t^3}{20}\bigl[\bigl(I^{\frac{3}{2}}z\bigr)^2 - \bigl(I^{\frac{3}{2}}y\bigr)^2\bigr] \\ &\quad \ge -\frac{t - t^2}{5}(z-y) \end{aligned}$$

and

$$f\bigl(t, z, I^{\alpha-\beta}z, I^{\alpha}z\bigr) - f\bigl(t, y, I^{\alpha-\beta}y, I^{\alpha}y\bigr) \le -\frac{t^2}{5}(z-y) + \frac{t^2}{5}I^1(z-y) + \frac{2t^3}{15\sqrt{\pi}}I^{\frac{3}{2}}(z-y).$$

Take $M(t) = \frac{t-t^2}{5}$, $N(t) = K(t) = \frac{t^2}{5}$, $L(t) = \frac{2t^3}{15\sqrt{\pi}}$. Through a simple calculation, we have

$$\lambda(t) = \frac{t^2}{5} + \frac{t^3}{5} + \frac{8t^{\frac{9}{2}}}{45\pi} < 1.$$

Then, all conditions of Theorem 2.3 are satisfied. In consequence, the problem (3.1) has a unique solution $u^* \in [0, \frac{4t^{\frac{3}{2}}}{3\sqrt{\pi}}]$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have equal contributions.

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