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Existence and multiplicity of solutions for a second-order impulsive differential equation via variational methods

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Abstract

In this paper, we study the existence and multiplicity of solutions for an impulsive differential equation via some critical point theory and the variational method. We extend and improve some recent results and reduce conditions.

Keywords: impulsive differential equation; critical point theory; variational structure

1 Introduction

As an important research field of study, the impulsive differential equation has been attracting the attention of several mathematicians. In the early years, the main way to resolve this kind of problems is based on the fixed point theory, the theorem of topological degree, the upper and lower solutions method coupled with the monotone iterative technique, and so on; see for example [1–3]. Recently, many authors have tried to use the variational method and some specific critical point theorems, such as mountain pass lemma, fountain theorem, linking theorem, symmetric mountain pass lemma, and so on, to study the existence (see [4–9]) and multiplicity (see [10–15]) of solutions for some impulsive differential equations.

In [5], authors have shown the variational structure of an impulsive differential equation and proved the existence of a solution by using the mountain pass lemma.

In [10], the authors studied the following equation:

$$\begin{cases} -(p(t)u'(t))' + r(t)u'(t) + q(t)u(t) = f(t, u(t)), & \text{a.e. } t \in J, \\ -\Delta(p(t)u'(t_i)) = I_i(u(t_i)), & i = 1, 2, \dots, p, \\ u(0) = 0, \quad \alpha_1 u(t) + u'(1) = 0, \end{cases}$$

where $J = [0, 1]$, $0 = t_0 < t_1 < t_2 < \dots < t_p < t_{p+1} = 1$, $f \in C[J \times \mathbb{R}, \mathbb{R}]$, $p \in C[J, \mathbb{R}^+]$, $q, r \in C[0, T]$.

Then *Zhang* in [11] proved the existence of two solutions and the existence of infinitely many solutions of problem given by

$$\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), & \text{a.e. } t \in [0, T], \\ \Delta u'(t_i) = I_i(u(t_i)), & i = 1, 2, \dots, p, \\ u(0) = u(T) = 0. \end{cases}$$

In this paper, we study the existence and multiplicity of solutions for the following non-linear impulsive problem:

$$\begin{cases} -u''(t) + r(t)u'(t) + \lambda u(t) = f(t, u(t)), & \text{a.e. } t \in J, \\ \Delta u'(t_i) = u'(t_i+) - u'(t_i-) = I_i(u(t_i)), & i = 1, 2, \dots, p, \\ u(0) = u(T) = 0, \end{cases} \quad (1)$$

where $J = [0, T]$, $0 = t_0 < t_1 < t_2 < \dots < t_p < t_{p+1} = T$, $r \in C[0, T]$, $I_i \in C[R, R]$, λ is a parameter, $f \in C[J \times R, R]$, with $F(t, u) = \int_0^u f(t, \xi) d\xi$.

We will prove that equation (1) has at least two classical solutions and infinitely many classical solutions under different conditions. Our main results extend the existing result in [5, 10, 11]. We prove the same impulsive problem in [5] cannot only have two solutions but also have infinitely many classical solutions. Compared with [10], we do not require the impulsive functions I_j and F to satisfy the sublinear growth condition and the superlinear growth condition about u^μ , which such that the problem more general. Different from [11] in which F is a negative function, in this paper, our results relax the restriction of F with a wider range of applications.

The rest of the paper is organized as follows: In Section 2, we give the variational structure and several important lemmas. The main theorems are formulated and proved in Section 3.

2 Preliminaries

Let $R(t) = \int_0^t r(s) ds$, $M = \max_{t \in [0, T]} e^{-R(t)}$, $m = \min_{t \in [0, T]} e^{-R(t)}$. Multiplying the first equation of (1) by $e^{-R(t)}$, we obtain

$$\begin{cases} -(e^{-R(t)}u'(t))' + \lambda e^{-R(t)}u(t) = e^{-R(t)}f(t, u(t)), & \text{a.e. } t \in J, \\ \Delta u'(t_i) = I_i(u(t_i)), & i = 1, 2, \dots, p, \\ u(0) = u(T) = 0. \end{cases} \quad (2)$$

Obviously, the solutions of equation (2) are solutions of equation (1). Consider the Hilbert space $H_0^1(0, T)$ with the inner product and norm

$$\begin{aligned} (u, v) &= \int_0^T e^{-R(t)} u'(t) v'(t) dt, \\ \|u\| &= \left(\int_0^T e^{-R(t)} (u'(t))^2 dt \right)^{1/2}. \end{aligned}$$

Multiply the first equation of (2) by $v \in H_0^1(0, T)$, integrate it on the interval $[0, T]$. It follows from the boundary conditions $u(0) = 0$, $u(T) = 0$ that

$$\begin{aligned} & \int_0^T e^{-R(t)} u'(t) v'(t) dt + \lambda \int_0^T e^{-R(t)} u(t) v(t) dt + \sum_{i=1}^p e^{-R(t_i)} I_i(u(t_i)) v(t_i) \\ &= \int_0^T e^{-R(t)} f(t, u(t)) v(t) dt. \end{aligned}$$

Now, define the function $\varphi : H_0^1(0, T) \rightarrow \mathbb{R}$

$$\begin{aligned} \varphi(u) &= L(u, u) + \sum_{i=1}^p e^{-R(t_i)} \int_0^{u(t_i)} I_i(s) ds \\ &\quad - \int_0^T e^{-R(t)} F(t, u(t)) dt, \end{aligned}$$

where

$$L(u, v) = \frac{1}{2} \int_0^T e^{-R(t)} u'(t) v'(t) dt + \frac{\lambda}{2} \int_0^T e^{-R(t)} u(t) v(t) dt,$$

and for all $v \in H_0^1(0, T)$, we have

$$\begin{aligned} \varphi'(u)v &= \int_0^T e^{-R(t)} u'(t) v'(t) dt + \lambda \int_0^T e^{-R(t)} u(t) v(t) dt \\ &\quad + \sum_{i=1}^p e^{-R(t_i)} I_i(u(t_i)) v(t_i) - \int_0^T e^{-R(t)} f(t, u(t)) v(t) dt. \end{aligned} \quad (3)$$

Definition 1 A weak solution of problem (1) is a function $u \in H_0^1(0, T)$ such that the integral equation (3) holds for all $v \in H_0^1(0, T)$.

Let $\lambda_k (k = 1, 2, \dots)$ be the eigenvalue of the following Dirichlet problem, where $\lambda_1 = \frac{\pi^2}{T^2}$ is the first eigenvalue:

$$\begin{cases} -u''(t) = \lambda u(t), & t \in [0, T], \\ u(0) = u(T) = 0. \end{cases} \quad (4)$$

We assume X_k is the feature space corresponding with λ_k , then $H_0^1(0, T) = \overline{\bigoplus_{i \in \mathbb{N}} X_i}$.

Lemma 1 *There exists a constant $C > 0$, such that $\|u\|_\infty \leq C\|u\|$, where $\|u\|_\infty = \max_{t \in [0, T]} |u(t)|$.*

Lemma 2 ([5]) *If $\lambda > -\frac{m\lambda_1}{M}$, then there exist $0 < a_1 < a_2$, such that*

$$a_1 \|u\|^2 \leq L(u, u) \leq a_2 \|u\|^2.$$

Remark 1 If $|\lambda| < \frac{m\lambda_1}{M}$, then $0 < a_1 < a_2 < 1$. In fact, if $\lambda > 0$, then let $a_1 = \frac{1}{2}$, and by the Poincaré inequality we have

$$\begin{aligned} L(u, u) &= \frac{1}{2} \int_0^T e^{-R(t)} (u'(t))^2 dt + \frac{\lambda}{2} \int_0^T e^{-R(t)} u^2(t) dt \\ &\leq \frac{1}{2} \|u\|^2 + \frac{\lambda M}{2} \int_0^T u^2(t) dt \\ &\leq \frac{1}{2} \|u\|^2 + \frac{\lambda M}{2\lambda_1} \int_0^T (u'(t))^2 dt \\ &\leq \frac{1}{2} \|u\|^2 + \frac{\lambda M}{2m\lambda_1} \int_0^T e^{-R(t)} (u'(t))^2 dt \\ &= \left(\frac{1}{2} + \frac{\lambda M}{2m\lambda_1} \right) \|u\|^2. \end{aligned}$$

Then letting $a_2 = \frac{1}{2} + \frac{\lambda M}{2m\lambda_1}$, we can get the result. Similarly, if $-\frac{m\lambda_1}{M} < \lambda < 0$, then let $a_2 = \frac{1}{2}$, and by the Poincaré inequality we have

$$\begin{aligned} L(u, u) &= \frac{1}{2} \int_0^T e^{-R(t)} (u'(t))^2 dt + \frac{\lambda}{2} \int_0^T e^{-R(t)} u^2(t) dt \\ &\geq \frac{1}{2} \|u\|^2 + \frac{\lambda M}{2} \int_0^T u^2(t) dt \\ &\geq \frac{1}{2} \|u\|^2 + \frac{\lambda M}{2\lambda_1} \int_0^T (u'(t))^2 dt \\ &\geq \frac{1}{2} \|u\|^2 + \frac{\lambda M}{2m\lambda_1} \int_0^T e^{-R(t)} (u'(t))^2 dt \\ &= \left(\frac{1}{2} + \frac{\lambda M}{2m\lambda_1} \right) \|u\|^2. \end{aligned}$$

We can also let $a_1 = \frac{1}{2} + \frac{\lambda M}{2m\lambda_1}$ to get the result.

Lemma 3 ([16]) *For the function $F : M \subseteq E \rightarrow \mathbb{R}$ with $M \neq \emptyset$, $\min_{u \in M} F(u) = \alpha$ has a solution in the case that the following hold:*

- (F₁) *E is a real reflexive Banach space,*
- (F₂) *M is bounded and weak sequentially closed,*
- (F₃) *F is weak sequentially lower semi-continuous on M , i.e., by definition, for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \rightarrow \infty$, we have $F(u) \leq \liminf_{n \rightarrow \infty} F(u_n)$.*

Lemma 4 ([6]) *Let E be a real Banach space with $E = V \oplus W$, where V is finite-dimensional. Suppose $\phi \in C'(E, \mathbb{R})$ satisfies P.S. condition, and:*

- (ϕ_1) *there are constants $\rho, \tau > 0$ such that $\phi|_{\partial B_\rho \cap W} \geq \tau$, and*
- (ϕ_2) *there is $e \in \partial B_1 \cap W$ and $R > \rho$ such that if $Q = (\overline{B_R} \cap V) \oplus \{re | 0 < r < R\}$, then $\phi|_{\partial Q} \leq 0$.*

Then ϕ possesses a critical value $c \geq \tau$ which can be characterized as $c = \inf_{h \in \Gamma} \max_{u \in Q} \phi(h(u))$, where $\Gamma = \{h \in C(\overline{Q}, E), h|_{\partial Q} = id\}$.

Lemma 5 ([17]) *Let E be an infinite-dimensional real Banach space and $\varphi \in C^1(E, \mathbb{R})$ be even, satisfying the P.S. condition and $u(0) = 0$. If $E = V \oplus W$, where V is finite-dimensional, and φ satisfies the following conditions:*

(φ_1) *there exist constants $\rho, \tau > 0$, such that $\varphi|_{\partial B_\rho \cap W} \geq \tau$,*

(φ_2) *for each finite-dimensional subspace $V_1 \subset E$, there is $R = R(V_1)$ such that, for all $u \in \{u \in V_1, \|u\| \geq R\}$, we have $\varphi(u) \leq 0$.*

Then φ has an unbounded sequence of critical values.

3 Main results

In this paper, we assume:

(H_1) There exists $\mu > 2$ such that $\mu F(t, u) \leq u f(t, u)$ and $I_i(u)u \leq \mu \int_0^u I_i(s) ds < 0$.

(H_2) There exists $\rho_0 > 0, \delta_i > 0$ such that $\int_0^u I_i(s) ds \geq -\delta_i |u|^\mu$ for all $\|u\| \leq \rho_0$.

(H_3) For all $u \in X_1 \cup X_2$, we have $\int_0^T F(t, u(t)) dt \geq \frac{a_2 \|u\|^2}{m}$.

The main results are the following theorems.

Theorem 1 *Assume (H_1)-(H_3) hold and $|\lambda| < \frac{m\lambda_1}{M}$, then the impulsive problem (1) has at least two weak solutions.*

Theorem 2 *Suppose that (H_1)-(H_2) are satisfied and $\lambda \geq -\frac{m\lambda_1}{M}$. If $f(t, u)$ and $I_i(u)$ are odd about u , then the problem (1) has infinitely many weak solutions.*

Next, we give the main lemma used in this paper.

Lemma 6 ([12]) *Suppose that (H_1) holds, then for all $t \in [0, T]$ we can obtain*

$$\begin{aligned} F(t, u) &\leq F\left(t, \frac{u}{|u|}\right) |u|^\mu, \quad 0 < |u| \leq 1, \\ F(t, u) &\geq F\left(t, \frac{u}{|u|}\right) |u|^\mu, \quad |u| \geq 1. \end{aligned} \quad (5)$$

Remark 2 For the convenience of the reader, we denote $M_1 = \sup\{F(t, u), t \in [0, T], |u| = 1\}$, $m_1 = \inf\{F(t, u), t \in [0, T], |u| = 1\}$.

Lemma 7 *Suppose that (H_1) and (H_2) hold, then φ satisfies the P.S. condition.*

Proof Let $\{u_n\} \in H_0^1(0, T)$ be such a sequence that $\{\varphi(u_n)\}$ is bounded and $\lim_{n \rightarrow \infty} \varphi'(u_n) = 0$, we will show that u_n has a convergent subsequence. In view of the given condition, there exists $C_1 > 0$ such that

$$|\varphi(u_n)| \leq C_1, \quad \|\varphi'(u_n)\| \leq C_1,$$

for all n . By the definition of φ and (3) we obtain

$$\begin{aligned} &\left(1 - \frac{2}{\mu}\right) \left(\int_0^T e^{-R(t)} (u'(t))^2 dt + \int_0^T e^{-R(t)} u^2(t) dt \right) \\ &= 2\varphi(u) - 2 \sum_{i=1}^p e^{-R(t_i)} \int_0^{u_n(t_i)} I_i(s) ds + 2 \int_0^T e^{-R(t)} F(t, u_n(t)) dt \end{aligned}$$

$$\begin{aligned}
& -\frac{2}{\mu}\varphi'(u_n)u_n + \frac{2}{\mu}\sum_{i=1}^p e^{-R(t)}I_i(u_n(t_i))u_n - \frac{2}{\mu}\int_0^T e^{-R(t)}f(t, u_n(t))u_n(t) dt \\
& = 2C_1 + \frac{2}{\mu}C_1\|u_n\| + 2\sum_{i=1}^p e^{-R(t)}\left[\frac{1}{\mu}I_i(u_n(t_i))u_n - \int_0^{u_n(t_i)} I_i(s) ds\right] \\
& \quad + 2\int_0^T e^{-R(t)}\left[F(t, u_n(t)) - \frac{1}{\mu}f(t, u_n(t))u_n(t)\right] dt.
\end{aligned}$$

Combining (H_1) , we have

$$\left(1 - \frac{2}{\mu}\right)2a_1\|u_n\|^2 \leq 2C_1 + \frac{2}{\mu}C_1\|u_n\|.$$

This implies that $\{u_n\}$ is bounded and, if necessary going to a sequence we can suppose that $u_n \rightharpoonup u \in H_0^1(0, T)$. By (3) and Lemma 2 we have

$$\begin{aligned}
(\varphi'(u_n) - \varphi'(u))(u_n - u) &= \int_0^T e^{-R(t)}(u'_n(t) - u'(t))^2 dt + \lambda \int_0^T e^{-R(t)}(u_n - u_n(t))^2 dt \\
&\quad + \sum_{i=1}^p e^{-R(t)}[I_i(u_n(t_i)) - I_i(u(t_i))](u_n(t) - u(t)) \\
&\quad - \int_0^T e^{-R(t)}[f(t, u_n(t)) - f(t, u(t))](u_n(t) - u(t)) dt \\
&\geq 2a_1\|u_n - u\|^2 + \sum_{i=1}^p e^{-R(t)}[I_i(u_n(t_i)) - I_i(u(t_i))](u_n(t) - u(t)) \\
&\quad - \int_0^T e^{-R(t)}[f(t, u_n(t)) - f(t, u(t))](u_n(t) - u(t)) dt. \quad (6)
\end{aligned}$$

Hence, $u_n \rightarrow u$ in $C[0, T]$. Furthermore

$$\begin{aligned}
& (\varphi'(u_n) - \varphi'(u))(u_n - u) \rightarrow 0, \\
& \sum_{i=1}^p e^{-R(t)}[I_i(u_n(t_i)) - I_i(u(t_i))](u_n(t) - u(t)) \rightarrow 0, \\
& \int_0^T e^{-R(t)}[f(t, u_n(t)) - f(t, u(t))](u_n(t) - u(t)) dt \rightarrow 0.
\end{aligned}$$

Combining with (6), we know $\|u_n - u\|^2 \rightarrow 0$. So, φ satisfies the P.S. condition. \square

Proof of Theorem 1 Because $H_0^1(0, T)$ is Hilbert space, $\overline{B_\rho}$ is bounded and weak sequentially closed for all $\rho > 0$. We will show φ is weak sequentially lower semi-continuous on $\overline{B_\rho}$. In fact suppose $u_n \rightharpoonup u$ in $H_0^1(0, T)$, then $\|u\| \leq \liminf_{n \rightarrow \infty} \|u_n\|$, and $u_n \rightarrow u$ in $C[0, T]$, so

$$\begin{aligned}
& \frac{\lambda}{2}\int_0^T e^{-R(t)}u_n^2(t) dt + \sum_{i=1}^p e^{-R(t)}\int_0^{u_n(t_i)} I_i(s) ds - \int_0^T e^{-R(t)}F(t, u_n(t)) dt \\
& \rightarrow \frac{\lambda}{2}\int_0^T e^{-R(t)}u^2(t) dt + \sum_{i=1}^p e^{-R(t)}\int_0^{u(t_i)} I_i(s) ds - \int_0^T e^{-R(t)}F(t, u(t)) dt.
\end{aligned}$$

By the definition of φ , we can obtain $\varphi(u) \leq \liminf_{n \rightarrow \infty} \varphi(u_n)$. From Lemma 3, there exists u_0 such that $\varphi(u_0) = \min\{\varphi(u), u \in \overline{B_\rho}\}$. Noting that $\varphi(0) = 0$, so $\varphi(u_0) \leq 0$. Combining with Lemma 2, Lemma 6, and (H_2) , there exists a small ρ with $\rho < \rho_0$ such that when $u(t) \in \partial B_\rho$ we have

$$\begin{aligned} \varphi(u) &\geq a_1 \|u\|^2 + \sum_{i=1}^p e^{-R(t)} \int_0^{u(t_i)} I_i(s) ds - \int_0^T e^{-R(t)} F(t, u(t)) dt \\ &\geq a_1 \|u\|^2 - \sum_{i=1}^p e^{-R(t)} \delta_i |u|^\mu - m_1 \int_0^T e^{-R(t)} |u(t)|^\mu dt \\ &\geq a_1 \|u\|^2 - \left(\sum_{i=1}^p \delta_i M + \delta M m_1 T \right) C \|u\|^\mu \\ &= a_1 \rho^2 - \left(\sum_{i=1}^p \delta_i M + \delta M m_1 T \right) C \rho^\mu = \tau > 0. \end{aligned}$$

$\varphi(u_0) \leq 0$ implies that φ possesses a critical point $u_0 \in B_\rho$.

Let $V = X_1 \oplus X_2$, $W = \bigoplus_{i=3}^\infty X_i$, then there exists $\rho, \tau > 0$, such that

$$\varphi|_{\partial B_\rho \cap W} \geq \tau.$$

Hence φ satisfies the condition (ϕ_1) .

Taking $e \in W$ such that $\|e\| = 1$. By Lemma 6, we claim that there exist $C_2, M_2 > 0$, such that

$$\|e\| \leq C_2 |e|_{L^2}, \quad \|u\| \leq C_2 |u|_{L^2}, \quad u \in V,$$

$$F(t, u) \geq C_2^2 |u|^2 - M_2, \quad (t, u) \in [0, T] \times R.$$

By (H_1) , for all $r > 0$, $u \in V$, we have

$$\begin{aligned} \varphi(re + u) &= L(re + u, re + u) + \sum_{i=1}^p e^{-R(t)} \int_0^{re+u} I_i(s) ds - \int_0^T e^{-R(t)} F(t, re + u) dt \\ &\leq a_2 \|re + u\|^2 - \int_0^T e^{-R(t)} (C_2^2 |re + u|^2 - M_2) dt \\ &\leq a_2 r^2 + a_2 \|u\|^2 + 2a_2 r \|u\| + TM_2 M - C_2^2 (|re|_{L^2}^2 + |u|_{L^2}^2 - 2|re|_{L^2} |u|_{L^2}) \\ &\leq a_2 r^2 + a_2 \|u\|^2 + 2a_2 r \|u\| + TM_2 M - r^2 - \|u\|^2 + 2r \|u\| \\ &= -(1 - a_2) r^2 - (1 - a_2) \|u\|^2 + 2(a_2 + 1) r \|u\| + TM_2 M. \end{aligned}$$

Combining with $a_2 < 1$, there exists $R > 0$, such that the following conclusions hold:

when $\|u\| \geq R$, $\varphi(re + u) < 0$ for all $0 < r < R$;

when $r \geq R$, $\varphi(re + u) < 0$ for all $\|u\| \leq R$.

For any $u(t) \in V$, $\|u\| \leq R$, we can obtain

$$\varphi(u) \leq a_2 \|u\|^2 - m \int_0^T F(t, u(t)) dt \leq 0.$$

Let $\Omega = (\overline{B_R} \cap V) \oplus \{re | 0 < r < R\}$, $\partial\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$, where

$$\Omega_1 = \{u \in V, \|u\| \leq R\}, \quad \Omega_2 = Re \oplus \{u \in V, \|u\| \leq R\},$$

$$\Omega_3 = \{re | 0 < r < R\} \oplus \{u \in V, \|u\| = R\}.$$

So $\varphi|_{\partial\Omega} \leq 0$. From Lemma 4 we know that φ has a critical point c with $c \geq \tau > 0$. In other words, there exists u_1 such that $\varphi(u_1) = c$. Hence u_0 and u_1 are classical solutions of the impulsive problem (1). \square

Proof of Theorem 2 Since $f(t, u)$ and I_i are odd about u , $\varphi(u)$ is even and $\varphi(0) = 0$. Lemma 7 shows that φ satisfies the P.S. condition. In the same way as in Theorem 1, we can verify φ satisfies the condition (φ_1) in Lemma 5. Finally we prove that φ also satisfies the condition (φ_2) . According to Lemma 6 and (H_1) , for every V_1 , $u \in V_1$, there exists $C_3, M_3 > 0$ such that

$$\begin{aligned} \varphi(u) &\leq a_2 \|u\|^2 + \sum_{i=1}^p e^{-R(t)} \int_0^{u(t_i)} I_i(s) ds - \int_0^T e^{-R(t)} F(t, u(t)) dt \\ &\leq a_2 \|u\|^2 - m \int_0^T (m_1 |u(t)|^\mu - C_3) dt \\ &\leq a_2 \|u\|^2 - M_3 \|u\|^\mu + mTC_3, \end{aligned}$$

then we can find $R > 0$ such that $\varphi(u) \leq 0$ when $\|u\| \geq R$. By Lemma 5, φ possesses infinitely many critical points. Hence, BVP (1) has infinitely many classical solutions. \square

Example Take $x(t) \in C[0, T]$, $t_1 \in (0, T)$ and $k > 0$. Consider the following impulsive problem:

$$\begin{cases} -u''(t) + tu'(t) + \lambda u(t) = e^{-t} x(t) u^{\frac{5}{3}}(t), & t \neq t_1, \\ \Delta u'(t_1) = u'(t_1+) - u'(t_1-) = -ku(t_1), \\ u(0) = u(T) = 0. \end{cases} \quad (7)$$

Let $\mu = 2.1$, $\delta_1 = 1$ and $\rho_0 = 1$, then the differential equation (7) has infinity many weak solutions by Theorem 2.

4 Conclusion

A second-order impulsive differential equation is considered in this paper. Some necessary conditions for the existence and multiplicity of solutions are presented by critical point theories and variational methods. We also proposed a numerical example to show the advantage.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have made equal contributions. All authors read and approved the final manuscript.

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