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Nonlinear impulsive differential and integral inequalities with integral jump conditions

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Abstract
Some new nonlinear impulsive differential inequalities and integral inequalities with integral jump conditions for discontinuous functions are established using the method of successive iteration. These jump conditions at a discontinuous point are related to the integral conditions of the past state, which can be used in the qualitative analysis of the solutions to certain nonlinear impulsive differential systems.

MSC: 34A37; 34A40

Keywords: integral inequalities; integral jump condition; discontinuous function; impulsive differential system

1 Introduction

Impulsive differential equations, that is, differential equations involving impulse effect, appear as a natural description of observed evolution phenomena of several real world problems. Many processes studied in applied sciences are represented by impulsive differential equations. However, the situation is quite different in many physical phenomena that have a sudden change in their states such as mechanical systems with impact, biological systems such as heart beats, blood flows, population dynamics theoretical physics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology processes, and so on (see [1–3] and [4] for details).

In spite of the importance of impulsive differential equations, the development of the theory of impulsive differential equations has been quite slow due to special features possessed by impulsive differential equations in general, such as pulse phenomena, confluence, and loss of autonomy. Among these results, differential inequalities and integral inequalities with impulsive effects play increasingly important roles in the study of quantitative properties of solutions of impulsive differential systems. However, most of these results involving impulsive effects are point-discontinuous, *i.e.*, impulsive effects are added at a sequence of discontinuous points (see [5–12] for details). For example, in 2004, Borysenko [13] considered the following integral inequality with impulsive effect:

$$u(t) \le a(t) + \int_{t_0}^{t} f(s)u(s)\,ds + \sum_{t_0<t_i<t} \alpha_i u^r(t_i - 0),$$

in 2007, Iovane [14] studied the following integral inequalities:

$$u(t) \le a(t) + \int_{t_0}^{t} f(s)u\big(\lambda(s)\big)\,ds + \sum_{t_0<t_i<t} \alpha_i u^r(t_i - 0),$$

$$\begin{aligned} u(t) \le{}& a(t) + q(t)\bigg[\int_{t_0}^{t} f(s)u\big(\alpha(s)\big)\,ds + \int_{t_0}^{t} f(s)\int_{t_0}^{s} g(t)u\big(\tau(t)\big)\,dt\,ds \\ &+ \sum_{t_0<t_i<t} \alpha_i u^r(t_i - 0)\bigg], \end{aligned}$$

in 2011, Wu-Sheng Wang [5] gave the upper bound for the nonlinear inequality

$$v^p(t) \le A_0(t) + \frac{p}{p-q}\int_{t_0}^{t} f(s)v^q\big(\tau(s)\big)\,ds + \sum_{t_0<t_i<t} \alpha_i v^q(t_i - 0).$$

As we know, most of the phenomena occurring in the natural world do not suddenly change, so the impulsive differential equations with integral jump conditions are more accurate than impulsive differential equations with stationary discontinuous points in characterizing the nature. In 2012, based on a well-known result given by Lakshmikantham *et al.* [1], Thiramanns and Tarboon [15] studied the following impulsive linear differential inequalities:

$$\begin{cases} m'(t) \le p(t)m(t) + q(t), & t \ne t_k, \\ m\big(t_k^+\big) \le d_k m(t_k) + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)\,ds + b_k, \end{cases} \tag{1.1}$$

and gave the upper-bound estimation of the unknown function $m(t)$.

Theorem 1.1 *Suppose that* (H_0) *and* (H_1) *hold. If* $p, q \in C[\mathbb{R}_+, \mathbb{R}]$ *and for* $k = 1, 2, \ldots, t \ge t_0$, *the impulsive linear differential inequality* (1.1) *holds, where* c_k; $d_k \ge 0$, $0 \le \sigma_k \le \tau_k \le t_k - t_{k-1}$, b_k *are constants. Then*

$$\begin{aligned} m(t) \le{}& \bigg\{m(t_0)\prod_{t_0<t_k<t}\bigg(d_k e^{\int_{t_{k-1}}^{t_k} p(\tau)\,d\tau} + c_k\int_{t_k-\tau_k}^{t_k-\sigma_k} e^{\int_{t_{k-1}}^{s} p(\tau)\,d\tau}\,ds\bigg) \\ &+ \sum_{t_0<t_k<t}\bigg[\prod_{t_k<t_j<t}\bigg(d_j e^{\int_{t_{j-1}}^{t_j} p(\tau)\,d\tau} + c_j\int_{t_j-\tau_j}^{t_j-\sigma_j} e^{\int_{t_{j-1}}^{s} p(\tau)\,d\tau}\,ds\bigg) \\ &\times\bigg(d_k\int_{t_{k-1}}^{t_k} q(s)e^{\int_s^{t_k} p(\tau)\,d\tau}\,ds \\ &+ c_k\int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{t_{k-1}}^{s} q(\nu)e^{\int_\nu^s p(\tau)\,d\tau}\,d\nu\,ds + b_k\bigg)\bigg]\bigg\}e^{\int_{t_i}^{t} p(\tau)\,d\tau} \\ &+ \int_{t_i}^{t} q(s)e^{\int_s^t p(\tau)\,d\tau}\,ds, \quad t \ge t_0. \end{aligned} \tag{1.2}$$

This result can be used to investigate the qualitative properties of certain linear impulsive differential equations.

A natural question arises, that is, how about the upper bound if the inequality is of nonlinearity? In this paper, under different jump conditions, we will study the upper-bound estimation of the nonlinear inequality

$$m'(t) \le p(t)m(t) + q(t)m^{\alpha}(t).$$

2 Main results

In this paper, let $0 \le t_0 < t_1 < t_2 < \cdots$ be a sequence. For $I \subset \mathbb{R}$, we denote by $\mathrm{PC}(I,\mathbb{R})$ the functions $u(t)$ defined on I, which is continuous for $t \ne t_k$, $u(0+)$, $u(t_k+)$, $u(t_k-)$ exist and $u(t)$ is left continuous at t_k, $k = 1,2,\ldots$, $\mathrm{PC}^1(I,\mathbb{R}_+)$ is the collection of functions $u(t)$ such that $u, u' \in \mathrm{PC}(I,\mathbb{R}_+)$. Throughout this paper, we assume the following hypotheses:

(H_0) the sequence $\{t_k\}$ satisfies $0 \le t_0 \le t_1 \le t_2 \le \cdots$, $\lim_{k\to\infty} t_k = +\infty$.
(H_1) $m \in \mathrm{PC}^1(I,\mathbb{R}_+)$, and $m(t)$ is left continuous at t_k, $k = 1,2,\ldots$.

Lemma 2.1 (see [11]) *Suppose that $a, b \in \mathbb{R}$, $p > 0$. Then*

$$\bigl(|a| + |b|\bigr)^p \le C_p\bigl(|a|^p + |b|^p\bigr),$$

where $C_p = 1$ for $0 < p \le 1$, and $C_p = 2^{p-1}$ for $p > 1$.

Theorem 2.1 *Suppose that (H_0) and (H_1) hold. If for $k = 1,2,\ldots,t \ge t_0$,*

$$m'(t) \le p(t)m(t) + q(t)m^{\alpha}(t), \quad t \ne t_k, \tag{2.1}$$

$$m^{1-\alpha}\bigl(t_k^+\bigr) \le d_k m^{1-\alpha}(t_k) + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m^{1-\alpha}(s)\,ds + b_k, \tag{2.2}$$

here $0 < \alpha < 1$, $p, q \in C[\mathbb{R}_+,\mathbb{R}]$, and for $k = 1,2,\ldots,t \ge t_0$, c_k; $d_k \ge 0$, $0 \le \sigma_k \le \tau_k \le t_k - t_{k-1}$, b_k are constants. We have the estimation

$$\begin{aligned} m(t) \le{}& \Biggl\{\Biggl[m^{1-\alpha}(t_0)\prod_{t_0<t_k<t} E_k + \sum_{t_0<t_k<t} G_k \prod_{t_k<t_j<t} E_j\Biggr] e^{\int_{t_i}^{t}(1-\alpha)p(\tau)\,d\tau} \\ &+ (1-\alpha)\int_{t_i}^{t} q(s)e^{\int_s^t(1-\alpha)p(\tau)\,d\tau}\,ds\Biggr\}^{\frac{1}{1-\alpha}}, \quad t \ge t_0, \end{aligned} \tag{2.3}$$

where

$$E_k = d_k e^{\int_{t_{k-1}}^{t_k}(1-\alpha)p(\tau)\,d\tau} + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} e^{\int_{t_{k-1}}^{s}(1-\alpha)p(\tau)\,d\tau}\,ds, \tag{2.4}$$

$$\begin{aligned} G_k ={}& d_k \int_{t_{k-1}}^{t_k}(1-\alpha)q(s)e^{\int_s^{t_k}(1-\alpha)p(\tau)\,d\tau}\,ds \\ &+ (1-\alpha)c_k \int_{t_k-\tau_k}^{t_k-\sigma_k}\int_{t_{k-1}}^{s} q(\nu)e^{\int_\nu^s(1-\alpha)p(\tau)\,d\tau}\,d\nu\,ds + b_k. \end{aligned} \tag{2.5}$$

Proof For $t \in [t_0, t_1]$, we have

$$\frac{d}{dt}\left[e^{\int_{t_0}^{t} -(1-\alpha)p(\tau)\,d\tau} m^{1-\alpha}(t)\right] \le (1-\alpha)q(t)e^{-\int_{t_0}^{t}(1-\alpha)p(\tau)\,d\tau}, \tag{2.6}$$

integrating (2.6) implies

$$m^{1-\alpha}(t) \le m^{1-\alpha}(t_0)e^{\int_{t_0}^{t}(1-\alpha)p(\tau)\,d\tau} + (1-\alpha)\int_{t_0}^{t} q(s)e^{\int_{s}^{t}(1-\alpha)p(\tau)\,d\tau}\,ds, \tag{2.7}$$

which shows that (2.3) holds for $t \in [t_0, t_1]$.

Now we suppose that (2.3) holds for $t \in [t_0, t_n]$, then we need only prove that (2.3) holds for $t \in (t_n, t_{n+1}]$ by mathematical induction. Since

$$\begin{aligned}
m^{1-\alpha}(t_n) \le{}& \left[m^{1-\alpha}(t_0)\prod_{t_0<t_k<t_n} E_k + \sum_{t_0<t_k<t_n} G_k \prod_{t_k<t_j<t_n} E_j\right] e^{\int_{t_{n-1}}^{t_n}(1-\alpha)p(\tau)\,d\tau} \\
&+ (1-\alpha)\int_{t_{n-1}}^{t_n} q(s)e^{\int_{s}^{t_n}(1-\alpha)p(\tau)\,d\tau}\,ds \\
={}& \left[m^{1-\alpha}(t_0)\prod_{i=1}^{n-1} E_i + \sum_{i=1}^{n-1} G_i \prod_{j=i+1}^{n-1} E_j\right] e^{\int_{t_{n-1}}^{t_n}(1-\alpha)p(\tau)\,d\tau} \\
&+ (1-\alpha)\int_{t_{n-1}}^{t_n} q(s)e^{\int_{s}^{t_n}(1-\alpha)p(\tau)\,d\tau}\,ds, \\
m^{1-\alpha}\left(t_n^{+}\right) \le{}& d_n m^{1-\alpha}(t_n) + c_n\int_{t_n-\tau_n}^{t_n-\sigma_n} m^{1-\alpha}(s)\,ds + b_n \\
\le{}& d_n\left\{\left[m^{1-\alpha}(t_0)\prod_{i=1}^{n-1} E_i + \sum_{i=1}^{n-1} G_i \prod_{j=i+1}^{n-1} E_j\right] e^{\int_{t_{n-1}}^{t_n}(1-\alpha)p(\tau)\,d\tau}\right. \\
&\left.+ (1-\alpha)\int_{t_{n-1}}^{t_n} q(s)e^{\int_{s}^{t_n}(1-\alpha)p(\tau)\,d\tau}\,ds\right\} \\
&+ c_n\int_{t_n-\tau_n}^{t_n-\sigma_n}\left[m^{1-\alpha}(t_0)\prod_{i=1}^{n-1} E_i + \sum_{i=1}^{n-1} G_i \prod_{j=i+1}^{n-1} E_j\right] e^{\int_{t_{n-1}}^{s}(1-\alpha)p(\tau)\,d\tau}\,ds \\
&+ c_n\int_{t_n-\tau_n}^{t_n-\sigma_n}(1-\alpha)\int_{t_{n-1}}^{s} q(\nu)e^{\int_{\nu}^{s}(1-\alpha)p(\tau)\,d\tau}\,d\nu\,ds + b_n \\
={}& \left[m^{1-\alpha}(t_0)\prod_{i=1}^{n-1} E_i + \sum_{i=1}^{n-1} G_i \prod_{j=i+1}^{n-1} E_j\right] \\
&\times\left[d_n e^{\int_{t_{n-1}}^{t_n}(1-\alpha)p(\tau)\,d\tau} + c_n\int_{t_n-\tau_n}^{t_n-\sigma_n} e^{\int_{t_{n-1}}^{s}(1-\alpha)p(\tau)\,d\tau}\,ds\right] \\
&+ d_n(1-\alpha)\int_{t_{n-1}}^{t_n} q(s)e^{\int_{s}^{t_n} p(\tau)\,d\tau}\,ds \\
&+ (1-\alpha)c_n\int_{t_n-\tau_n}^{t_n-\sigma_n}\int_{t_{n-1}}^{s} q(\nu)e^{\int_{\nu}^{s}(1-\alpha)p(\tau)\,d\tau}\,d\nu\,ds + b_n \\
={}& \left[m^{1-\alpha}(t_0)\prod_{i=1}^{n-1} E_i + \sum_{i=1}^{n-1} G_i \prod_{j=i+1}^{n-1} E_j\right] E_n + G_n
\end{aligned}$$

$$
\begin{aligned}
&= m^{1-\alpha}(t_0)\prod_{i=1}^{n} E_i + \left(\sum_{i=1}^{n-1} G_i \prod_{j=i+1}^{n-1} E_j\right) E_n + G_n \\
&= m^{1-\alpha}(t_0)\prod_{i=1}^{n} E_i + \sum_{i=1}^{n} G_i \prod_{j=i+1}^{n} E_j, \qquad (2.8)
\end{aligned}
$$

substituting (2.8) into (2.7), with t_0 being replaced by t_n^+, we obtain, for $t \in (t_n, t_{n+1}]$,

$$
\begin{aligned}
m^{1-\alpha}(t) &\leq m^{1-\alpha}\left(t_n^+\right) e^{\int_{t_n}^{t}(1-\alpha)p(\tau)\,d\tau} + (1-\alpha)\int_{t_n}^{t} q(s) e^{\int_s^t (1-\alpha)p(\tau)\,d\tau}\,ds \\
&\leq \left[m^{1-\alpha}(t_0)\prod_{i=1}^{n} E_i + \sum_{i=1}^{n} G_i \prod_{j=i+1}^{n} E_j\right] e^{\int_{t_n}^{t}(1-\alpha)p(\tau)\,d\tau} \\
&\quad + (1-\alpha)\int_{t_n}^{t} q(s) e^{\int_s^t (1-\alpha)p(\tau)\,d\tau}\,ds \\
&= \left[m^{1-\alpha}(t_0)\prod_{t_0<t_k<t_{n+1}} E_k + \sum_{t_0<t_k<t_{n+1}} G_k \prod_{t_k<t_j<t_{n+1}} E_j\right] e^{\int_{t_n}^{t}(1-\alpha)p(\tau)\,d\tau} \\
&\quad + (1-\alpha)\int_{t_n}^{t} q(s) e^{\int_s^t (1-\alpha)p(\tau)\,d\tau}\,ds.
\end{aligned}
$$

This completes the proof of Theorem 2.1. □

If $d_k \equiv 0$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.1 *Suppose that* (H_0) *and* (H_1) *hold,* $p, q \in C[\mathbb{R}_+, \mathbb{R}]$ *and for* $k = 1, 2, \ldots, t \geq t_0$,

$$
\begin{aligned}
&m'(t) \leq p(t)m(t) + q(t)m^{\alpha}(t), \quad 0 < \alpha < 1, \\
&m^{1-\alpha}\left(t_k^+\right) \leq c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m^{1-\alpha}(s)\,ds + b_k,
\end{aligned}
$$

where c_k, b_k, σ_k, τ_k *are defined as in Theorem* 2.1, *then we have*

$$
\begin{aligned}
m(t) \leq \Bigg\{ &\Bigg[m^{1-\alpha}(t_0)\prod_{t_0<t_k<t} c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} e^{\int_{t_{k-1}}^{s}(1-\alpha)p(\tau)\,d\tau}\,ds \\
&+ \sum_{t_0<t_k<t} (1-\alpha)c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \int_{t_{k-1}}^{s} q(\nu) e^{\int_\nu^s (1-\alpha)p(\tau)\,d\tau}\,d\nu\,ds \\
&\times \prod_{t_k<t_j<t} c_j \int_{t_j-\tau_j}^{t_j-\sigma_j} e^{\int_{t_{j-1}}^{s}(1-\alpha)p(\tau)\,d\tau}\,ds \Bigg] e^{\int_{t_i}^{t}(1-\alpha)p(\tau)\,d\tau} \\
&+ (1-\alpha)\int_{t_i-1}^{t} q(s) e^{\int_s^t (1-\alpha)p(\tau)\,d\tau}\,ds \Bigg\}^{\frac{1}{1-\alpha}}.
\end{aligned}
$$

If $d_k \equiv 1$, we obtain the following theorem.

Theorem 2.2 *Suppose that* (H_0) *and* (H_1) *hold. If, for* $k = 1, 2, \ldots, t \geq t_0$,

$$\begin{cases} m'(t) \leq p(t)m(t) + q(t)m^{\alpha}(t), & t \neq t_k, \\ \Delta m^{1-\alpha}(t_k) \leq c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m^{1-\alpha}(s)\,ds + b_k, \end{cases} \tag{2.9}$$

where $0 < \alpha < 1$, $p, q \in C[\mathbb{R}_+, \mathbb{R}]$, *and for* $k = 1, 2, \ldots, t \geq t_0$, $c_k \geq 0$, $0 \leq \sigma_k \leq \tau_k \leq t_k - t_{k-1}$, b_k *are constants,* $\Delta m^{1-\alpha}(t_k) = m^{1-\alpha}(t_k^+) - m^{1-\alpha}(t_k)$. *We have the estimation*

$$\begin{aligned} m(t) \leq{}& \biggl\{\biggl[m^{1-\alpha}(t_0)\prod_{t_0<t_k<t} E_k + \sum_{t_0<t_k<t} H_k \prod_{t_k<t_j<t} E_j\biggr] e^{\int_{t_i}^{t}(1-\alpha)p(\tau)\,d\tau} \\ &+ (1-\alpha)\int_{t_0}^{t} q(s)e^{\int_s^t(1-\alpha)p(\tau)\,d\tau}\,ds\biggr\}^{\frac{1}{1-\alpha}}, \quad t \geq t_0, \end{aligned} \tag{2.10}$$

where E_k *is defined as* (2.4) (*with* $d_k \equiv 1$),

$$H_k = (1-\alpha)c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \int_{t_0}^{s} q(\nu)e^{\int_\nu^s(1-\alpha)p(\tau)\,d\tau}\,d\nu\,ds + b_k.$$

Proof As the proof of Theorem 2.1, we prove (2.7) holds for $t \in [t_0, t_1]$, which means that (2.10) holds for $t \in [t_0, t_1]$. Now suppose that (2.10) holds for $t \in [t_0, t_n]$, then

$$\begin{aligned} m^{1-\alpha}(t_n) \leq{}& \biggl[m^{1-\alpha}(t_0)\prod_{t_0<t_k<t} E_k + \sum_{t_0<t_k<t} H_k \prod_{t_k<t_j<t} E_j\biggr] e^{\int_{t_{n-1}}^{t_n}(1-\alpha)p(\tau)\,d\tau} \\ &+ (1-\alpha)\int_{t_0}^{t_n} q(s)e^{\int_s^{t_n}(1-\alpha)p(\tau)\,d\tau}\,ds \\ ={}& \biggl[m^{1-\alpha}(t_0)\prod_{i=1}^{n-1} E_i + \sum_{i=1}^{n-1} H_i \prod_{j=i+1}^{n-1} E_j\biggr] e^{\int_{t_{n-1}}^{t_n}(1-\alpha)p(\tau)\,d\tau} \\ &+ (1-\alpha)\int_{t_0}^{t_n} q(s)e^{\int_s^{t_n}(1-\alpha)p(\tau)\,d\tau}\,ds. \end{aligned}$$

So

$$\begin{aligned} m^{1-\alpha}\bigl(t_n^+\bigr) \leq{}& m^{1-\alpha}(t_n) + c_n \int_{t_n-\tau_n}^{t_n-\sigma_n} m^{1-\alpha}(s)\,ds + b_n \\ \leq{}& \biggl[m^{1-\alpha}(t_0)\prod_{i=1}^{n-1} E_i + \sum_{i=1}^{n-1} H_i \prod_{j=i+1}^{n-1} E_j\biggr] e^{\int_{t_{n-1}}^{t_n}(1-\alpha)p(\tau)\,d\tau} \\ &+ (1-\alpha)\int_{t_0}^{t_n} q(s)e^{\int_s^{t_n}(1-\alpha)p(\tau)\,d\tau}\,ds \\ &+ c_n \int_{t_n-\tau_n}^{t_n-\sigma_n} \biggl[m^{1-\alpha}(t_0)\prod_{i=1}^{n-1} E_i + \sum_{i=1}^{n-1} H_i \prod_{j=i+1}^{n-1} E_j\biggr] e^{\int_{t_{n-1}}^{s}(1-\alpha)p(\tau)\,d\tau}\,ds \\ &+ c_n \int_{t_n-\tau_n}^{t_n-\sigma_n} (1-\alpha)\int_{t_0}^{s} q(\nu)e^{\int_\nu^s(1-\alpha)p(\tau)\,d\tau}\,d\nu\,ds + b_n \end{aligned}$$

$$
\begin{aligned}
&= \left[m^{1-\alpha}(t_0) \prod_{i=1}^{n-1} E_i + \sum_{i=1}^{n-1} H_i \prod_{j=i+1}^{n-1} E_j \right] \left[e^{\int_{t_{n-1}}^{t_n} (1-\alpha) p(\tau)\, d\tau} \right. \\
&\quad \left. + c_n \int_{t_n-\tau_n}^{t_n-\sigma_n} e^{\int_{t_{n-1}}^{s} (1-\alpha)p(\tau)\, d\tau}\, ds \right] \\
&\quad + (1-\alpha) \int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha)p(\tau)\, d\tau}\, ds + H_n \\
&= m^{1-\alpha}(t_0) \prod_{i=1}^{n} E_i + \left(\sum_{i=1}^{n-1} H_i \prod_{j=i+1}^{n-1} E_j \right) E_n \\
&\quad + H_n + (1-\alpha) \int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha)p(\tau)\, d\tau}\, ds \\
&= m^{1-\alpha}(t_0) \prod_{i=1}^{n} E_i + \sum_{i=1}^{n} H_i \prod_{j=i+1}^{n} E_j \\
&\quad + (1-\alpha) \int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha)p(\tau)\, d\tau}\, ds.
\end{aligned}
$$

Using (2.7) (with t_0 being replaced by t_n^+), we obtain, for $t \in (t_n, t_{n+1}]$,

$$
\begin{aligned}
m^{1-\alpha}(t) &\le \left[m^{1-\alpha}(t_0) \prod_{i=1}^{n} E_i + \sum_{i=1}^{n} H_i \prod_{j=i+1}^{n} E_j + (1-\alpha) \int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha)p(\tau)\, d\tau}\, ds \right] \\
&\quad \times e^{\int_{t_n}^{t} (1-\alpha)p(\tau)\, d\tau} + (1-\alpha) \int_{t_n}^{t} q(s) e^{\int_s^{t} (1-\alpha)p(\tau)\, d\tau}\, ds \\
&= \left[m^{1-\alpha}(t_0) \prod_{i=1}^{n} E_i + \sum_{i=1}^{n} H_i \prod_{j=i+1}^{n} E_j \right] e^{\int_{t_n}^{t} (1-\alpha)p(\tau)\, d\tau} \\
&\quad + (1-\alpha) \int_{t_0}^{t} q(s) e^{\int_s^{t} (1-\alpha)p(\tau)\, d\tau}\, ds \\
&= \left[m^{1-\alpha}(t_0) \prod_{t_0<t_k<t_{n+1}} E_k + \sum_{t_0<t_k<t_{n+1}} H_k \prod_{t_k<t_j<t_{n+1}} E_j \right] e^{\int_{t_n}^{t} (1-\alpha)p(\tau)\, d\tau} \\
&\quad + (1-\alpha) \int_{t_0}^{t} q(s) e^{\int_s^{t} (1-\alpha)p(\tau)\, d\tau}\, ds.
\end{aligned}
$$

This completes the proof. □

Remark 2.1 If $\alpha = 0$, then Theorem 2.1 reduces to Theorem 1.1, and Theorem 2.2 improves Theorem 1.1.

If $p(t) \equiv 0$ in Theorem 2.2, we obtain the following useful corollary.

Corollary 2.2 *If* (H_0) *and* (H_1) *hold and for* $k = 1, 2, \ldots, t \ge t_0$,

$$
\begin{cases}
m'(t) \le q(t) m^{\alpha}(t), \\
\Delta m^{1-\alpha}(t) \le c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m^{1-\alpha}(s)\, ds + b_k,
\end{cases}
\tag{2.11}
$$

then

$$m(t) \le \Big\{ m^{1-\alpha}(t_0) \prod_{t_0<t_k<t} \big(1 + c_k(\tau_k - \sigma_k)\big) + \sum_{t_0<t_k<t} \Big[(1-\alpha)c_k \Big(\int_{t_k-\tau_k}^{t_k-\sigma_k} \int_{t_{k-1}}^{s} q(v)\,dv\,ds + b_k \Big) \prod_{t_k<t_j<t} \big(1 + c_j(\tau_j - \sigma_j)\big) \Big] + (1-\alpha) \int_{t_0}^{t} q(s)\,ds \Big\}^{\frac{1}{1-\alpha}}.$$

Next, we will give another kind of nonlinear impulsive differential inequalities.

Theorem 2.3 *Suppose that* (H_0) *holds, and* $m \in \mathrm{PC}^1[\mathbb{R}_+, \mathbb{R}_+]$, $m(t)$ *is left continuous at* t_k, $k = 1, 2, \ldots, p(t)$, $q(t) \in C[\mathbb{R}_+, \mathbb{R}_+]$. *Assume*

$$\begin{cases} m'(t) \le p(t)m(t) + q(t)m^{\alpha}(t), & t \ne t_k, \\ \Delta m(t_k) \le c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)\,ds + b_k, \end{cases} \tag{2.12}$$

where $\Delta m(t_k) = m(t_k^+) - m(t_k)$, $0 < \alpha < 1$, $c_k \ge 0$, $0 \le \sigma_k \le \tau_k \le t_k - t_{k-1}$, b_k *are constants. We have the estimation*

$$m^{1-\alpha}(t) \le \Big(m(t_0) \prod_{t_0<t_k<t} F_k + \sum_{t_0<t_k<t} R_k \prod_{t_k<t_j<t} F_j \Big)^{1-\alpha} e^{\int_{t_i}^{t}(1-\alpha)p(\tau)\,d\tau} + 2^{(k-1)\alpha}(1-\alpha) \int_{t_0}^{t} q(s) e^{\int_s^t (1-\alpha)p(\tau)\,d\tau}\,ds, \quad t \ge t_0, \tag{2.13}$$

where

$$F_k = 2^{\frac{\alpha}{1-\alpha}} \Big[e^{\int_{t_{k-1}}^{t_k} p(\tau)\,d\tau} + c_k \int_{t_k-\tau_k}^{t_k-\sigma_k} e^{\int_{t_{k-1}}^{s} p(\tau)\,d\tau}\,ds \Big],$$

$$R_k = c_k 2^{\frac{(k-1)\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \int_{t_k-\tau_k}^{t_k-\sigma_k} \Big(\int_{t_0}^{v} q(s) e^{\int_s^v (1-\alpha)p(\tau)\,d\tau}\,ds \Big)^{\frac{1}{1-\alpha}}\,dv + b_k.$$

Proof Obviously, (2.13) holds for $t \in [t_0, t_1]$ as (2.7). Now we suppose (2.13) holds for $t \in [t_0, t_n]$, then by mathematical induction, we see that

$$\begin{aligned} m^{1-\alpha}(t_n) &\le \Big(m(t_0) \prod_{t_0<t_k<t_n} F_k + \sum_{t_0<t_k<t_n} R_k \prod_{t_k<t_j<t_n} F_j \Big)^{1-\alpha} e^{\int_{t_{n-1}}^{t_n}(1-\alpha)p(\tau)\,d\tau} \\ &\quad + 2^{(n-1)\alpha}(1-\alpha) \int_{t_0}^{t_n} q(s) e^{\int_s^{t_n}(1-\alpha)p(\tau)\,d\tau}\,ds \\ &= \Big(m(t_0) \prod_{i=1}^{n-1} F_i + \sum_{i=1}^{n-1} R_i \prod_{j=i+1}^{n-1} F_j \Big)^{1-\alpha} e^{\int_{t_{n-1}}^{t_n}(1-\alpha)p(\tau)\,d\tau} \\ &\quad + 2^{(n-1)\alpha}(1-\alpha) \int_{t_0}^{t_n} q(s) e^{\int_s^{t_n}(1-\alpha)p(\tau)\,d\tau}\,ds. \end{aligned}$$

Since $\frac{1}{1-\alpha} > 1$, by Lemma 2.1,

$$\begin{aligned}
m(t_n) &\le 2^{\frac{\alpha}{1-\alpha}} \left(m(t_0) \prod_{i=1}^{n-1} F_i + \sum_{i=1}^{n-1} R_i \prod_{j=i+1}^{n-1} F_j \right) e^{\int_{t_{n-1}}^{t_n} p(\tau)\,d\tau} \\
&\quad + 2^{\frac{\alpha}{1-\alpha}} 2^{\frac{(n-1)\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \left(\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha) p(\tau)\,d\tau}\,ds \right)^{\frac{1}{1-\alpha}}, \\
m\bigl(t_n^+\bigr) &\le m(t_n) + c_n \int_{t_n-\tau_n}^{t_n-\sigma_n} m(s)\,ds + b_n \\
&\le 2^{\frac{\alpha}{1-\alpha}} \left(m(t_0) \prod_{i=1}^{n-1} F_i + \sum_{i=1}^{n-1} R_i \prod_{j=i+1}^{n-1} F_j \right) e^{\int_{t_{n-1}}^{t_n} p(\tau)\,d\tau} \\
&\quad + 2^{\frac{n\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \left(\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha) p(\tau)\,d\tau}\,ds \right)^{\frac{1}{1-\alpha}} \\
&\quad + c_n \int_{t_n-\tau_n}^{t_n-\sigma_n} 2^{\frac{\alpha}{1-\alpha}} \left(m(t_0) \prod_{i=1}^{n-1} F_i + \sum_{i=1}^{n-1} R_i \prod_{j=i+1}^{n-1} F_j \right) e^{\int_{t_{n-1}}^{s} p(\tau)\,d\tau}\,ds \\
&\quad + c_n 2^{\frac{n\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \int_{t_n-\tau_n}^{t_n-\sigma_n} \left(\int_{t_0}^{v} q(s) e^{\int_s^{v} (1-\alpha) p(\tau)\,d\tau}\,ds \right)^{\frac{1}{1-\alpha}} dv + b_n \\
&= \left(m(t_0) \prod_{i=1}^{n-1} F_i + \sum_{i=1}^{n-1} R_i \prod_{j=i+1}^{n-1} F_j \right) \\
&\quad \times \left[2^{\frac{\alpha}{1-\alpha}} \left(e^{\int_{t_{n-1}}^{t_n} p(\tau)\,d\tau} + c_n \int_{t_n-\tau_n}^{t_n-\sigma_n} e^{\int_{t_{n-1}}^{s} p(\tau)\,d\tau}\,ds \right) \right] \\
&\quad + R_n + 2^{\frac{n\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \left(\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha) p(\tau)\,d\tau}\,ds \right)^{\frac{1}{1-\alpha}} \\
&= \left(m(t_0) \prod_{i=1}^{n-1} F_i + \sum_{i=1}^{n-1} R_i \prod_{j=i+1}^{n-1} F_j \right) F_n + R_n \\
&\quad + 2^{\frac{n\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \left(\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha) p(\tau)\,d\tau}\,ds \right)^{\frac{1}{1-\alpha}} \\
&= m(t_0) \prod_{i=1}^{n} F_i + \sum_{i=1}^{n} R_i \prod_{j=i+1}^{n} F_j \\
&\quad + 2^{\frac{n\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \left(\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n} (1-\alpha) p(\tau)\,d\tau}\,ds \right)^{\frac{1}{1-\alpha}}.
\end{aligned}$$

So for $t \in (t_n, t_{n+1}]$, since $0 < 1-\alpha < 1$, by Lemma 2.1 and (2.7) (with t_0 being replaced by t_n^+), we obtain

$$\begin{aligned}
m^{1-\alpha}(t) &\le m^{1-\alpha}\bigl(t_n^+\bigr) e^{\int_{t_n}^{t} (1-\alpha) p(\tau)\,d\tau} \\
&\quad + (1-\alpha) \int_{t_n}^{t} q(s) e^{\int_s^{t} (1-\alpha) p(\tau)\,d\tau}\,ds
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg[m(t_0)\prod_{i=1}^{n} F_i + \sum_{i=1}^{n} R_i \prod_{j=i+1}^{n} F_j \\
&\quad + 2^{\frac{n\alpha}{1-\alpha}}(1-\alpha)^{\frac{1}{1-\alpha}} \left(\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n}(1-\alpha)p(\tau)\,d\tau}\,ds \right)^{\frac{1}{1-\alpha}} \Bigg]^{1-\alpha} \\
&\quad \times e^{\int_{t_n}^{t}(1-\alpha)p(\tau)\,d\tau} + (1-\alpha)\int_{t_n}^{t} q(s) e^{\int_s^{t}(1-\alpha)p(\tau)\,d\tau}\,ds \\
&\le \Bigg[\left(m(t_0)\prod_{i=1}^{n} F_i + \sum_{i=1}^{n} R_i \prod_{j=i+1}^{n} F_j \right)^{1-\alpha} \\
&\quad + 2^{n\alpha}(1-\alpha)\int_{t_0}^{t_n} q(s) e^{\int_s^{t_n}(1-\alpha)p(\tau)\,d\tau}\,ds \Bigg] \\
&\quad \times e^{\int_{t_n}^{t}(1-\alpha)p(\tau)\,d\tau} + (1-\alpha)\int_{t_n}^{t} q(s) e^{\int_s^{t}(1-\alpha)p(\tau)\,d\tau}\,ds \\
&\le \left(m(t_0)\prod_{i=1}^{n} F_i + \sum_{i=1}^{n} R_i \prod_{j=i+1}^{n} F_j \right)^{1-\alpha} e^{\int_{t_n}^{t}(1-\alpha)p(\tau)\,d\tau} \\
&\quad + 2^{n\alpha}(1-\alpha)\int_{t_0}^{t} q(s) e^{\int_s^{t}(1-\alpha)p(\tau)\,d\tau}\,ds \\
&= \left(m(t_0)\prod_{t_0<t_k<t_{n+1}} F_k + \sum_{t_0<t_k<t_{n+1}} R_k \prod_{t_k<t_j<t_{n+1}} F_j \right)^{1-\alpha} e^{\int_{t_n}^{t}(1-\alpha)p(\tau)\,d\tau} \\
&\quad + 2^{n\alpha}(1-\alpha)\int_{t_0}^{t} q(s) e^{\int_s^{t}(1-\alpha)p(\tau)\,d\tau}\,ds,
\end{aligned}
$$

which shows that (2.13) holds for $k = n + 1$. This completes the proof. □

Now we give an upper-bound estimation of a nonlinear integral inequality with integral jump conditions.

Theorem 2.4 *Suppose that* (H_0) *holds, and suppose* $m, p, q \in C[\mathbb{R}_+, \mathbb{R}_+]$. *For* $t \ge t_0$, *if*

$$
m(t) \le c + \int_{t_0}^{t} p(s)m(s)\,ds + \int_{t_0}^{t} q(s)m^{\alpha}(s)\,ds + \sum_{t_0<t_k<t} \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)\,ds, \tag{2.14}
$$

where $\alpha_k \ge 0$, $0 \le \sigma_k \le \tau_k \le t_k - t_{k-1}$, $c \ge 0$, $0 < \alpha < 1$ *are constants. Then we have the estimation*

$$
\begin{aligned}
m(t) &\le \Bigg\{ \left(c\prod_{t_0<t_k<t} F_k + \sum_{t_0<t_k<t} R_k \prod_{t_k<t_j<t} F_j \right)^{1-\alpha} e^{\int_{t_i}^{t}(1-\alpha)p(\tau)\,d\tau} \\
&\quad + 2^{(k-1)\alpha}(1-\alpha)\int_{t_0}^{t} q(s) e^{\int_s^{t}(1-\alpha)p(\tau)\,d\tau}\,ds \Bigg\}^{1/(1-\alpha)},
\end{aligned}
\tag{2.15}
$$

where F_k *and* R_k *are defined as that in Theorem* 2.3, *with* c_k *being replaced by* α_k.

Proof Defined the right-hand side of (2.14) as a new function $v(t)$, we have $m(t) \le v(t)$ and $v(t_0) = c$. Since

$$\begin{aligned} &v'(t) = p(t)m(t) + q(t)m^\alpha(t), \quad t \neq t_k, \\ &v(t_k+) = v(t_k) + \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} m(s)\,ds, \end{aligned}$$

we obtain further

$$\begin{aligned} &v'(t) \le p(t)v(t) + q(t)v^\alpha(t), \quad t \neq t_k, \\ &v(t_k+) = v(t_k) + \alpha_k \int_{t_k-\tau_k}^{t_k-\sigma_k} v(s)\,ds. \end{aligned}$$

Then using Theorem 2.3 implies the estimation of $v(t)$, the estimation of the unknown function $m(t)$ is obtained since $m(t) \le v(t)$, and this completes the proof. □

3 Application to impulsive differential equations

As an application of Theorem 2.4, we give an upper-bound estimation of certain nonlinear impulsive differential equation as follows:

$$\begin{cases} v'(t) = f(t,v), \quad t \neq t_k, \\ \Delta v(t_k) = I_k\bigl(\int_{t_k-\tau_k}^{t_k-\sigma_k} v(s)\,ds\bigr), \quad t \in [t_0,\infty), \\ v(t_0) = v_0, \end{cases} \tag{3.1}$$

where $f \in C(\mathbb{R} \times \mathbb{R},\mathbb{R})$, $I_k \in C(\mathbb{R},\mathbb{R})$, $0 < t_0 < t_1 < \cdots$, $\lim_{t\to\infty} t_k = +\infty$, $0 \le \sigma_k \le \tau_k \le t_k - t_{k-1}$, $k = 1,2,\ldots$. If there exists $L > 0$ such that

$$\bigl|f(t,v)\bigr| \le L|v|^\alpha, \quad 0 < \alpha < 1; \tag{3.2}$$

and there exist $\iota_k \ge 0$, such that

$$\bigl|I_k(v)\bigr| \le \iota_k|v|, \quad k = 1,2,\ldots, \tag{3.3}$$

then for any solution $v(t)$ of (3.1), we have

$$\begin{aligned} \bigl|v(t)\bigr| \le \Biggl\{ \Biggl(&|v_0| \prod_{t_0<t_k<t} 2^{\alpha/(1-\alpha)}\bigl(1 + \iota_k(\tau_k - \sigma_k)\bigr) \\ &+ \sum_{t_0<t_k<t} 2^{\frac{(k-1)\alpha}{1-\alpha}} \bigl(L(1-\alpha)\bigr)^{\frac{1}{1-\alpha}} \iota_k(t_k - t_0)^{\frac{2-\alpha}{1-\alpha}} \\ &\times \prod_{t_k<t_j<t} 2^{\frac{\alpha}{1-\alpha}} \bigl(1 + \iota_j(\tau_j - \sigma_j)\bigr)\Biggr)^{1-\alpha} \\ &+ 2^{(k-1)\alpha}(1-\alpha)L(t - t_0)\Biggr\}^{1/(1-\alpha)}. \end{aligned} \tag{3.4}$$

Proof Suppose $v = v(t)$ is a solution of (3.1), we integrate (3.1) to obtain

$$\begin{aligned} v(t) &= v(t_0) + \int_{t_0}^{t} f\bigl(s, v(s)\bigr)\,ds + \sum_{t_0<t_k<t} I_k\biggl(\int_{t_k-\tau_k}^{t_k-\sigma_k} v(s)\,ds\biggr) \\ &= v_0 + \int_{t_0}^{t} f\bigl(s, v(s)\bigr)\,ds + \sum_{t_0<t_k<t} I_k\biggl(\int_{t_k-\tau_k}^{t_k-\sigma_k} v(s)\,ds\biggr). \end{aligned} \tag{3.5}$$

By (3.2) and (3.3), we obtain

$$\begin{aligned} \bigl|v(t)\bigr| &\le |v_0| + \int_{t_0}^{t} \bigl|f\bigl(s, v(s)\bigr)\bigr|\,ds + \sum_{t_0<t_k<t} \biggl|I_k\biggl(\int_{t_k-\tau_k}^{t_k-\sigma_k} v(s)\,ds\biggr)\biggr| \\ &\le |v_0| + L\int_{t_0}^{t} \bigl|v(s)\bigr|^{\alpha}\,ds + \sum_{t_0<t_k<t} \iota_k \int_{t_k-\tau_k}^{t_k-\sigma_k} \bigl|v(s)\bigr|\,ds. \end{aligned} \tag{3.6}$$

Then by Theorem 2.4, we compute that

$$\begin{aligned} F_k &= 2^{\alpha/(1-\alpha)}\bigl(1 + \iota_k(\tau_k - \sigma_k)\bigr); \\ R_k &= \iota_k 2^{\frac{(k-1)\alpha}{1-\alpha}} (1-\alpha)^{\frac{1}{1-\alpha}} \int_{t_k-\tau_k}^{t_k-\sigma_k} \biggl(\int_{t_0}^{v} L\,ds\biggr)^{1/(1-\alpha)} dv \\ &= \frac{1-\alpha}{2-\alpha} 2^{\frac{(k-1)\alpha}{1-\alpha}} \bigl(L(1-\alpha)\bigr)^{\frac{1}{1-\alpha}} \iota_k\bigl[(t_k - \sigma_k - t_0)^{\frac{2-\alpha}{1-\alpha}} - (t_k - \tau_k - t_0)^{\frac{2-\alpha}{1-\alpha}}\bigr] \\ &\le 2^{\frac{(k-1)\alpha}{1-\alpha}} \bigl(L(1-\alpha)\bigr)^{\frac{1}{1-\alpha}} \iota_k (t_k - t_0)^{\frac{2-\alpha}{1-\alpha}}. \end{aligned}$$

Substituting F_k and R_k in (2.15), we obtain (3.4). This completes the proof. □

Competing interests
The authors declare that there are no competing interests.

Authors' contributions
JS gave the main theorems; FM gave some useful comments and revised the paper. All authors have read and approved the final manuscript.

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