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ABOUT THE FREE-VIBRATION MODE SHAPES OF ELASTOPLASTIC DISSIPATIVE SYSTEMS

Aleksander N. Potapov

South Ural State University (National Research University), Chelyabinsk, RUSSIA

Abstract: The author presents the conditions of the generalized orthogonality of the free-vibration mode shapes of an elastic dissipative system, for which traditional classical orthogonality conditions are a private case. As opposed to these conditions, the above ratios contain the mass matrix, the damping matrix, and the diagonal form of the spectral characteristics (damping coefficients and mode-shape frequencies). Within the theory of time analysis, free-vibration mode shapes of an elastoplastic system are built on the basis of using a schematized diagram of strain with hardening. The author proposes a design scheme that reduces the process of nonlinear vibrations to a sequence of processes flowing according to a linear scenario within the time intervals called quasilinear. In these intervals, the parameters of the dynamic model (elements of the stiffness matrix and the damping matrix) remain unchanged, all the changes occur only when passing through the critical points. As a result, the author formulated the condition for the nondegenerate state of an elastoplastic dissipative system. According to the condition, local plastic zones characterized by the size, the number and location of the zones on the design scheme of the structure correspond to each quasilinear interval. Since within the intervals, the parameters of the plastic zones are unchanged, the conditions of the generalized orthogonality of the mode shapes of the elastoplastic system are satisfied by analogy with the vibration mode shapes of an elastic dissipative system. The free-vibration motion of a hinged beam with three degrees of freedom are analyzed taking into account local plastic zones with different lengths and the location of zones in different nodes. It is shown that the configuration of the forms of elastoplastic oscillations differs qualitatively from the configuration of the corresponding forms of elastic vibrations.

Keywords: elastoplastic system, dissipative system, free-vibration mode shapes, plastic zone, orthogonality, stiffness matrix, damping matrix

О ФОРМАХ СОБСТВЕННЫХ КОЛЕБАНИЙ УПРУГОПЛАСТИЧЕСКИХ ДИССИПАТИВНЫХ СИСТЕМ

А.Н. Потапов

Южно-Уральский государственный университет (национальный исследовательский университет), г. Челябинск, РОССИЯ

Аннотация: Приведены условия обобщённой ортогональности форм собственных колебаний упругой диссипативной системы, для которых традиционные классические условия ортогональности являются частным случаем. В отличие от этих условий приведённые соотношения содержат связанные между собой матрицы масс, демпфирования и диагональную форму спектральных характеристик (коэффициентов демпфирования и частот собственных колебаний). В рамках теории временного анализа осуществлено построение форм собственных колебаний упругопластической системы на основе использования схематизированной диаграммы деформирования с упрочнением. Предложена расчётная схема, сводящая процесс нелинейных колебаний к последовательности процессов, протекающих по линейному сценарию на интервалах времени, называемых квазилинейными. На данных интервалах параметры расчетной динамической модели (элементы матриц жесткости и демпфирования) остаются неизменными, все изменения происходят только при переходе через критические точки. В результате сформулировано условие невырожденного состояния упругопластической диссипативной системы. Согласно условию, каждому квазилинейному интервалу соответствуют локальные пластические зоны, характеризуемые размерами, количеством и расположением зон на расчётной схеме сооружения. Так как внутри интервалов параметры пластических зон – неизменны, то условия обобщённой ортогональности собственных форм колебаний упругопластической системы выполняются по аналогии с формами колебаний упругой диссипативной системы. Проведён анализ собственных колебаний шарнирной балки с тремя степенями свободы с учётом локальных пластических зон с различной длиной и различным расположением зон в узлах. Показано, что конфигурация собственных форм упругопластических колебаний качественно отличается от конфигурации соответствующих форм упругих колебаний.

Ключевые слова: упругопластическая система, диссипативная система, форма собственных колебаний, пластическая зона, ортогональность, матрица жёсткости, матрица демпфирования

1. INTRODUCTION

Modern structures operate in the conditions of complex dynamic influences, when maximum stresses in the main bearing elements often exceed the value of the elastic limit. Off the scale stresses must be considered, for example, in the design of seismic buildings and structures [1, 2]. In this case, the appearance of nonlinear strains in frame buildings is not only considered admissible, but also justified by special calculations for the organization of so-called "plasticity zones" generally located in horizontal bearing elements (crossbars). These zones are created in order to absorb the seismic action energy and its subsequent diversion (dissipation) into the environment, which ensures the integrity of the system in general. The developments in this field are reflected in national regulatory documents [3, 4] and in advanced foreign studies and regulations [5-7].

The presence of plastic zones in the bearing elements of structures naturally influences the system stiffness parameters, and, consequently, its internal dynamic characteristics. [8, 9] studied the behavior of the beams made of an elastoplastic material under impulse actions. [8] obtained an approximate solution of the problem of residual deflections of an impulse loaded rigidly fixed beam at its nonlinear vibrations, taking into account the sensitivity of the material to the strain rate in the plastic zone.[9]gives an original solution of the problem of free vibrations of a hinged beam beyond the elastic limit under the influence of a distributed impulse simulating an explosion, based on the analytical approach. The article presents algorithm of formation of mode frequencies for an ideal elastoplastic material and demonstrates a decrease of their values with an increasing length of the plastic zones.

Using computer simulation of frame buildings within of the elastoplastic analysis, frequency-damped characteristics and free-vibration mode shapes were determined on the basis of Prandtl diagram [10, 11]. A frame building was presented in the form of a discrete dissipative system (DDS), for which it was also shown that an increase in the yield in the frame floors leads to a decrease in the frequency spectrum.

The present work deals with the issues relating to the free-vibration motion in the DDS taking into account elastoplastic strains. We obtained the conditions, under which the fundamental properties of the orthogonality of free-vibration mode shapes are satisfied in a dissipative system in the yield state associated with the appearance of local plastic zones. The qualitative nature of the influence of a nonlinear process on the configurations of mode shapes is exemplified by a hinged beam.

2. BASIC RATIOS

Free-vibration mode shapes of any system (linear or nonlinear) are determined by its strained state caused by the action of inertial and dissipative forces. Free mode shapes are unchanged in the process of vibrations in a linearly strained system, so their configuration does not depend on time. It is manifested in a known property: the ratio of any two ordinates of the k-th mode shape is a constant value: p_{jk} / p_{ik} = const.

In the elastoplastic system, the appearance of yield in structural elements is accompanied by a decrease in its stiffness parameters. The presence of plastic strain zones in the structure leads to a redistribution of internal forces and stresses and, as a consequence, to a change in the configuration of mode shapes. Therefore, the ratio of the ordinates p_{jk} and p_{ik} of an arbitrary k-th mode shape of an elastoplastic system taken at different instants of time will no longer be a constant value: $p_{jk} / p_{ik} \neq$const.

Let us denote the free-vibration mode shapes of a discrete system through P_k (k= 1, ... , n). These mode shapes have some properties known as orthogonality conditions. For an elastic conservative system, the mode shapes are orthogonal in relation to the mass matrix

$$M = \mathrm{diag}\,(m_1, \ldots, m_n)$$

and to the stiffness matrix $K = K^{\mathrm{T}}$ [12, 13]:

$$P_j^{\mathrm{T}} M P_k = 0,\ \ P_j^{\mathrm{T}} K P_k = 0 \qquad (1)$$
$$(k, j = 1, \ldots, n;\ k \neq j).$$

In case of the DDS vibrations, the damping matrix C is generally chosen so that the equations of motion could be reduced to a diagonal form. In this case, the matrix C corresponds to a proportional damping model [14], and one more orthogonality condition, including the damping matrix C, is added to the orthogonality conditions, which take the same form as for the conservative system:

$$P_l^{\mathrm{T}} C P_k = 0\ (k \neq l). \qquad (2)$$

The presence of conditions (1), (2) is a consequence of the separation of all the three forces: inertia, elasticity, and damping in the normal coordinates.

The equations of motion are not reduced to normal coordinates or dissipative structures with an arbitrary damping matrix. In this case, building of orthogonality ratios faces considerable difficulties [15-17], in particular, in [16, 17] the orthogonality conditions of the free-vibration mode shapes are obtained in convolutions.

The time analysis theory [11] introduces matrix conditions for the generalized orthogonality of free elastic vibration mode shapes of the DDS with an arbitrary symmetric damping matrix C. These ratios have a rather simple mathematical form (represented by equations (10), (12)) due to the developed apparatus associated with the analysis of the characteristic matrix quadratic equation (the equation of motion of mode shapes):

$$MS^2 + CS + K = 0. \qquad (3)$$

Equation (3) in the basis consisting of the mode-shape vectors of the matrix

$$S = P \Lambda P^{-1}$$

has the form:

$$MP\Lambda^2 + CP\Lambda + KP = 0, \qquad (4)$$

where S – matrix of internal dynamic characteristics of the system;

$$P = [P_1, \ldots, P_n]$$

– matrix of the mode-shape vectors (vibration mode shapes);

$$\Lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$$

– diagonal matrix of spectral characteristics.

Each vector P_k determining the k-th mode shape of damped vibrations corresponds to the characteristic complex number $\lambda_k = -\varepsilon_k + i\omega_k$, which contains the damping factor $\varepsilon_k$in the real part and the mode frequency ω_k in the imaginary part.

The solution of equation (3) is presented in the form of the root pair

$$S_{1,2} = M^{-1}(-C + V \pm U)/2, \qquad (5)$$

where the matrices V and U have the properties of oblique and direct symmetry:

$$V = -V^{\mathrm{T}},\ U = U^{\mathrm{T}},$$

and Vieta's formulas are valid for the root pair S_1, S_2

$$S_l^{\mathrm{T}} M + MS_m = -C,\quad S_l^{\mathrm{T}} MS_m = K \qquad (6)$$
$$(m, l = 1, 2;\ m \neq l).$$

Under the condition of a small dissipation, the elements of the matrices V and iU (i is the imaginary unit) belong to the real number field [11], therefore, the matrix roots (5) are complex conjugate $S_1 = S$, $S_2 = \bar{S}$. Then, ratios (6) are converted into the form:

$$S^*M + MS = -C, \quad S^*MS = K, \tag{7}$$

where $S^* = \bar{S}^{\mathrm{T}}$.

If the mode-shape matrix P is normalized, the symmetrical matrix U is determined by the formula

$$U = (PP^{\mathrm{T}})^{-1}.$$

3. GENERALIZED ORTHOGONALITY OF THE FREEVIBRATION MODE SHAPES OF AN ELASTIC DDS

Equation (3) is reduced to a system of equivalent equations [11]:

$$S^{\mathrm{T}}M + MS + C = U, \quad S^{\mathrm{T}}MS - K = US,$$

which after the transformation in the new basis are converted into the form:

$$\Lambda P^{\mathrm{T}}MP + P^{\mathrm{T}}MP\Lambda + P^{\mathrm{T}}CP = E, \tag{8}$$

$$\Lambda P^{\mathrm{T}}MP\Lambda - P^{\mathrm{T}}KP = \Lambda. \tag{9}$$

Matrix equation (8) is the ratio of the generalized orthogonality of the matrix P. Unlike traditional orthogonality conditions (1), (2), this equation contains the matrices M, C and the diagonal form Λ. Equation (9) is a diagonal form of writing the equation of motion of the free mode shapes (3). The left-hand side of this equation is an algebraic sum total of two terms, each of which is not a diagonal matrix in the general case.

Passing from equation (8) to the system of n^2 scalar equations, we obtain respectively $n(n-1)$ conditions of the generalized orthogonality of the mode shapes P_k, P_j (at $k \neq j$) and n conditions of their normalization (at $k = j$):

$$P_j^{\mathrm{T}}[M(\lambda_j + \lambda_k) + C]P_k = 0 (k \neq j), \tag{10}$$

$$P_k^{\mathrm{T}}[2M\lambda_k + C]P_k = 1 (k = j) \tag{11}$$

$$(k, j = 1, \dots, n).$$

n^2 scalar rational so follow from the diagonal form equation (9), wherein $n(n-1)$ conditions analogous to the orthogonality ratios result at $k \neq j$ and n conditions at $k = j$:

$$P_j^{\mathrm{T}}[M\lambda_j\lambda_k - K]P_k = 0 (k \neq j), \tag{12}$$

$$P_k^{\mathrm{T}}[M\lambda_k^2 - K]P_k = \lambda_k (k = j) \tag{13}$$

$$(k, j = 1, \dots, n).$$

In the private case of the conservative system ($C = 0$, $\varepsilon_j = \varepsilon_k = 0$), formulas (10), (12), due to

$(\lambda_j + \lambda_k) = i(\omega_k + \omega_j) \neq 0$ and $\lambda_j \cdot \lambda_k = -\omega_k \cdot \omega_j \neq 0$,

are reduced to ratios (1).

The conditions, under which the ratios of the generalized orthogonality (10), (12) become applicable to an elastoplastic system, are shown below.

4. NONDEGENERATE STATE OF THE ELASTOPLASTIC DDS

Let us consider a fragment of the design scheme (to be specific–a beam model) shown in Figure 1. Suppose that plastic strains develop near the j-th node forming a local zone with the sizes a_j and b_j (respectively, left and right of the node). In relative quantities, the length of this zone is determined by the parameters

$$\alpha_j = a_j/d, \quad \beta_j = b_j / d,$$

where d represents the distance between the adjacent nodes. At the boundaries of the plastic zone, the stress value corresponds to the yield limit σ_{yl}. According to the schematic diagram of strain with hardening (Figure 2), the maximum stress $\sigma_{\max}$ in the j-th node should not exceed the strength limit σ_u.

[11, 18] analyze elastoplastic systems using Prandtl diagram and the bilinear diagram.

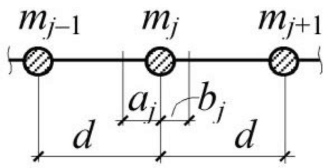

Figure 1. *Fragment of the CMD with a plastic zone (a_i, b_i) near the j-th node.*

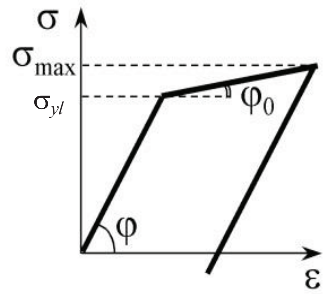

Figure 2. *"σ-ε" bilinear diagram.*

The process of calculating the inelastic response according to the mathematical vibration models was divided into a series of successive (quasilinear) intervals $t \in [t_i, t_{i+1}]$ (i = 0, 1, 2, …), within which the parameters of the design scheme (elements of the stiffness and damping matrices) remained unchanged. The elements of the mass matrix were taken constant during the entire elastoplastic strain process.

According to the proposed scheme of analysis, a certain local plastic zone (one or several) with fixed parameters, characterized by the sizes, number and location of the zones in the design scheme of the structure corresponds to the i-th quasilinear interval $t \in [t_i, t_{i+1}]$. Any non-linear events occurring in the structural elements of the system, including the beginning or the end of non-linear strains (yield and unloading), as well as changes in the parameters of the local plastic zones, are considered to occur only at the boundaries of these intervals–at the critical points t_i, t_{i+1}, ... As a result of this simulation, the inelastic analysis is reduced to a successive series of discretely varying systems that are implemented in these intervals according to the elastic design scheme.

The value of the determinant of the matrix $K(t_i) > 0$, where t_i is the time at the beginning of the i-th interval, is used to evaluate the nondegenerate state of a quasilinear system. The condition of the nondegenerate state is formulated as the following statement [11].

The stiffness matrix $K(t_i)$ of a quasilinear system is *nondegenerate if and only if both matrices of internal dynamical characteristics* S_1, S_2 in (5) *are nondegenerate:*

$$\det K(t_i) > 0 \Leftrightarrow \det S_m \neq 0 \quad (m = 1, 2). \qquad (14)$$

The fact of non-degeneracy of both matrix roots S_m follows from the second Vieta's formula in (6). Let us demonstrate the need for the condition. Suppose that the matrix $K(t_i)$ is nondegenerate in the quasilinear interval $t \in [t_i, t_{i+1}]$, then

$$\det K(t_i) = \det (S_l^{\mathrm{T}} M S_m) =$$
$$= \det (S_l) \cdot \det (S_m) \cdot \det (M) > 0 \; (m \neq l).$$

Hence, $\det S_m \neq 0$ (m = 1, 2) follows for both matrices. The need is demonstrated. The sufficiency of the statement is also obvious.

It follows from the above result, under the condition of a small dissipation, that matrices (5) in the considered interval, as well as for the elastic solution, are presented in a complex conjugate form. Therefore, conditions (6) take the form analogous to (7):

$$S^* M + MS = -C(t_i), \; S^* MS = K(t_i). \qquad (15)$$

The following inequality results from the second ratio in (15) for the lower boundary of the matrix S:

$$\|S\| \geq \sqrt{\|M\|^{-1} \cdot \|K(t_i)\|},$$

where $\|\cdot\|$ – norm of the corresponding matrix. A decrease in the stiffness parameters with a growing yield leads to a decrease in the norm$\|K(t_i)\|$, and, consequently, $\|S\|$. At the same time, the weaker the norm of the matrix S, the smaller the modules of its characteristic numbers

$$\lambda_k = -\varepsilon_k + i\omega_k (k = 1, \ldots, n).$$

Since under the usual conditions of the elastic system vibrations, the damping coefficients ${}_k(t_0)$are much less than the corresponding mode frequencies $\omega_k(t_0)$,these are the frequencies that are most sensitive to the change in the elements of the matrix K. Therefore, a decrease in the modules $\lambda_k$is initially due to a decrease in the frequency spectrum $\omega_k(t)$ ($k = 1, \dots , n$) (Figure 3).

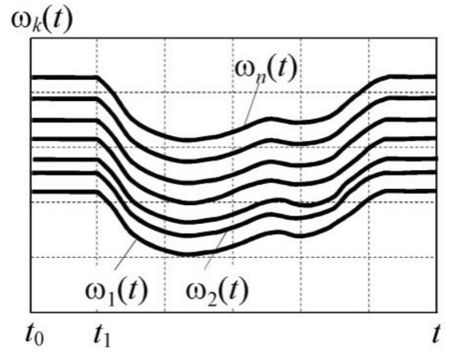

Figure 3. Frequency spectrum of a nondegenerate elastoplastic system in the course of vibrations.

Consequently, when condition (14) is met, the frequency spectrum is strictly positive and below the elastic one $0<\omega_k(t)\leq\omega_k(t_0)$, where $\omega_k(t_0)$ is the mode frequency in the elastic state of the calculation model at $t\in [t_0, t_1]$.
Hence, it follows that all the mode shapes have a harmonic law of motion

$$\varphi_k(t) = (A_k \cos \omega_k t + B_k \sin \omega_k t)\, e^{-\varepsilon_j t} ,$$

where A_k, B_k represents arbitrary constants. For a conservative system ($C = 0$), there is no damping term in the equations of mode shapes.

5. GENERALIZED ORTHOGONALITY OF THE MODE SHAPES OF ANELASTOPLASTIC DDS

An important consequence of the above result is that, under the condition of a nondegenerate state of the elastoplastic system (14), equalities (8) – (13) will be satisfied together with ratios (15). In particular, the orthogonality conditions in the matrix (8) and scalar (10), (11) form, respectively, take the form:

$$\Lambda P^{\mathrm{T}}MP + P^{\mathrm{T}}MP\Lambda + P^{\mathrm{T}}C(t_i)P = E, \qquad (16)$$

$$P_j^{\mathrm{T}}[M(\lambda_j + \lambda_k) + C(t_i)]P_k= 0 \;\; (k\neq j), \qquad (17)$$

$$P_k^{\mathrm{T}}[2M\lambda_k + C(t_i)]P_k= 1 \; (k = j) \qquad (18)$$

$$(k, j= 1, \dots , n).$$

These conditions will be valid only if two arbitrary mode shapes P_k, P_j ($k\neq j$) belong to the same quasilinear interval $t\in [t_i, t_{i+1}]$ or, which is the same, *correspond to the plastic zone (or zones) with the same parameters.*

6. EXAMPLE

Let us consider an example of elastoplastic vibrations of a hinged beam with three degrees of freedom ($n = 3$) under a dynamic action. The parameters of the design scheme of the beam are: I-beam No. 70, A516 steel, $E = 2{,}1\cdot10^5$ MPa, stiffness of the beam $EJ = 2{,}83\cdot10^9$ kN·cm^2, $d = 300$ cm (Figure 4a). The characteristics of the strength and deformability of the beam material: $\sigma_{yl} = 305$ MPa, $\sigma_u = 440$ MPa, $\delta_{yl}= \sigma_{yl}/E = 0{,}00145$, $\delta_u= 0{,}21$.
The mass matrix and stiffness matrix of the elastic system were:

$$M = \mathrm{diag}\,(0{,}5\;\; 0{,}6\;\; 0{,}5)\cdot10^{-2}\mathrm{kN\cdot s^2/cm},$$

$$K = \begin{bmatrix} 1{,}0319 & -0{,}9871 & 0{,}4038 \\ -0{,}9871 & 1{,}4357 & -0{,}9871 \\ 0{,}4038 & -0{,}9871 & 1{,}0319 \end{bmatrix}\cdot10^3\ (\mathrm{kN/cm}).$$

A nonproportional damping model was used to build the damping matrix [11]:

$$C(t_i) = [K(t_i)T + TK(t_i)]/2,$$

where: $T = \gamma W_0^{-1}$, $W_0 = \mathrm{diag}(w_{01}, \dots , w_{0n})$, $w_{0i} = \sqrt{r_{jj}\;/\;m_j}$ (r_{jj},m_j– diagonal elements of the matrices $K(t_i)$ and M), $\gamma= \delta/\pi$, $\delta = 0{,}07$.
In the elastoplastic vibration mod shape, the stiffness matrix was formed using the flexibility matrix $L(t_i) = K(t_i)^{-1}$. The displacements$\delta_{kj}(t_i)$– elements of the flexibility matrix were determined by the known methods of structural mechanics, taking into account the presence of the plastic

zones in the nodal sections of the beam: $\delta_{kj}(t_i) = \delta_{kj} f_{kj}(t_i)$.Here,$\delta_{kj}$– elements of the flexibility matrix L of the elastic system; $f_{kj}(t_i)$–correction functions that take into account non-linear strains, depend on the size and location of the plasticity zones, on the value $\xi = E_0/E$, where$E$= tgφ, E_0= tgφ$_0$(ξ = 0,003). The functions $f_{kj}(t_i)$ (k, j= 1, 2, 3) for the plastic zones near nodes 2 (Figure 4*b*) and 3 (Figure 4*c*) are shown below.

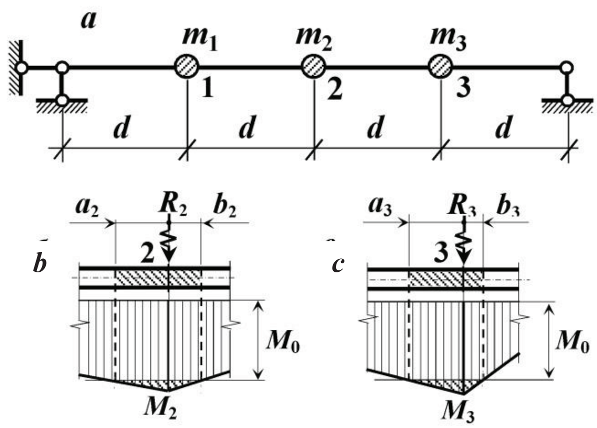

Figure 4. Design scheme of the beam (a); fragments of bending-moment curves for the plastic zones near: (b) – node 2; (c) – node 3.

Node 2 ($\alpha = a_2 / d$, $\beta = b_2 / d$):

$$f_{11} = 1 + \nu\cdot[\psi_1(\alpha) + \psi_2(\beta)]/36;$$
$$f_{22} = 1 + \nu\cdot[\psi_2(\alpha) + \psi_2(\beta)]/16;$$
$$f_{12} = f_{21} = 1 + \nu\cdot[\psi_3(\alpha) + \psi_2(\beta)]/22;$$
$$f_{33} = 1 + \nu\cdot[\psi_2(\alpha) + \psi_1(\beta)]/36;$$
$$f_{13} = f_{31} = 1 + \nu\cdot[\psi_3(\alpha) + \psi_3(\beta)]/28;$$
$$f_{23} = f_{32} = 1 + \nu\cdot[\psi_2(\alpha) + \psi_3(\beta)]/22,$$

where $\psi_1(\gamma) = 12\gamma + 6\gamma^2 + \gamma^3$, $\psi_2(\gamma) = 12\gamma - 6\gamma^2 + \gamma^3$, $\psi_3(\gamma) = 12\gamma - \gamma^3$($\gamma = \alpha, \beta$),$\nu = (1/\xi - 1)$.

Node 3 ($\alpha = a_3 / d$, $\beta = b_3 / d$):

$$f_{11} = 1 + \nu\cdot[\eta_1(\alpha) + \eta_3(\beta)]/36;$$
$$f_{22} = 1 + \nu\cdot[\eta_1(\alpha) + \eta_3(\beta)]/16;$$
$$f_{12} = f_{21} = 1 + \nu\cdot[\eta_1(\alpha) + \eta_3(\beta)]/22;$$
$$f_{33} = 1 + \nu\cdot[(27\alpha - 9\alpha^2 + \alpha^3) + 9\eta_3(\beta)]/36;$$
$$f_{13} = f_{31} = 1 + \nu\cdot[\eta_2(\alpha) + 3\eta_3(\beta)]/28;$$
$$f_{23} = f_{32} = 1 + \nu\cdot[\eta_2(\alpha) + 3\eta_3(\beta)]/22,$$

where: $\eta_1(\alpha) = 3\alpha + 3\alpha^2 + \alpha^3$, $\eta_2(\alpha) = 9\alpha + 3\alpha^2 - \alpha^3$, $\eta_3(\beta) = 3\beta - 3\beta^2 + \beta^3$.

In Figure 4: M_0 – value of the limit moment corresponding to the yield limit; R_j– restoring force at the j-th node.

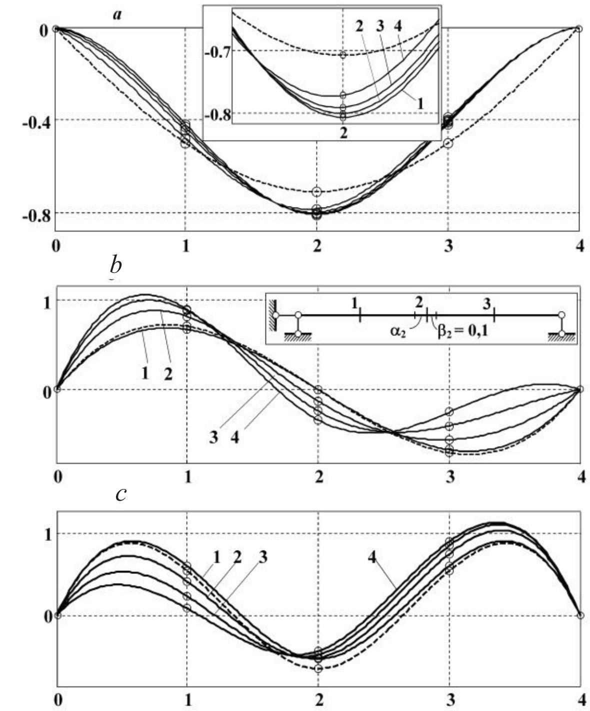

Figure 5. First (a), second (b) and third (c) free mode shapes for the yield zones in node 2 at α_2: 1 – 0,1; 2 – 0,2; 3 – 0,3; 4 – 0,5 (β_2 = 0,1).

We considered various vibration modes of the calculation beam model beyond the elastic limit. Figure 5 shows the free-vibration mode shapes P_k (k = 1, 2, 3) for local plastic zones of different lengths near node 2. Figures 6 and 7 show, respectively, the first and the second elastoplastic mode shapes of the beam for the plastic strains zones near nodes 2 and 3.

It follows from the analysis of Figures 5 – 7 that the configuration of the elastoplastic mode shapes differs sharply from the corresponding mode shapes of elastic vibrations, which are shown by a dotted line in all the figures. The configuration of the elasticoplastic modes depends on many factors: on the intensity of the yield strains development, which influences the sizes (length) of the plastic zones in the dangerous section (Figure 5); on the number of sections with the nonlinear work of the material; on the location of the zones in the design scheme (Figures 6, 7).

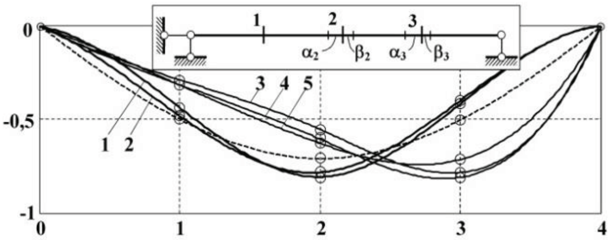

Figure 6. *First free mode shape for the yield zones: 1 – $\alpha_2 = \beta_2 = 0{,}1$;*
2 (5) – α_2 (α_3)=0,5; β_2 (β_3) = 0,1;
3 – $\alpha_3 = 0{,}05$; $\beta_3 = 0{,}005$;
4 – $\alpha_3 = 0{,}1$; $\beta_3 = 0{,}002$.

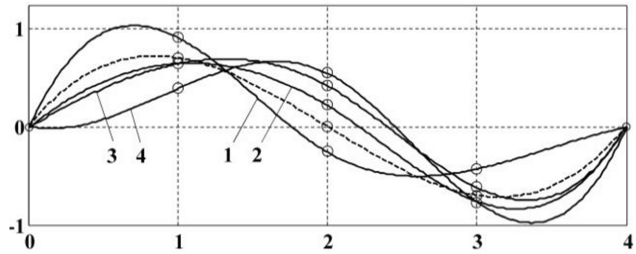

Figure 7. *Second free mode shape for the yield zones: 1 – $\alpha_2 = 0{,}3$; $\beta_2 = 0{,}1$;*
2 – $\alpha_3 = 0{,}002$; $\beta_3 = 0{,}002$;
3 – $\alpha_3 = 0{,}01$; $\beta_3 = 0{,}001$;
4 – $\alpha_3 = 0{,}5$; $\beta_3 = 0{,}002$.

The curvature of the rod axis sharply increases in the yield zones in comparison with the elastic sections of the design scheme; therefore, even in case of small sizes of the plastic zones, the free mode shapes can differ significantly from the corresponding elastic mode shapes. An example is the second mode shape in Figure 7 (position 2) built at the yield in the 3rd node for a local zone with the total length of 0,004d. In percentage terms, this length is (0,004d/4d)100% = 0,1% of the beam span l = 4d, i.e. 1/1000 part of the span, where the beam material works beyond the elastic limit. A comparison of this mode shape with its "double" –a mode shape obtained at the yield in the second node (Figure 7, position 1) with the length of the plastic zone 0,4d (10% of the beam span) shows that these mode shapes differ significantly by the configuration not only from the mode shape of elastic vibrations, but also from each other.
As it has already been noted, the critical zones in the elastoplastic process can change in size, disappear and appear again (due to multiple unloadings and loadings), change their location on the structure, etc. Since all this is connected with the appearance of the zones with a concentration of infinitely large curvatures, it will lead to a significant "distortion" of their mode shapes with respect to their elastic analogs and, eventually, to the modification of these mode shapes that cease to be similar to themselves in the process of nonlinear vibrations.
It should be noted that during a nonlinear analysis of structures using numerical methods of the stepwise integration, the dynamic response is generally built using the original (i.e., elastic) mode shapes of undamped vibrations. It is considered to be permissible, "if inelastic strains do not lead to major changes in the nature of their deflections ..." [12]. However, in practice, such estimates are difficult to implement, and their implementation provides for the creation of more sophisticated mathematical models that ensure building of free-vibration mode shape staking into account inelastic strains.
Resorting to the previous example, we will give a comparative analysis of the movements of the third node at elastoplastic beam vibrations caused by the action of the force R_3 (Figure 4c). For a plastic zone with the following parameters: α_3 = 0,002; β_3 = 0,001,the node movements are 1,73 times larger than the corresponding elastic movements, i.e., the local zone with the total length of 0,003d = 0,9 cm leads to a change in the node movements by almost 2 times. If the zone sizes are doubled: α_3 = 0,004; β_3 = 0,002, the ratio of the movements increases already in 2,45 times.
Thus, the proposed algorithm allows us to obtain not only qualitative but also quantitative estimates of the influence of the inelastic strains on the characteristics of the strain-stress state in the structure. Since even small plastic zones (less than 1 cm) can significantly alter the strained state of structural elements, it suggests the need to create new approaches that allow us to consider nonlinear strains when formation of mode shapes. In this connection, there arises a question

on the correctness of the methods based on the use of elastic vibration mode shapes in a nonlinear analysis, and an estimate of the error degree of such an analysis.

The final part of the paper gives an example of meeting the orthogonality conditions (16) for the free-vibration mode shapes of a beam with a local plastic zone in the second node (Figure 5). The correction functions $f_{kj}(t_i)$ are built for the parameters: $\alpha_2 = 0,2$; $\beta_2 = 0,1$. As a result, the values of the stiffness matrix and damping matrix were:

$$K(t_i) = \begin{bmatrix} 407,96 & -261,45 & 83,56 \\ -261,45 & 319,26 & -347,94 \\ 83,56 & -347,94 & 592,77 \end{bmatrix} \text{(kN/cm)},$$

$$C(t_i) = \begin{bmatrix} 0.0318 & -0.0228 & 0.00596 \\ -0.0228 & 0.0308 & -0.0281 \\ 0.00596 & -0.0281 & 0.0384 \end{bmatrix} \text{(kN·s/cm)}.$$

From the solution of (3) we obtained matrix roots (5) $S_1 = S$, $S_2 = \bar{S}$, which allows us to calculate the spectral characteristics – the mode-shape matrix P and the diagonal form Λ (the values of the matrices P and Λ correspond to the matrix S_1):

$$S_{1,2} = \begin{bmatrix} -3,18 \pm i258,77 & 2,02 \mp i132,44 & -0,59 \mp i0,62 \\ +2,12 \mp i110,37 & -2,57 \pm i136,67 & 2,75 \mp i128,78 \\ -0,60 \mp i0,62 & 2,31 \mp i154,53 & -3,84 \pm i314,05 \end{bmatrix} (c^{-1}),$$

$$P = \begin{bmatrix} 0.431 & 0.819 & 0.420 + i0,39*10^{-2} \\ 0.803 & -0.127 + i0,12*10^{-2} & -.509 - i0,46*10^{-2} \\ 0.411 & -0.560 - i0,25*10^{-2} & 0.752 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$

where: $\lambda_1 = -0,01 + i11,5$; $\lambda_2 = -2,91 + i279,8$; $\lambda_3 = -6,68 + i418,19$ (c^{-1}).

For comparison, λ_k corresponding to the elastic solution are shown: $\lambda_1 = -0,17 + i85,07$; $\lambda_2 = -3,08 + i354,43$; $\lambda_3 = -12,32 + i720,45$ (c^{-1}).

The above values of the matrices satisfy ratios (15), (16). Thus, the calculation of the left-side of ratio (16) leads to the identity matrix:

$$\Lambda P^T MP + P^T MP\Lambda + P^T CP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thegeneralizedorthogonalityconditionspresentedbythescalarformoftheratios (17) are realized for off-diagonal elements of this matrix. The diagonal elements of the matrix are obtained by normalizing the mode-shape matrix P using the diagonal matrix D:

$$D = \begin{bmatrix} 1.962 - i1.963 & 0 & 0 \\ 0 & 0.422 - i0.423 & 0 \\ 0 & 0 & 0.336 - i0.339 \end{bmatrix},$$

which ensures the fulfillment of the conditions (18).

7. CONCLUSION

1. It is shown that:
- traditional (classical) ratios of the orthogonality for the free-vibration mode shapes are a private case of more general conditions of the generalized orthogonality of the mode shapes valid for an elastic dissipative system;
- the free mode shapes of an elastoplastic system in a quasilinear interval corresponding to the same parameters of the plastic zones have the property of the generalized orthogonality, which is determined by the equations analogous to the equations of an elastic dissipative system.

2. For the free-vibrations motion of a hinged beam with three degrees of freedom, taking into account the internal friction, it is shown that:
- elastoplastic mode shapes are qualitatively different from the corresponding mode shapes

of elastic vibrations in their configuration;
- arbitrary k-th free-vibration mode shapes formed for the plastic zones located in different nodes of the calculation model, also differ from each other in their configuration.

3. The proposed algorithm allows us to obtain not only qualitative but also quantitative estimates of the influence of the inelastic strains on the strain-stress state characteristics of the structure. The development of this approach to formating of mode shapes, taking into account the elastoplastic strains, opens the possibility of creating more sophisticated structural analysis methods at nonlinear vibrations.

REFERENCES

1. **Kilimnik L.Sh**. K Razrabotke Metodiki Otsenki Predel'nykh Sostoianii Mnogoetazhnykh Karkasnykh Zdanii pri Seismicheskikh Vozdeistviiakh [To the Development of a Procedure for Estimating the Limiting States of Multi-Story Frame Buildings under Seismic Influence]. // Proceedings of the Central Research Institute of Building Constructions named after V.A. Kucherenko, 1975, Issue 44, pp. 66-82.
2. **Newmark N., Rosenblueth E.** Osnovy Seismostoikogo Stroitel'stva [Fundamentals of Earthquake Engineering]. Moscow, Stroyizdat, 1980, 344 pages.
3. Svod pravil SP 14.13330.2014 Stroitel'stvo v Seismicheskikh Raionakh SNiP II-7-81* [Rules and Regulations SP 14.13330.2014 / Construction in Seismic Regions SNiP II-7-81*]. Moscow, 126 pages.
4. **Sosnin A.V.** Ob Osobennostiakh Metodologii Nelineinogo Staticheskogo Analiza i Ego Soglasovannosti s Bazovoi Normativnoi Metodikoi Rascheta Zdanii i Sooruzhenii na Deistvie Seismicheskikh Sil [On the Features of the Methodology of the Non-Linear Static Analysis and its Consistency with the Basic Normative Methodology for Structural Engineering for the Action of Seismic Forces]. // Bulletin of the SUSU. Series "Construction and Architecture", 2016, Vol. 16, No. 1, pp. 12-19.
5. **Paulay T., Priestley M.J.N.** Seismic Design of Reinforced Concrete and Masonry Buildings. New York: John Wiley&Sons, Inc., 1992. – 744 p.
6. **Inel M., Ozmen H.B.** Effects of Plastic Hinge Properties in Nonlinear Analysis of Reinforced Concrete Buildings // Engineering Structures. No. 28. 2006. PP. 1494-1502.
7. **Gorji M.S., Moghaddam M.Z.** Analysis of the Resistance of Joints in Reinforced Concrete Strengthened by FRP Sheets against Released Forces of Earthquake. // Journal of American Science, 2011, No. 7(9), pp. 995-999.
8. **Naida A.A.** Ostatochnye Deformatsii Impul'sno Nagruzhennykh Balok, Dopuskaiushchikh Bol'shie Progiby v Predelakh Uprugostil [Residual Strains of Impulse Loaded Beams Allowing Large Bends Within the Elasticity Range]. // Scientific notes of the Central Aerohydrodynamic Institute, 1987, V. XVIII, No. 5, pp. 106-112.
9. **Vitvitskiy N.K.** Vychislenie Chastot Sobstvennykh Izgibnykh Kolebanii Balok za Predelom Uprugosti pri Nalichii Plasticheskikh Zon [Calculation of the Frequencies of Free Bending Vibrations of Beams Beyond the Elastic Limit in the Presence of Plastic Zones]. // News of S.M. Kirov's Tomsk Polytechnic Institute of the Order of the Red Banner of Labor, 1967, Vol. 154, pp. 28-34.
10. **Potapov A.N.** Analiz Vnutrennikh Dinamicheskikh Parametrov Konstruktsii pri Neuprugikh Kolebaniiakh [Analysis of Internal Dynamic Parameters of Structures at Inelastic Vibrations]. // News of higher educational institutions. Construction, 2000, No. 6, pp. 31-37.
11. **Potapov A.N.** Dinamicheskii Analiz Diskretnykh Dissipativnykh Sistem pri Nestatsionarnykh Vozdeistviiakh [Dynamic Analysis of Discrete Dissipative Systems

under Nonstationary Influences: monograph]. Chelyabinsk, Publishing House of the SUSU, 2003, 167 pages.
12. **Clough R., Penzien D.J.** Dinamika Sooruzhenii [Dynamics of Structures]. Moscow, Stroyizdat, 1979, 320 pages.
13. **Kiselev V.A.** Stroitel'naia Mekhanika [Structural Mechanics]. // Special course: Dynamics and Stability of Structures. Moscow, Stroyizdat, 1980, 616 pages.
14. **Sorokin E.S.** O Pogreshnostiakh Obshcheizvestnogo Metoda Teorii Kolebanii Dissipativnykh Sistem v Primenenii k Neodnorodnomu Dempfirovaniiu [On the Errors of the Well-Known Method of the Theory of Vibrations in Dissipative Systems as Applied to Inhomogeneous Damping]. Structural Mechanics and Analysis of Constructions, 1984, pp. 29-34.
15. **Harris S.M., Krid C.I.** Spravochnik po Udarnym Nagruzkam [Handbook on Shock Loads]. London, Shipbuilding, 1980, 360 pages.
16. **Pashkov I.A.** Ob Ortogonal'nosti Sobstvennykh Form Kolebanii Viazkouprugogo Tela [On the Orthogonality of Free-Vibration Mode Shapes of a Viscoelastic Body]. // News of the Academy of Sciences of the USSR. Solid Mechanics. 1989, No. 4, pp. 104-111.
17. **Pashkov I.A., Troyanovsky I.E.** Metod Razlozhenii po Sobstvennym Formam Kolebanii Uprugogo Tela s Vnutrennim i Vneshnim Treniem [Method of Splitting by the Free-Vibration Mode Shapes of an Elastic Body with Internal and External Friction]. // Applied Mathematics and Mechanics, 1991, Vol. 55, Issue 6, pp. 972-981.
18. **Potapov A.N., Zyambaev N.A.** Postroenie Matematicheskoi Modeli Fizicheski Nelineinykh Kolebanii Mnogoetazhnogo Karkasa [Building of a Mathematical Model of Physically Nonlinear Vibrations of a Multistoried Frame]. // Bulletin of the SUSU. Series “Construction and Architecture”, 2017, Vol. 17, No. 3, pp. 12-17.

СПИСОК ЛИТЕРАТУРЫ

1. **Килимник Л.Ш.** К разработке методики оценки предельных состояний многоэтажных каркасных зданий при сейсмических воздействиях. // Труды Центрального научно-исследовательского институт строительных конструкций им. В.А. Кучеренко, 1975, Вып. 44, с. 66-82.
2. **Ньюмарк Н., Розенблюэт Э.** Основы сейсмостойкого строительства. – М.: Стройиздат, 1980. – 344 с.
3. **Свод правил** СП 14.13330.2014 / Строительство в сейсмических районах СНиП II-7-81*. – М., 126 с.
4. **Соснин А.В.** Об особенностях методологии нелинейного статического анализа и его согласованности с базовой нормативной методикой расчета зданий и сооружений на действие сейсмических сил. // Вестник ЮУрГУ. Серия «Строительство и архитектура», 2016, Том 16, №1, с. 12-19.
5. **Paulay T. Priestley M.J.N.** Seismic Design of Reinforced Concrete and Masonry Buildings. New York, John Wiley&Sons Inc., 1992, 744 pages.
6. **Inel M., Ozmen H.B.** Effects of Plastic Hinge Properties in Nonlinear Analysis of Reinforced Concrete Buildings. // Engineering Structures, 2006, No. 28, pp. 1494-1502.
7. **Gorji M.S., Moghaddam M.Z.** Analysis of the Resistance of Joints in Reinforced Concrete Strengthened by FRP Sheets against Released Forces of Earthquake. // Journal of American Science, 2011, No. 7(9), pp. 995-999.
8. **Найда А.А.** Остаточные деформации импульсно нагруженных балок, допускающих большие прогибы в пределах упругости. // Ученые записки ЦАГИ, 1987, Том XVIII, №5, с. 106-112.
9. **Витвицкий Н.К.** Вычисление частот собственных изгибных колебаний балок за пределом упругости при наличии пластических зон. // Известия Томского ор-

дена трудового красного знамени политехнического института им. С.М. Кирова, 1967, Том 154, с. 28-34.
10. **Потапов А.Н.** Анализ внутренних динамических параметров конструкций при неупругих колебаниях. // Известия вузов. Строительство, 2000, №6, с. 31-37.
11. **Потапов А.Н.** Динамический анализ дискретных диссипативных систем при нестационарных воздействиях. – Челябинск: Издательство ЮУрГУ, 2003. – 167 с.
12. **Клаф Р., Пензиен Д.Ж.** Динамика сооружений. – М.: Стройиздат, 1979. – 320 с.
13. **Киселев В.А.** Строительная механика. Специальный курс: Динамика и устойчивость сооружений. – М.: Стройиздат, 1980. – 616 с.
14. **Сорокин Е.С.** О погрешностях общеизвестного метода теории колебаний диссипативных систем в применении к неоднородному демпфированию. // Строительная механика и расчёт сооружений, 1984, с. 29-34.
15. **Харрис С.М., Крид Ч.И.** Справочник по ударным нагрузкам. – Л.: Судостроение, 1980. – 360 с.
16. **Пашков И.А.** Об ортогональности собственных форм колебаний вязкоупругого тела. // Известия АН СССР. Механика твердого тела, 1989, №4, с. 104-111.
17. **Пашков И.А., Трояновский И.Е.** Метод разложений по собственным формам колебаний упругого тела с внутренним и внешним трением. // Прикладная математика и механика, 1991, Том 55, Вып. 6, с. 972-981.
18. **Потапов А.Н., Зямбаев Н.А.** Построение математической модели физически нелинейных колебаний многоэтажного каркаса. // Вестник ЮУрГУ. Серия «Строительство и архитектура», 2017, Том 17, №3, с. 12-17.

Alexander N. Potapov, Professor, Dr.Sc., Department “Construction production and theory of structures”, South-Ural State University (National Research University); 76, Lenina Avenue, Chelyabinsk, 454080, Russia;
phone: +7(351)267-91-83, +7(919)343-71-29;
E-mail: potapov.alni@gmail.com.

Потапов Александр Николаевич, профессор, доктор технических наук, кафедра «Строительное производство и теория сооружений», Южно-Уральский государственный университет (национальный исследовательский университет); 454080, Россия, г. Челябинск, проспект Ленина, дом 76;
тел.: +7(351) 267-91-83, +7(919)343-71-29
E-mail: potapov.alni@gmail.com.