Kiataramkul et al. *Boundary Value Problems* (2016) 2016:217
DOI 10.1186/s13661-016-0725-1

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Generalized Sturm-Liouville and Langevin equations via Hadamard fractional derivatives with anti-periodic boundary conditions

Chanakarn Kiataramkul[1], Sotiris K Ntouyas[2,3], Jessada Tariboon[1*] and Atthapol Kijjathanakorn[1]

*Correspondence:
jessada.t@sci.kmutnb.ac.th
[1]Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce a new class of anti-periodic boundary value problems by combining Sturm-Liouville and Langevin fractional differential equations of Hadamard type. Existence and uniqueness results are proved via fixed point theorems. Examples illustrating the obtained results are also presented.

MSC: 34A08; 34B15; 34B24

Keywords: Sturm-Liouville problem; Langevin equation; fractional differential equations; Hadamard fractional derivative; boundary value problems; fixed point theorems

1 Introduction

The Sturm-Liouville problem has many applications in different areas of science, for example, engineering and mathematics. The classical Sturm-Liouville problem for a linear differential equation of second order is a boundary value problem as the following one:

$$\begin{cases} -\frac{d}{dt}[p(t)\frac{dx}{dt}] + v(t)x = \lambda r(t)x, \quad t \in [a,b], \\ a_1 x(a) + a_2 x'(a) = 0, \\ b_1 x(b) + b_2 x'(b) = 0. \end{cases} \tag{1.1}$$

Recently in [1] the authors proposed an approach to the fractional version of the Sturm-Liouville problem. They investigated the eigenvalues and eigenfunctions associated to these operators and also theirs properties, with the objective of applying this generalized Sturm-Liouville theory to fractional partial differential equations.

Fractional differential equations have attracted the attention of many researchers working in a variety of disciplines, due to the development and applications of these equations in many fields such as engineering, mathematics, physics, chemistry, etc. For recent developments of the topic, we refer the reader to [2–13]. However, it has been noticed that most of the work on the topic is concerned with Riemann-Liouville- or Caputo-type fractional differential equations. Besides these fractional derivatives, another kind of fractional derivatives found in the literature is the fractional derivative due to Hadamard, introduced in 1892 [14], which differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains a logarithmic function of arbitrary exponent. A detailed description of the Hadamard fractional derivative and integral can be found in [3, 15–19].

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [20]. For some new developments on the fractional Langevin equation, see, for example, [21–29].

In this paper we introduce a new class of boundary value problems by combining Sturm-Liouville and Langevin fractional differential equations. More precisely, we initiate the study of the existence and uniqueness of solutions for the generalized Sturm-Liouville and Langevin fractional differential equations of Hadamard type, with anti-periodic boundary conditions of the form

$$\begin{cases} D^{\beta}([p(t)D^{\alpha}+r(t)]x(t)) = g(t,x(t)), & 1<t<T, \\ x(1) = -x(T), \qquad D^{\alpha}x(1) = -D^{\alpha}x(T), \end{cases} \tag{1.2}$$

where D^{ρ} denotes the Caputo-type Hadamard fractional derivative of order ρ, $\rho \in \{\alpha,\beta\}$ with $0<\alpha,\beta<1$, $p \in C([1,T],\mathbb{R})$ with $|p(t)| \geq K > 0$, $r \in C([1,T],\mathbb{R})$, and $g \in C([1,T] \times \mathbb{R},\mathbb{R})$.

Note that:

- If $r(t) \equiv 0$ for all $t \in [1,T]$, then the problem (1.2) is reduced to the Sturm-Liouville fractional boundary value problem of Hadamard type of the form

$$\begin{cases} D^{\beta}(p(t)D^{\alpha}x(t)) = g(t,x(t)), & 1<t<T, \\ x(1) = -x(T), \qquad D^{\alpha}x(1) = -D^{\alpha}x(T). \end{cases} \tag{1.3}$$

- If $p(t) \equiv 1$ and $r(t) \equiv \lambda$, $\lambda \in \mathbb{R}$, for $t \in [1,T]$, then the problem (1.2) is reduced to

$$\begin{cases} D^{\beta}(D^{\alpha}+\lambda)x(t) = g(t,x(t)), & 1<t<T, \\ x(1) = -x(T), \qquad D^{\alpha}x(1) = -D^{\alpha}x(T), \end{cases} \tag{1.4}$$

  which is the Langevin fractional boundary value problem.

This paper is organized as follows. In Section 2, some necessary definitions and lemmas that will be used to prove our main result are shown. In Section 3, we prove our main results. By applying the Banach contraction mapping principle an existence and uniqueness result is proved. Moreover, two existence results are established via Leray-Schauder nonlinear alternative and Krasnosleskii's fixed point theorem. Illustrative examples are presented in Section 4.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [2, 3] and present preliminary results needed in our proofs later.

Definition 2.1 For an at least n-times differentiable function $f:[1,\infty) \to \mathbb{R}$, the Caputo-type Hadamard derivative of fractional order α is defined as

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} \delta^n f(s)\frac{ds}{s}, \quad n-1<\alpha<n, n=[\alpha]+1,$$

where $\delta = t\frac{d}{dt}$, $\log(\cdot) = \log_e(\cdot)$, $[\alpha]$ denotes the integer part of the real number α.

Definition 2.2 The Hadamard fractional integral of order α is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}, \quad \alpha > 0,$$

provided the integral exists.

Lemma 2.1 ([30]) *Let $u \in AC_\delta^n[a,b]$ or $C_\delta^n[a,b]$ and $\alpha \in \mathbb{C}$, where $X_\delta^n[a,b] = \{f : [a,b] \to \mathbb{C} : \delta^{n-1}f(t) \in X[a,b]\}$. Then we have*

$$\left(I^\alpha D^\alpha u\right)(t) = u(t) - \sum_{k=0}^{n-1} c_k(\log t)^k,$$

where $c_k \in \mathbb{R}$, $k = 0,1,2,\dots,n-1$, $(n = [\alpha]+1)$.

For the sake of convenience, we set the constants

$$\mu = \frac{p(1)}{p(T)}, \qquad \xi = \mu r(T) - r(1) \quad \text{and} \quad \eta = I^\alpha\left(\frac{1}{p}\right)(T). \tag{2.1}$$

Observe that $\mu > 0$ and $\eta \neq 0$. For $g \in C([1,T] \times \mathbb{R}, \mathbb{R})$, we use the following notation:

$$I^\alpha(g_x)(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} g\big(s,x(s)\big)\frac{ds}{s}.$$

Lemma 2.2 *The problem* (1.2) *is equivalent to the following fractional integral equation*:

$$\begin{aligned} x(t) = {} & I^\alpha\left(\frac{1}{p}I^\beta g_x\right)(t) - I^\alpha\left(\frac{r}{p}x\right)(t) \\ & + \left(\frac{-\mu}{\mu+1}I^\beta(g_x)(T) + \frac{\xi}{\mu+1}x(T)\right)I^\alpha\left(\frac{1}{p}\right)(t) \\ & - \frac{1}{2}\left[I^\alpha\left(\frac{1}{p}I^\beta g_x\right)(T) - I^\alpha\left(\frac{r}{p}x\right)(T)\right. \\ & \left. + \left(\frac{-\mu}{\mu+1}I^\beta(g_x)(T) + \frac{\xi}{\mu+1}x(T)\right)\eta\right], \end{aligned} \tag{2.2}$$

where μ, ξ, and η are defined by (2.1).

Proof Taking the Hadamard fractional integral of order β to both sides of the problem (1.2) and applying Lemma 2.1, we obtain

$$p(t)D^\alpha x(t) + r(t)x(t) = I^\beta(g_x)(t) + c_0, \quad c_0 \in \mathbb{R},$$

which yields

$$D^\alpha x(t) = \frac{I^\beta(g_x)(t) - r(t)x(t) + c_0}{p(t)}. \tag{2.3}$$

The boundary condition $D^{\alpha}x(1) = -D^{\alpha}x(T)$ implies

$$c_0 = \frac{-\mu}{\mu+1} I^{\beta}(g_x)(T) + \frac{\xi}{\mu+1}x(T).$$

Using the Hadamard fractional integral of order α to both sides of (2.3) and applying Lemma 2.1 again, we have

$$x(t) = I^{\alpha}\left(\frac{1}{p}I^{\beta}g_x\right)(t) - I^{\alpha}\left(\frac{r}{p}x\right)(t) + c_0 I^{\alpha}\left(\frac{1}{p}\right)(t) + c_1, \tag{2.4}$$

where $c_1 \in \mathbb{R}$. By utilizing the anti-periodic boundary condition $x(1) = -x(T)$, it follows that

$$c_1 = -\frac{1}{2}\left[I^{\alpha}\left(\frac{1}{p}I^{\beta}g_x\right)(T) - I^{\alpha}\left(\frac{r}{p}x\right)(T) + c_0 I^{\alpha}\left(\frac{1}{p}\right)(T)\right].$$

Substituting the constants c_0 and c_1 into (2.4), we get the fractional integral equation (2.2) as desired.

Conversely, it can easily be shown by direct computation that the integral equation (2.2) satisfies the problem (1.2). This completes the proof. □

3 Main results

Let $\mathcal{C} = C([1,T],\mathbb{R})$ be the Banach space of all continuous functions from $[1,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [1,T]\}$. Define an operator $\mathcal{A}:\mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}\mathcal{A}x(t) = {} & I^{\alpha}\left(\frac{1}{p}I^{\beta}g_x\right)(t) - I^{\alpha}\left(\frac{r}{p}x\right)(t) \\ & + \left(\frac{-\mu}{\mu+1}I^{\beta}(g_x)(T) + \frac{\xi}{\mu+1}x(T)\right)I^{\alpha}\left(\frac{1}{p}\right)(t) \\ & - \frac{1}{2}\left[I^{\alpha}\left(\frac{1}{p}I^{\beta}g_x\right)(T) - I^{\alpha}\left(\frac{r}{p}x\right)(T)\right. \\ & \left. + \left(\frac{-\mu}{\mu+1}I^{\beta}(g_x)(T) + \frac{\xi}{\mu+1}x(T)\right)\eta\right].\end{aligned} \tag{3.1}$$

Observe that the problem (1.2) has solutions if and only if the operator $\mathcal{A}$ has fixed points.

Define the constants $p_* = \inf_{t\in[1,T]}|p(t)|$ and $r^* = \sup_{t\in[1,T]}|r(t)|$ and set

$$\Omega_1 = \frac{3(\log T)^{\alpha+\beta}}{2p_*\Gamma(\alpha+\beta+1)} + \frac{3\mu|\eta|(\log T)^{\beta}}{2(\mu+1)\Gamma(\beta+1)}, \tag{3.2}$$

$$\Omega_2 = \frac{3r^*(\log T)^{\alpha}}{2p_*\Gamma(\alpha+1)} + \frac{3|\xi||\eta|}{2(\mu+1)}, \tag{3.3}$$

$$\Omega_3 = \Omega_1 L + \Omega_2. \tag{3.4}$$

3.1 Existence and uniqueness result

In this subsection we give one existence and uniqueness result, by using the Banach contraction mapping principle.

Theorem 3.1 *Assume that*:

(H_1) *there exists a constant* $L > 0$ *such that* $|g(t,x)-g(t,y)| \leq L|x-y|$, *for each* $t \in [1,T]$ *and* $x, y \in \mathbb{R}$.

If

$$\Omega_3 < 1, \tag{3.5}$$

where Ω_3 *is defined by* (3.3), *then the problem* (1.2) *has a unique solution on* $[1,T]$.

Proof To prove that the problem (1.2) has a unique solution, we consider a fixed point problem, $x = \mathcal{A}x$, where the operator $\mathcal{A}$ is defined as in (3.1). To accomplish our purpose, we apply the Banach contraction mapping principle to show that $\mathcal{A}$ has a unique fixed point.

We define $\sup_{t\in[1,T]} |g(t,0)| = M < \infty$, and choose

$$R \geq \frac{\Omega_1 M}{1-\Omega_3}, \tag{3.6}$$

where Ω_1 and Ω_3 are defined by (3.2) and (3.4), respectively. Now, we show that $\mathcal{A}B_R \subset B_R$, where $B_R = \{x \in \mathcal{C} : \|x\| \leq R\}$. For any $x \in B_R$, we have

$$\begin{aligned}
\bigl|\mathcal{A}x(t)\bigr| \leq{}& I^{\alpha}\left(\frac{1}{|p|}I^{\beta}\bigl(|g_x|\bigr)\right)(t) + I^{\alpha}\left(\frac{|r|}{|p|}|x|\right)(t) \\
&+ \left(\frac{\mu}{(\mu+1)}I^{\beta}\bigl(|g_x|\bigr)(T) + \frac{|\xi|}{(\mu+1)}\bigl|x(T)\bigr|\right)I^{\alpha}\left(\frac{1}{|p|}\right)(t) \\
&+ \frac{1}{2}\Biggl[I^{\alpha}\left(\frac{1}{|p|}I^{\beta}|g_x|\right)(T) + I^{\alpha}\left(\frac{|r|}{|p|}|x|\right)(T) \\
&+ \left(\frac{\mu}{(\mu+1)}I^{\beta}\bigl(|g_x|\bigr)(T) + \frac{|\xi|}{(\mu+1)}\bigl|x(T)\bigr|\right)|\eta|\Biggr] \\
\leq{}& I^{\alpha}\left(\frac{1}{|p|}I^{\beta}\bigl(|g_x - g_0| + |g_0|\bigr)\right)(T) + I^{\alpha}\left(\frac{|r|}{|p|}|x|\right)(T) \\
&+ \left(\frac{\mu}{(\mu+1)}I^{\beta}\bigl(|g_x - g_0| + |g_0|\bigr)(T) + \frac{|\xi|}{(\mu+1)}\bigl|x(T)\bigr|\right)I^{\alpha}\left(\frac{1}{|p|}\right)(T) \\
&+ \frac{1}{2}I^{\alpha}\left(\frac{1}{|p|}I^{\beta}\bigl(|g_x - g_0| + |g_0|\bigr)\right)(T) + \frac{1}{2}I^{\alpha}\left(\frac{|r|}{|p|}|x|\right)(T) \\
&+ \frac{1}{2}\left(\frac{\mu}{(\mu+1)}I^{\beta}\bigl(|g_x - g_0| + |g_0|\bigr)(T) + \frac{|\xi|}{(\mu+1)}\bigl|x(T)\bigr|\right)|\eta| \\
\leq{}& \frac{3}{2}\left(\frac{RL+M}{p_*}\right)\frac{(\log T)^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{3}{2}\left(\frac{Rr^*}{p_*}\right)\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \\
&+ \frac{3}{2}\left(\frac{\mu|\eta|}{(\mu+1)}\right)(RL+M)\frac{(\log T)^{\beta}}{\Gamma(\beta+1)} + \frac{3|\xi||\eta|}{2(\mu+1)}R \\
={}& \Omega_1 M + \Omega_3 R \leq R.
\end{aligned}$$

This implies that $\mathcal{A}B_R \subset B_R$.

By (H_1), for any $x, y \in B_R$, we have

$$\begin{aligned}
&\left|\mathcal{A}x(t) - \mathcal{A}y(t)\right| \\
&\leq \frac{1}{p_*} I^{\alpha+\beta}\left(|g_x - g_y|\right)(T) + \frac{r^*}{p_*}\|x-y\| I^{\alpha}(1)(T) + \frac{|\eta|\mu}{(\mu+1)} I^{\beta}\left(|g_x - g_y|\right)(T) \\
&\quad + |\eta|\|x-y\|\frac{|\xi|}{(\mu+1)} + \frac{1}{2p_*} I^{\alpha+\beta}\left(|g_x - g_y|\right)(T) + \frac{r^*}{2p_*}\|x-y\| I^{\alpha}(1)(T) \\
&\quad + \frac{|\eta|\mu}{2(\mu+1)} I^{\beta}\left(|g_x - g_y|\right)(T) + \frac{|\eta||\xi|}{2(\mu+1)}\|x-y\| \\
&\leq \frac{L\|x-y\|(\log T)^{\alpha+\beta}}{p_*\Gamma(\alpha+\beta+1)} + \frac{r^*\|x-y\|(\log T)^{\alpha}}{p_*\Gamma(\alpha+1)} + \frac{|\eta|\mu}{(\mu+1)}\frac{L\|x-y\|(\log T)^{\beta}}{\Gamma(\beta+1)} \\
&\quad + \frac{|\eta||\xi|}{(\mu+1)}\|x-y\| + \frac{L\|x-y\|(\log T)^{\alpha+\beta}}{2p_*\Gamma(\alpha+\beta+1)} + \frac{r^*\|x-y\|(\log T)^{\alpha}}{2p_*\Gamma(\alpha+1)} \\
&\quad + \frac{|\eta|\mu}{2(\mu+1)}\frac{L\|x-y\|(\log T)^{\beta}}{\Gamma(\beta+1)} + \frac{|\eta||\xi|}{2(\mu+1)}\|x-y\| \\
&= \Omega_3\|x-y\|.
\end{aligned}$$

As $\Omega_3 < 1$, $\mathcal{A}$ is a contraction. Therefore, we see from the Banach contraction mapping principle that the operator $\mathcal{A}$ has a fixed point which is the unique solution of the boundary value problem (1.2). This completes the proof. □

If $r(t) \equiv 0$ for $t \in [1, T]$, then we have $\xi = 0$ and $r^* = 0$ and we also get the following result.

Corollary 3.1 *Suppose that the condition* (H_1) *holds. If* $\Omega_1 L < 1$, *where* Ω_1 *is defined by* (3.2), *then the problem* (1.3) *has a unique solution on* $[1, T]$.

If $p(t) \equiv 1$ and $r(t) \equiv \lambda$ for $t \in [1, T]$ and $\lambda \in \mathbb{R}$, then we obtain $p_* = 1$, $r^* = |\lambda|$, $\mu = 1$, $\xi = 0$, $\eta = \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}$, and the following corollary.

Corollary 3.2 *Assume that the condition* (H_1) *is satisfied. If*

$$\frac{3}{2}\left(\frac{1}{\Gamma(\alpha+\beta+1)} + \frac{1}{2\Gamma(\alpha+1)\Gamma(\beta+1)}\right)L(\log T)^{\alpha+\beta} + \frac{3}{2}|\lambda|\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} < 1,$$

then the problem (1.4) *has a unique solution on* $[1, T]$.

3.2 Existence results

Now we give an existence result via Leray-Schauder nonlinear alternative.

Theorem 3.2 (Nonlinear alternative for single valued maps [31]) *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $\mathcal{A} : \bar{U} \to C$ is a continuous, compact* (*that is, $\mathcal{A}(\bar{U})$ is a relatively compact subset of C*) *map. Then either*

(i) *$\mathcal{A}$ has a fixed point in $\bar{U}$, or*

(ii) *there is a $x \in \partial U$* (*the boundary of U in C*) *and $\lambda \in (0, 1)$ with $x = \lambda\mathcal{A}(x)$.*

Theorem 3.3 *Assume that*:

(H_2) *there exist a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $\varphi \in C([1,T],\mathbb{R}^+)$ *such that*

$$\bigl|g(t,x)\bigr| \le \varphi(t)\psi\bigl(|x|\bigr) \quad \text{for each } (t,x) \in [1,T] \times \mathbb{R}; \tag{3.7}$$

(H_3) *there exists a constant* $M > 0$ *such that*

$$\frac{(1-\Omega_2)M}{\|\varphi\|\psi(M)\Omega_1} > 1, \qquad \Omega_2 < 1, \tag{3.8}$$

where Ω_1, Ω_2 *are defined by* (3.2) *and* (3.3), *respectively.*

Then the boundary value problem (1.2) *has at least one solution on* $[1,T]$.

Proof Let the operator $\mathcal{A}$ be defined as in (3.1). Now, we are going to prove that the operator $\mathcal{A}$ *maps bounded sets (balls) into bounded sets in* $C([1,T],\mathbb{R})$. For $\rho > 0$, we define a bounded ball $B_\rho = \{x \in C([1,T],\mathbb{R}) : \|x\| \le \rho\}$. Then, for $t \in [1,T]$, we have

$$\begin{aligned}
\bigl|(\mathcal{A}x)(t)\bigr| \le{}& I^\alpha\left(\frac{1}{|p|}I^\beta\bigl(|g_x|\bigr)\right)(t) + I^\alpha\left(\frac{|r|}{|p|}|x|\right)(t) + \left(\frac{\mu}{(\mu+1)}I^\beta\bigl(|g_x|\bigr)(T)\right.\\
&\left.+ \frac{|\xi|}{(\mu+1)}\bigl|x(T)\bigr|\right)I^\alpha\left(\frac{1}{|p|}\right)(t) + \frac{1}{2}\left[I^\alpha\left(\frac{1}{|p|}I^\beta|g_x|\right)(T)\right.\\
&\left.+ I^\alpha\left(\frac{|r|}{|p|}|x|\right)(T) + \left(\frac{\mu}{(\mu+1)}I^\beta\bigl(|g_x|\bigr)(T) + \frac{|\xi|}{(\mu+1)}\bigl|x(T)\bigr|\right)|\eta|\right]\\
\le{}& \frac{\|\varphi\|\psi(|x|)}{p_*}I^{\alpha+\beta}(1)(T) + \frac{\rho r^*}{p_*}I^\alpha(1)(T) + \frac{|\eta|\mu\|\varphi\|\psi(|x|)}{(\mu+1)}I^\beta(1)(T)\\
&+ \frac{|\eta||\xi|\rho}{(\mu+1)} + \frac{\|\varphi\|\psi(|x|)}{2p_*}I^{\alpha+\beta}(1)(T) + \frac{\rho r^*}{2p_*}I^\alpha(1)(T)\\
&+ \frac{|\eta|\mu\|\varphi\|\psi(|x|)}{2(\mu+1)}I^\beta(1)(T) + \frac{|\eta||\xi|\rho}{2(\mu+1)}\\
\le{}& \|\varphi\|\psi(\rho)\Omega_1 + \rho\Omega_2,
\end{aligned}$$

which leads to

$$\|\mathcal{A}x\| \le \|\varphi\|\psi(\rho)\Omega_1 + \rho\Omega_2. \tag{3.9}$$

Next we will show that the operator $\mathcal{A}$ *maps bounded sets into equicontinuous sets of* $C([0,T],\mathbb{R})$. Let $\tau_1,\tau_2 \in [1,T]$ such that $\tau_1 < \tau_2$ and $x \in B_\rho$. Then we have

$$\begin{aligned}
&\bigl|(\mathcal{A}x)(\tau_2) - (\mathcal{A}x)(\tau_1)\bigr|\\
&\quad\le \frac{\|\varphi\|\psi(\rho)}{p_*\Gamma(\alpha+\beta)}\left[\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha+\beta-1}\frac{ds}{s} - \int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha+\beta-1}\frac{ds}{s}\right]\\
&\qquad+ \frac{\rho r^*}{p_*\Gamma(\alpha)}\left[\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{ds}{s} - \int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{ds}{s}\right]\\
&\qquad+ \left(\frac{\mu\|\varphi\|\psi(\rho)(\log T)^\beta}{(\mu+1)\Gamma(\beta+1)} + \frac{\rho|\xi|}{(\mu+1)}\right)\cdot\frac{1}{p_*\Gamma(\alpha)}
\end{aligned}$$

$$\times \left[\int_1^{\tau_2}\left(\log\frac{\tau_2}{s}\right)^{\alpha-1}\frac{ds}{s}-\int_1^{\tau_1}\left(\log\frac{\tau_1}{s}\right)^{\alpha-1}\frac{ds}{s}\right]$$
$$= \frac{\|\varphi\|\psi(\rho)}{p_*\Gamma(\alpha+\beta+1)}\left|(\log\tau_2)^{\alpha+\beta}-(\log\tau_1)^{\alpha+\beta}\right| + \frac{\rho r^*}{p_*\Gamma(\alpha+1)}\left|(\log\tau_2)^{\alpha}-(\log\tau_1)^{\alpha}\right|$$
$$+ \left(\frac{\mu\|\varphi\|\psi(\rho)(\log T)^{\beta}}{(\mu+1)\Gamma(\beta+1)}+\frac{\rho|\xi|}{(\mu+1)}\right)\cdot\frac{1}{p_*\Gamma(\alpha+1)}\cdot\left|(\log\tau_2)^{\alpha}-(\log\tau_1)^{\alpha}\right|.$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero independently of $x \in B_\rho$. Therefore by the Arzelá-Ascoli theorem the operator $\mathcal{A} : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Theorem 3.2) once we have proved the boundedness of the set of the solutions to the equations $x = \nu\mathcal{A}x$ for $\nu \in (0,1)$.

Let x be a solution of the operator equation $x = \mathcal{A}x$. Then, for $t \in [1,T]$, by directly computation, we have

$$\left|x(t)\right| \le \|\varphi\|\psi\left(\|x\|\right)\Omega_1 + \|x\|\Omega_2, \tag{3.10}$$

which yields

$$\frac{(1-\Omega_2)\|x\|}{\|\varphi\|\psi(\|x\|)\Omega_1} \le 1,$$

where the constants Ω_1 and Ω_2 are defined by (3.2) and (3.3), respectively. From (H_3), there exists a positive constant M such that $\|x\| \ne M$. Let us set

$$U = \left\{x \in C\left([1,T],\mathbb{R}\right) : \|x\| < M\right\}.$$

We observe that the operator $\mathcal{A} : \overline{U} \to C([1,T],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \nu\mathcal{A}x$ for some $\nu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Theorem 3.2), we see that the operator $\mathcal{A}$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (1.2). The proof is completed. □

Corollary 3.3 *Suppose that the condition* (H_2) *is satisfied. If there exists a positive constant* M *such that*

$$\frac{M}{\|\varphi\|\psi(M)\Omega_1} > 1, \tag{3.11}$$

then the problem (1.3) *has at least one solution on* $[1,T]$.

Corollary 3.4 *Assume that the condition* (H_2) *is fulfilled. If there exists a positive constant* M *such that*

$$\frac{(1-\frac{3}{2}|\lambda|\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)})M}{\frac{3}{2}\|\varphi\|\psi(M)(\frac{1}{\Gamma(\alpha+\beta+1)}+\frac{1}{2\Gamma(\alpha+1)\Gamma(\beta+1)})(\log T)^{\alpha+\beta}} > 1, \tag{3.12}$$

with

$$\frac{3}{2}|\lambda|\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} < 1, \tag{3.13}$$

then the problem (1.4) *has at least one solution on* $[1,T]$.

Our last existence result is based on Krasnoselskii's fixed point theorem.

Theorem 3.4 (Krasnoselskii's fixed point theorem [32]) *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be operators such that*
(a) *$Ax + By \in M$ where $x, y \in M$*;
(b) *A is compact and continuous*;
(c) *B is a contraction mapping.*
Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 3.5 *Let $g : [1,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying (H_1) in Theorem* 3.1. *In addition, assume that*:

(H_4) $|g(t,x)| \le \omega(t)$, $\forall (t,x) \in [1,T] \times \mathbb{R}$, *and* $\omega \in C([1,T],\mathbb{R}^+)$.

If

$$L\Omega_1 < 1 \quad \textit{and} \quad \Omega_2 < 1, \tag{3.14}$$

where Ω_1 and Ω_2 are defined by (3.2) *and* (3.3), *respectively, then the boundary value problem* (1.2) *has at least one solution on* $[1,T]$.

Proof We decompose the operator $\mathcal{A}$ defined in (3.1), into two operators $\mathcal{A}_1$ and $\mathcal{A}_2$ on $B_\varrho = \{x \in \mathcal{C} : \|x\| \le \varrho\}$ by

$$\begin{aligned}
\mathcal{A}_1 x(t) &= -I^{\alpha}\left(\frac{r}{p}x\right)(t) + \frac{\xi}{\mu+1}x(T)I^{\alpha}\left(\frac{1}{p}\right)(t) + \frac{1}{2}I^{\alpha}\left(\frac{r}{p}x\right)(T) \\
&\quad - \frac{\eta\xi}{2(\mu+1)}x(T), \\
\mathcal{A}_2 x(t) &= I^{\alpha}\left(\frac{1}{p}I^{\beta}g_x\right)(t) + \frac{-\mu}{\mu+1}I^{\beta}(g_x)(T)I^{\alpha}\left(\frac{1}{p}\right)(t) - \frac{1}{2}I^{\alpha}\left(\frac{1}{p}I^{\beta}g_x\right)(T) \\
&\quad + \frac{\eta\mu}{2(\mu+1)}I^{\beta}(g_x)(T),
\end{aligned}$$

with ϱ satisfying

$$\varrho \ge \frac{\|\omega\|\Omega_1}{1-\Omega_2}, \tag{3.15}$$

and $\|\omega\| = \sup_{t\in[1,T]}|\omega(t)|$. Note that the ball B_ϱ is a closed, bounded, and convex subset of the Banach space $\mathcal{C}$.

To prove that $\mathcal{A}_1 x + \mathcal{A}_2 y \in B_\varrho$, we let $x, y \in B_\varrho$. Then we get

$$\begin{aligned}
&\bigl|\mathcal{A}_1 x(t) + \mathcal{A}_2 y(t)\bigr| \\
&\quad \le I^{\alpha}\left(\frac{|r|}{|p|}|x|\right)(t) + \frac{|\xi|}{(\mu+1)}\bigl|x(T)\bigr|I^{\alpha}\left(\frac{1}{|p|}\right)(t) + \frac{1}{2}I^{\alpha}\left(\frac{|r|}{|p|}|x|\right)(T)
\end{aligned}$$

$$\begin{aligned}
&+\frac{|\eta||\xi|}{2(\mu+1)}\left|x(T)\right|+I^{\alpha}\left(\frac{1}{|p|}I^{\beta}\left(|g_y|\right)\right)(t)+\frac{\mu}{(\mu+1)}I^{\beta}\left(|g_y|\right)(T)I^{\alpha}\left(\frac{1}{|p|}\right)(t)\\
&+\frac{1}{2}I^{\alpha}\left(\frac{1}{|p|}I^{\beta}\left(|g_y|\right)\right)(T)+\frac{|\eta|\mu}{2(\mu+1)}I^{\beta}\left(|g_y|\right)(T)\\
&\le \varrho\Omega_2+\|\omega\|\Omega_1\le\varrho.
\end{aligned}$$

It follows that $\mathcal{A}_1x+\mathcal{A}_2y\in B_\varrho$. Thus condition (a) of Theorem 3.4 is satisfied. To prove that $\mathcal{A}_2$ is a contraction mapping, for $x,y\in B_\varrho$, we have

$$\begin{aligned}
\left|\mathcal{A}_2x(t)-\mathcal{A}_2y(t)\right| &\le I^{\alpha}\left(\frac{1}{|p|}I^{\beta}\left(|g_x-g_y|\right)\right)(t)+\frac{\mu}{(\mu+1)}I^{\beta}\left(|g_x-g_y|\right)(T)I^{\alpha}\left(\frac{1}{|p|}\right)(t)\\
&\quad+\frac{1}{2}I^{\alpha}\left(\frac{1}{|p|}I^{\beta}\left(|g_x-g_y|\right)\right)(T)+\frac{|\eta|\mu}{2(\mu+1)}I^{\beta}\left(|g_x-g_y|\right)(T)\\
&\le L\Omega_1\|x-y\|,
\end{aligned}$$

which is a contraction, since $L\Omega_1<1$. Therefore, the condition (c) of Theorem 3.4 is fulfilled.

By using the continuity of the function g, we deduce that the operator $\mathcal{A}_1$ is continuous. For $x\in B_\varrho$, it follows that

$$\|\mathcal{A}_1x\|\le\varrho\Omega_2,$$

which implies that the operator $\mathcal{A}_1$ is uniformly bounded on B_ϱ. Now we are going to prove that $\mathcal{A}_1$ is equicontinuous. For $\tau_1,\tau_2\in[1,T]$ such that $\tau_1<\tau_2$ and for $x\in B_\varrho$, we have

$$\begin{aligned}
&\left|\mathcal{A}_1x(\tau_2)-\mathcal{A}_1x(\tau_1)\right|\\
&\quad\le\frac{\varrho r^*}{p_*\Gamma(\alpha+1)}\left|(\log\tau_2)^{\alpha}-(\log\tau_1)^{\alpha}\right|+\frac{\varrho|\xi|}{p_*(\mu+1)\Gamma(\alpha+1)}\left|(\log\tau_2)^{\alpha}-(\log\tau_1)^{\alpha}\right|,
\end{aligned}$$

which is independent of x and tends to zero as $\tau_1\to\tau_2$. Hence $\mathcal{A}_1$ is equicontinuous. Therefore $\mathcal{A}_1$ is relatively compact on B_ϱ, and by Arzelá-Ascoli theorem, $\mathcal{A}_1$ is compact on B_ϱ. Thus the condition (b) of Theorem 3.4 is fulfilled. Therefore all conditions of Theorem 3.4 are satisfied, and consequently, the problem (1.2) has at least one solution on $[1,T]$. This completes the proof. □

Corollary 3.5 *Suppose that the conditions* (H_1) *and* (H_4) *are satisfied. If* $\Omega_1L<1$, *where* Ω_1 *is defined by* (3.2), *then the problem* (1.3) *has at least one solution on* $[1,T]$.

Corollary 3.6 *Assume that the conditions* (H_1) *and* (H_4) *are fulfilled. If*

$$\frac{3}{2}\left(\frac{1}{\Gamma(\alpha+\beta+1)}+\frac{1}{2\Gamma(\alpha+1)\Gamma(\beta+1)}\right)L(\log T)^{\alpha+\beta}<1\quad\textit{and}\quad\frac{3}{2}|\lambda|\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)}<1,$$

then the problem (1.4) *has at least one solution on* $[1,T]$.

4 Examples

In this section, we present some examples to illustrate our results.

Example 4.1 Consider the following generalized Sturm-Liouville and Langevin equations via Hadamard fractional derivatives with anti-periodic boundary conditions:

$$\begin{cases} D^{2/3}([(t^{3/2}+5)D^{1/2}+(t^{1/2}+1)]x(t)) = g(t,x(t)), & t \in [1,2], \\ x(1) = -x(2), \qquad D^{1/2}x(1) = -D^{1/2}x(2). \end{cases} \tag{4.1}$$

Here $\alpha = 1/2$, $\beta = 2/3$, $p(t) = t^{3/2} + 5$, $r(t) = t^{1/2} + 1$, and $T = 2$. From the given information, we find that $\mu = 0.7664374855$, $|\xi| = 0.149656228$, $\eta = 0.1334073445$, $p_* = 6$, $r^* = 2.414213562$, $\Omega_1 = 0.2259466330$, and $\Omega_2 = 0.5839543729$.

(i) Let $g : [1,2] \times \mathbb{R} \to \mathbb{R}$ with

$$g(t,x) = \frac{3}{4}\left(\frac{x^2+|x|}{|x|+1}\right)\cos^2 \pi t + \frac{1}{2}. \tag{4.2}$$

Then we have $|g(t,x) - g(t,y)| \leq (3/2)|x-y|$ and (H_1) is satisfied with $L = 3/2$. Thus $\Omega_3 = 0.9228743224 < 1$. Hence, by Theorem 3.1, the problem (4.1) with (4.2) has a unique solution on $[1,2]$.

(ii) Let now $g : [1,2] \times \mathbb{R} \to \mathbb{R}$ with

$$g(t,x) = 4\left(\frac{|x|}{|x|+1}\right)\cos^2 \pi t + \frac{3}{2}. \tag{4.3}$$

From (4.3), we have $|g(t,x) - g(t,y)| \leq 4|x-y|$ with $L = 4$ and $|g(t,x)| \leq 4\cos^2 \pi t + 3/2$ for all $(t,x) \in [1,T] \times \mathbb{R}$, and thus (H_1) and (H_4) are satisfied. Therefore, we have

$$L\Omega_1 = 0.9037865320 < 1 \quad \text{and} \quad \Omega_2 = 0.5839543729.$$

Thus all assumptions of Theorem 3.5 are satisfied, and consequently the problem (4.1) with g given by (4.3) has at least one solution on $[1,2]$.

Remark 4.1 Theorem 3.1 cannot be applied to the problem (4.1) with g given by (4.3) since $\Omega_3 = 1.487740905 > 1$.

Example 4.2 Consider the following generalized Sturm-Liouville and Langevin equations via Hadamard fractional derivatives with anti-periodic boundary conditions:

$$\begin{cases} D^{3/4}([p(t)D^{2/3}+r(t)]x(t)) = \frac{1}{t^2+5}(\frac{x^2(t)}{4(1+|x(t)|)}+\frac{3}{8}), & t \in [1,e], \\ x(1) = -x(e), \qquad D^{2/3}x(1) = -D^{2/3}x(e). \end{cases} \tag{4.4}$$

Here $\alpha = 2/3$, $\beta = 3/4$, $T = e$, and $g(t,x) = (1/(t^2+5))((x^2/(4(1+|x|))) + (3/8))$. Since $|g(t,x)| \leq (1/(t^2+5))((|x|/4) + (3/8))$, we set $\varphi(t) = 1/(t^2+5)$ and $\psi(|x|) = (1/4)|x| + (3/8)$.

(i) *The Sturm-Liouville case.* Let

$$r(t) \equiv 0 \quad \text{and} \quad p(t) = 2\sqrt{t} + 3. \tag{4.5}$$

We can find that $\mu = 0.7939731035$, $\xi = 0$, $\eta = 0.1942851753$, and $\Omega_1 = 0.3792202416$. Therefore, there exists a constant $M > 0.02408177749$ satisfying inequality (3.11). Thus, by Corollary 3.3, the problem (4.4) with (4.5) has at least one solution on $[1,e]$.

(ii) *The Langevin case*. For $t \in [1, e]$, let

$$r(t) \equiv \frac{2}{7} \quad \text{and} \quad p(t) \equiv 1. \tag{4.6}$$

Then there exists a positive constant $M > 0.2995449526$, which satisfies the inequality (3.12). Therefore, by Corollary 3.4, the problem (4.4) with (4.6) has at least one solution on $[1, e]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this article. They read and approved the final manuscript.

Author details
[1]Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, 10800, Thailand. [2]Department of Mathematics, University of Ioannina, Ioannina, 451 10, Greece. [3]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia.

Acknowledgements
This research was funded by Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Thailand. Contract no. 5942108.

Received: 18 February 2016 Accepted: 28 November 2016 Published online: 08 December 2016

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