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Positive and negative solutions of a boundary value problem for a fractional q-difference equation

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Abstract

In this work, we study a boundary value problem for a fractional q-difference equation. By using the monotone iterative technique and lower-upper solution method, we get the existence of positive or negative solutions under the nonlinear term is local continuity and local monotonicity. The results show that we can construct two iterative sequences for approximating the solutions.

MSC: 34B18; 33D05; 39A13

Keywords: fractional q-difference equation; positive and negative solutions; iterative method; lower and upper solutions

1 Introduction

In this paper, we consider the following nonlinear boundary value problem for a fractional q-difference equation:

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = D_q u(0) = D_q u(1) = 0, \end{cases} \tag{1.1}$$

where $0 < q < 1$, $2 < \alpha < 3$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$, D_q^α is the fractional q-derivative of the Riemann-Liouville type.

As a branch of mathematical analysis, the q-difference calculus has developed into a demonstrated popular tool for the description of experimental data refer to the mathematical model. In the early twentieth century, Jackson [1], the first scholar to develop q-calculus in a systematic way, introduced the notion of the definite q-integral and some classical concepts. Furthermore, the fractional q-difference calculus was initially proposed by Al-Salam [2] and Agarwal [3], and one can find more details in [4–6]. Since then, there has appeared much work on the theory of fractional q-difference calculus and nonlinear fractional q-difference equation boundary value problems; see [4–15] for example. Moreover, fractional q-difference equations have wide applications in several fields such as engineering, economy, chemistry, physic, *etc*. Compared with the classical fractional differential equations involving the Caputo fractional derivative (see [16–18] for instance), fractional q-difference equations involving the Caputo fractional derivative are general by allowing for q-derivatives. So boundary value problems for fractional q-difference equations have become of importance and the existence of positive solutions has been studied by a great number of researchers; see [19–29] and references therein. The methods used in these papers are mainly Krasnoselskii fixed point theorem, the lower and upper solution method, the critical point theory and fixed point theorems on cones and so on. For instance, by using some fixed point theorems on cones associated with some properties of the Green's function, several existence results of positive solution for nonlinear q-fractional boundary value problem are obtained by Liu [26]. Yang [29] investigated the existence of positive solutions for fractional q-difference equation boundary value problems with Φ-Laplacian operator by means of the lower-upper solution method coupled with the Schauder fixed point theorem. More recently, Ahmad *et al.* [30] obtained the existence of solutions for Caputo fractional q-difference inclusions with q-antiperiodic boundary conditions by using a fixed point theorem for upper semi-continuous compact map. In [19], the authors considered the problem (1.1) and obtained the existence of at least one or two positive solutions by using the Krasnoselskii fixed point theorem. Different from [19, 30], we will discuss the existence of positive solutions or negative solutions for the problem (1.1). Our main tool is monotone iterative via lower and upper solutions. Here we do not require that the nonlinear term $f(t,x)$ is nonnegative, global monotone and global continuous. We establish the existence of positive solutions or negative solutions under the conditions of local monotone and local continuous. Moreover, we can approximate the positive solution or negative solution by constructing two iterative sequences.

2 Preliminaries

In this section, we list some well-known concepts on fractional q-calculus.

Let q be a real number with $0 < q < 1$ and I be a real interval containing 0. The definition of the q-analog for $a \in \mathbf{R}$ is

$$[a]_q := \frac{1-q^a}{1-q}.$$

Let $u : I \to \mathbf{R}$ with $u'(0)$ exist, the q-derivative $D_q[u]$ of u is defined by

$$D_q[u](t) := \begin{cases} \frac{u(qt)-u(t)}{(q-1)t}, & t \neq 0, \\ u'(0), & t = 0. \end{cases}$$

Clearly, if $u'(t)$ exists, then $\lim_{q\to 1} D_q[u](t) = u'(t)$. q-derivatives of higher order are defined by

$$D_q^0[u] := u, \qquad D_q^n[u] := D_q\bigl[D_q^{n-1}[u]\bigr], \quad n \in \mathbf{N}.$$

Let Γ_q be the q-gamma function defined by

$$\Gamma_q(z) := (1-q)^{1-z} \prod_{n=0}^{\infty} \frac{1-q^{n+1}}{1-q^{n+z}}, \quad z \in \mathbf{R} \setminus \{0, -1, -2, \ldots\}.$$

It is clear that

$$\Gamma_q(1) = 1, \qquad \Gamma_q(z+1) = [z]_q \Gamma_q(z).$$

The q-analog of the power function $(a-b)^{(\alpha)}$ with real exponent $\alpha \in \mathbf{R}$ is defined by

$$(a-b)^{(\alpha)} := a^{\alpha} \prod_{n=0}^{\infty} \frac{1-(b/a)q^n}{1-(b/a)q^{n+\alpha}}, \quad a, b \in \mathbf{R}. \tag{2.1}$$

Then $(a-b)^{(0)} = 1$, $a^{(\alpha)} = a^{\alpha}$, when $b = 0$. Further,

$$\bigl(a(t-s)\bigr)^{(\alpha)} = a^{\alpha}(t-s)^{(\alpha)}.$$

We can see that the q-gamma function can also be written by

$$\Gamma_q(z) = \frac{(1-q)^{(z-1)}}{(1-q)^{z-1}}, \quad \text{for } z \in \mathbf{R} \setminus \{0, -1, -2, \ldots\}.$$

Remark 2.1 (See [8]) If $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$.

Let $a, b \in I$ and $a < b$. For $u : I \to R$, the q-integral of u from a to b is defined by

$$\int_a^b u(s)\, d_qs := \int_0^b u(s)\, d_qs - \int_0^a u(s)\, d_qs,$$

here

$$\int_0^t u(s)\, d_qs = t(1-q) \sum_{n=0}^{\infty} q^n u\bigl(tq^n\bigr), \quad t \in I, \tag{2.2}$$

provided that the series converges at $t = a$ and $t = b$. Set

$$I_q[x](t) := \int_0^t x(s)\, d_qs.$$

Then the two operators I_q, D_q are inverse of each other, in the sense that $D_q[I_q[u]] = u$. Suppose that u is continuous at $t = 0$, then

$$I_q\bigl[D_q[u]\bigr] = u(t) - u(0).$$

Assume that u is continuous at $t = 0$ and, by using the last equality,

$$\int_a^b D_q[u](t)\, d_qt = u(b) - u(a), \quad a, b \in I.$$

It is also clear that the operator I_q is a linear operator. Let $D_q[\tau \to f(\tau, t)](t)$ be the q-derivative of a function f with respect to τ. Note that $D_q[\tau \to \tau^n](t) = [n]_q t^{n-1}$, for $n \in \mathbf{N}$, which is similar to the ordinary derivative of t^n. Then some properties of the q-integral are given below:

(i) $\int_a^b u(s) D_q[v](s)\, d_qs = [u(s)v(s)]_a^b - \int_a^b v(sq) D_q[u](s)\, d_qs$;

(ii) $\int_0^t s^{\lambda}\, d_qs = \frac{t^{\lambda+1}}{[\lambda+1]_q}$, $\lambda > -1$;

(iii) $D_q[\tau \to \int_0^{\tau} g(s,t)\, d_qs](t) = \int_0^t D_q[\tau \to g(s,\tau)](t)\, d_qs + g(t, qt)$.

Definition 2.1 (See [3]) Let $\alpha \geq 0$ and u be a function defined on $[0,1]$. The fractional q-integral of Riemann-Liouville type is $I_q^0[u](t) = u(t)$ and

$$I_q^{\alpha}[u](t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} u(s)\, d_q s, \quad \alpha > 0. \tag{2.3}$$

Note that when $\alpha = 1$, $I_q^{\alpha}[u] = I_q[u]$.

Lemma 2.1 (See [9]) *If $\alpha \geq 1$ and $u : [0,1] \to \mathbf{R}$ is continuous, then $I_q^{\alpha}[u]$ is a continuous function.*

Lemma 2.2 *If $\alpha \geq 1$ and u, v are continuous on $[0,1]$ with $u(t) \leq v(t)$, then $I_q^{\alpha}[u](t) \leq I_q^{\alpha}[v](t)$.*

Proof From the definitions (2.2) and (2.3), we get

$$\begin{aligned} I_q^{\alpha}[u](t) - I_q^{\alpha}[v](t) &= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} u(s)\, d_q s - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} v(s)\, d_q s \\ &= \frac{1}{\Gamma_q(\alpha)} t(1-q) \sum_{n=0}^{\infty} q^n \bigl(t - tq^{n+1}\bigr)^{(\alpha-1)} \bigl[u\bigl(tq^n\bigr) - v\bigl(tq^n\bigr)\bigr] \\ &= \frac{1}{\Gamma_q(\alpha)} t^{\alpha}(1-q) \sum_{n=0}^{\infty} q^n \bigl(1 - q^{n+1}\bigr)^{(\alpha-1)} \bigl[u\bigl(tq^n\bigr) - v\bigl(tq^n\bigr)\bigr]. \end{aligned}$$

Since $0 < q^{n+1} < 1$, $u(tq^n) \leq v(tq^n)$, we have $I_q^{\alpha}[u](t) - I_q^{\alpha}[v](t) \leq 0$. The proof is completed. □

Lemma 2.3 *If $u(t)$ is continuous on $[0,1]$. Then $|\int_0^1 u(s)\, d_q s| \leq \int_0^1 |u(s)|\, d_q s$.*

Proof Since $u(t)$ is continuous, $\int_0^1 u(s)\, d_q s$, $\int_0^1 |u(s)|\, d_q s$ exist. Hence, the series $\sum_{n=0}^{\infty} q^n u(q^n)$, $\sum_{n=0}^{\infty} q^n |u(q^n)|$ convergence. Since $0 < q < 1$, we have

$$\left| \sum_{n=0}^{\infty} q^n u\bigl(q^n\bigr) \right| \leq \sum_{n=0}^{\infty} q^n \bigl|u\bigl(q^n\bigr)\bigr|.$$

In view of $1 - q > 0$, we get

$$\left| (1-q) \sum_{n=0}^{\infty} q^n u\bigl(q^n\bigr) \right| \leq (1-q) \sum_{n=0}^{\infty} q^n \bigl|u\bigl(q^n\bigr)\bigr|,$$

and from the definition (2.2), we have $|\int_0^1 u(s)\, d_q s| \leq \int_0^1 |u(s)|\, d_q s$. □

Definition 2.2 (See [3]) The fractional q-derivative of the Riemann-Liouville type of $u : I \to \mathbf{R}$ is defined by $D_q^0[u](t) = u(t)$ and

$$D_q^{\alpha}[u](t) = D_q^{\lceil \alpha \rceil} \bigl[I_q^{\lceil \alpha \rceil - \alpha}[u]\bigr](t), \quad \alpha > 0,$$

where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α. It is easy to prove that if $\alpha = 1$, $D_q^{\alpha}[u] = D_q[u]$ and $D_q^{\alpha}[I_q^{\alpha}[u]] = u$.

Lemma 2.4 *Let $\alpha \geq 0$ and $a_m \to a$ as $m \to \infty$. Then*

$$(a_m - b)^{(\alpha)} \overset{m\to\infty}{\longrightarrow} (a-b)^{(\alpha)}, \quad a, b \in \mathbf{R}.$$

Proof From the definition (2.1), we get

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{n+\alpha}} = \lim_{n\to\infty} a^\alpha \prod_{k=0}^{n-1} \frac{a - bq^n}{a - bq^{n+\alpha}}$$

and

$$(a_m-b)^{(\alpha)} = a_m^\alpha \prod_{n=0}^{\infty} \frac{a_m - bq^n}{a_m - bq^{n+\alpha}} = \lim_{n\to\infty} a_m^\alpha \prod_{k=0}^{n-1} \frac{a_m - bq^n}{a_m - bq^{n+\alpha}}.$$

Since $\lim_{n\to\infty} a_m = a$, we obtain

$$\begin{aligned} \lim_{m\to\infty} (a_m - b)^{(\alpha)} &= \lim_{m\to\infty} \lim_{n\to\infty} a_m^\alpha \prod_{k=0}^{n-1} \frac{a_m - bq^n}{a_m - bq^{n+\alpha}} \\ &= \lim_{n\to\infty} \lim_{m\to\infty} a_m^\alpha \prod_{k=0}^{n-1} \frac{a_m - bq^n}{a_m - bq^{n+\alpha}} = (a-b)^{(\alpha)}. \end{aligned}$$

The proof is complete. □

Now we present the following fixed point theorem, which will be the main tool for our analysis.

Lemma 2.5 (See [31]) *Assume that X is a Banach space and K is a normal cone in X, $T : [u_0, v_0] \to X$ is a completely continuous increasing operator which satisfies $u_0 \leq Tu_0$, $Tv_0 \leq v_0$. Then T has a minimal fixed point u_* and a maximal fixed point v^* with $u_0 \leq u_* \leq v^* \leq v_0$. In addition,*

$$u_* = \lim_{n\to\infty} T^n u_0, \qquad v^* = \lim_{n\to\infty} T^n v_0,$$

where $\{T^n u_0\}_{n=1}^{\infty}$ is an increasing sequence, $\{T^n v_0\}_{n=1}^{\infty}$ is a decreasing sequence.

3 Existence of q-fractional positive solutions for problem (1.1)

Lemma 3.1 (See [19]) *Assume $g \in C[0,1]$, then the following boundary value problem*:

$$\begin{cases} D_q^\alpha u(t) + g(t) = 0, \quad 0 < t < 1, \\ u(0) = D_q u(0) = D_q u(1) = 0, \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G(t, qs) g(t) \, d_q s,$$

where

$$G(t,qs) = \frac{1}{\Gamma_q(\alpha)} \begin{cases} (1-qs)^{(\alpha-2)}t^{\alpha-1} - (t-qs)^{(\alpha-1)}, & 0 \le qs \le t \le 1, \\ (1-qs)^{(\alpha-2)}t^{\alpha-1}, & 0 \le t \le qs \le 1. \end{cases}$$

Lemma 3.2 (See [19]) *The function $G(t,qs)$ has the following properties*:

(i) $G(t,qs) \ge 0$, $G(t,qs) \le G(1,qs)$, $0 \le t,s \le 1$;

(ii) $G(t,qs) \ge t^{\alpha-1}G(1,qs)$, $0 \le t,s \le 1$.

Remark 3.1 The function $G(t,qs)$ has some other properties:

(a) $G(t,qs) \le \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}t^{\alpha-1} \le \frac{1}{\Gamma_q(\alpha)}$, $0 \le t,s \le 1$.

(b) We can obtain the following inequalities:

(i) For $0 \le t_1 \le t_2 \le qs \le 1$, we get

$$\begin{aligned} \left|G(t_2,qs) - G(t_1,qs)\right| &= \left|\frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}t_2^{\alpha-1} - \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}t_1^{\alpha-1}\right| \\ &= \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}\left|t_2^{\alpha-1} - t_1^{\alpha-1}\right| \\ &\le \frac{1}{\Gamma_q(\alpha)}\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right). \end{aligned}$$

(ii) For $0 \le t_1 \le qs \le t_2 \le 1$, we get

$$\begin{aligned} \left|G(t_2,qs) - G(t_1,qs)\right| &= \left|\frac{1}{\Gamma_q(\alpha)}\left[(1-qs)^{(\alpha-2)}t_2^{\alpha-1} - (t_2-qs)^{(\alpha-1)}\right]\right. \\ &\quad \left. - \frac{1}{\Gamma_q(\alpha)}(1-qs)^{(\alpha-2)}t_1^{\alpha-1}\right| \\ &= \frac{1}{\Gamma_q(\alpha)}\left|(1-qs)^{(\alpha-2)}\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) - (t_2-qs)^{(\alpha-1)}\right| \\ &\le \frac{1}{\Gamma_q(\alpha)}\left|(1-qs)^{(\alpha-2)}\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)\right| + \left|(t_2-qs)^{(\alpha-1)}\right| \\ &\le \frac{1}{\Gamma_q(\alpha)}\left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) + (t_2-t_1)^{(\alpha-1)}\right]. \end{aligned}$$

(iii) For $0 \le qs \le t_1 \le t_2 \le 1$, we get

$$\begin{aligned} \left|G(t_2,qs) - G(t_1,qs)\right| &= \left|\frac{1}{\Gamma_q(\alpha)}\left[(1-qs)^{(\alpha-2)}t_2^{\alpha-1} - (t_2-qs)^{(\alpha-1)}\right.\right. \\ &\quad \left.\left. - (1-qs)^{(\alpha-2)}t_1^{\alpha-1} + (t_1-qs)^{(\alpha-1)}\right]\right| \\ &\le \frac{1}{\Gamma_q(\alpha)}\left[\left|(1-qs)^{(\alpha-2)}\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)\right|\right. \\ &\quad \left. + \left|(t_1-qs)^{(\alpha-1)} - (t_2-qs)^{(\alpha-1)}\right|\right] \\ &\le \frac{1}{\Gamma_q(\alpha)}\left[\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right) + (t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)}\right]. \end{aligned}$$

(c) $G(t,qs) > 0$ for $(t,s) \in (0,1) \times (0,1)$ (see [16]).

Let $X = C[0,1]$, the Banach space of all continuous functions on $[0,1]$, with norm $\|u\| = \max\{|u(t)| : t \in [0,1]\}$. In our considerations, we need the standard cone $K \subset X$ by

$$K = \{u \in [0,1] : u(t) \geq 0, 0 \leq t \leq 1\}.$$

It is clear that the cone K is normal. In addition, we always use Lemmas 2.2-2.4 in the following analysis.

Theorem 3.1 *Assume that*

(F_1) *there exist a real number $d > 0$ and $g \in L^1[0,1]$, such that*

(i_1) *$f : [0,1] \times [0,d] \to [0,+\infty)$ is continuous, $f(t,u) \leq g(t)$ for $(t,u) \in [0,1] \times [0,d]$ and $f(t,u) \leq f(t,v)$ for $0 \leq t \leq 1$, $0 \leq u \leq v \leq d$;*

(i_2) *the following inequality holds*:

$$\frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} f\bigl(s, ds^{\alpha-1}\bigr)\, d_qs \leq d;$$

(F_2) *there exists $c \in (0,d)$ such that*

$$\int_0^1 G(1,qs) f\bigl(s, cs^{\alpha-1}\bigr)\, d_qs \geq c.$$

Then the problem (1.1) *has two positive solutions $u^*, v^* \in D$, where $D = \{u \in C[0,1] \mid ct^{\alpha-1} \leq u(t) \leq dt^{\alpha-1}, t \in [0,1]\}$. In addition, let $u_0(t) = ct^{\alpha-1}$, $v_0(t) = dt^{\alpha-1}$ and construct the following sequences*:

$$u_{n+1} = \int_0^1 G(t,qs) f\bigl(s, u_n(s)\bigr)\, d_qs, \qquad v_{n+1} = \int_0^1 G(t,qs) f\bigl(s, v_n(s)\bigr)\, d_qs,$$

$n = 0,1,2,\ldots$, *one has* $\lim_{n\to\infty} u_n = u^*$, $\lim_{n\to\infty} v_n = v^*$.

Proof From the non-negativeness and continuity of G and f, we can define an operator $T : C[0,1] \to C[0,1]$ by

$$Tu(t) = \int_0^1 G(t,qs) f\bigl(s, u(s)\bigr)\, d_qs, \quad 0 \leq t \leq 1.$$

From Lemma 3.1, we can see that u is the solution of the problem (1.1) if and only if u is the fixed point of T. We will show that T has fixed points in the order interval $[u_0, v_0]$.

We need to show that $T : [u_0, v_0] \to C[0,1]$ is a completely continuous operator. For $u \in [u_0, v_0]$, we have $0 \leq ct^{\alpha-1} \leq u(t) \leq dt^{\alpha-1} \leq d$, $0 \leq t \leq 1$. Since $G(t,qs)$ is continuous, it follows from the hypothesis (F_1)-(i_1) that T is continuous. So we only prove T is compact. Let $M = \int_0^1 g(s)\, d_qs$, then $0 \leq M < +\infty$. From the hypothesis (F_1)-(i_2) and Lemma 3.2, we get

$$\begin{aligned} \|Tu\| &= \max_{0\leq t\leq 1} \left| \int_0^1 G(t,qs) f\bigl(s, u(s)\bigr)\, d_qs \right| \\ &\leq \max_{0\leq qs\leq 1} G(1,qs) \int_0^1 f\bigl(s, u(s)\bigr)\, d_qs \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma_q(\alpha)} \int_0^1 g(s)\,d_q s \\
&= \frac{M}{\Gamma_q(\alpha)}.
\end{aligned}$$

This shows that the set $T([u_0, v_0])$ is uniform bounded in $C[0,1]$. After that, for given $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, and $u \in [u_0, v_0]$, we obtain

$$\begin{aligned}
\bigl|Tu(t_1) - Tu(t_2)\bigr| &\leq \int_0^1 \bigl|G(t_1, qs) - G(t_2, qs)\bigr| f\bigl(s, u(s)\bigr)\,d_q s \\
&\leq \max_{0 \leq qs \leq 1} \bigl|G(t_1, qs) - G(t_2, qs)\bigr| \int_0^1 g(s)\,d_q s \\
&= M \max_{0 \leq qs \leq 1} \bigl|G(t_1, qs) - G(t_2, qs)\bigr|.
\end{aligned}$$

In view of Remark 3.1(b) and Lemma 2.4, one has $Tu(t_1) \to Tu(t_2)$, as $t_1 \to t_2$. So we claim that the set $T([u_0, v_0])$ is equi-continuous in $C[0,1]$. By means of the Arzela-Ascoli theorem, $T : [u_0, v_0] \to C[0,1]$ is a completely continuous operator. By the hypothesis (F_1)-(i_1), T is an increasing operator.

From (F_1), (F_2) and Lemma 3.2, for any $t \in [0,1]$, one can see that

$$\begin{aligned}
Tu_0(t) &= \int_0^1 G(t, qs) f\bigl(s, u_0(s)\bigr)\,d_q s = \int_0^1 G(t, qs) f\bigl(s, cs^{\alpha-1}\bigr)\,d_q s \\
&\geq \int_0^1 t^{\alpha-1} G(1, qs) f\bigl(s, cs^{\alpha-1}\bigr)\,d_q s \geq t^{\alpha-1} c = u_0(t)
\end{aligned}$$

and

$$\begin{aligned}
Tv_0(t) &= \int_0^1 G(t, qs) f\bigl(s, v_0(s)\bigr)\,d_q s = \int_0^1 G(t, qs) f\bigl(s, ds^{\alpha-1}\bigr)\,d_q s \\
&\leq \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-2)} f\bigl(s, ds^{\alpha-1}\bigr)\,d_q s \leq t^{\alpha-1} d = v_0(t).
\end{aligned}$$

Hence, we get $Tu_0 \geq u_0$, $Tv_0 \leq v_0$. We construct the following sequences:

$$u_{n+1} = \int_0^1 G(t, qs) f\bigl(s, u_n(s)\bigr)\,d_q s, \qquad v_{n+1} = \int_0^1 G(t, qs) f\bigl(s, v_n(s)\bigr)\,d_q s,$$

$n = 0, 1, 2, \ldots$. From the monotonicity of T, we have $u_{n+1} \geq u_n$, $v_{n+1} \leq v_n$, $n = 1, 2, \ldots$. By using Lemma 2.5, we know that the operator T has two positive solutions $u^*, v^* \in C[0,1]$ with $u_0 \leq u^* \leq v^* \leq v_0$, that is, $ct^{\alpha-1} \leq u^*(t) \leq v^*(t) \leq dt^{\alpha-1}$, $0 < t \leq 1$. In addition, $\lim_{n\to\infty} u_n = u^*$, $\lim_{n\to\infty} v_n = v^*$. □

Theorem 3.2 *Assume that*

(F_3) *there exist a real number* $d > 0$ *and* $g \in L^1[0,1]$, *such that*

(i_3) $f : [0,1] \times [0,d] \to \mathbf{R}$ *is continuous,* $|f(t,u)| \leq g(t)$ *for* $(t,u) \in [0,1] \times [0,d]$ *and* $f(t,u) \leq f(t,v)$ *for* $0 \leq t \leq 1$, $0 \leq u \leq v \leq d$;

(i_4) *the following inequality holds*:

$$\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)} \max\{f(s,ds^{\alpha-1}),0\}\,d_qs + \int_0^1 G(1,qs)\min\{f(s,ds^{\alpha-1}),0\}\,d_qs \leq d;$$

(F_4) *there exists* $c \in (0,d)$ *such that*

$$\int_0^1 G(1,qs)\max\{f(s,cs^{\alpha-1}),0\}\,d_qs + \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)}\min\{f(s,cs^{\alpha-1}),0\}\,d_qs \geq c.$$

Then the problem (1.1) *has two positive solutions* $u^*, v^* \in D$, *where* $D = \{u \in C[0,1] \mid ct^{\alpha-1} \leq u(t) \leq dt^{\alpha-1}, t \in [0,1]\}$. *In addition, let* $u_0(t) = ct^{\alpha-1}$, $v_0(t) = dt^{\alpha-1}$ *and construct the following sequences*:

$$u_{n+1} = \int_0^1 G(t,qs)f(s,u_n(s))\,d_qs, \qquad v_{n+1} = \int_0^1 G(t,qs)f(s,v_n(s))\,d_qs,$$

$n = 0,1,2,\ldots$, *one has* $\lim_{n\to\infty} u_n = u^*$, $\lim_{n\to\infty} v_n = v^*$.

Proof Consider the same operator $T : C[0,1] \to C[0,1]$ as defined in the proof of Theorem 3.1:

$$Tu(t) = \int_0^1 G(t,qs)f(s,u(s))\,d_qs, \quad 0 \leq t \leq 1.$$

We also show that T has fixed points in the order interval $[u_0, v_0]$.

Similar to the proof of Theorem 3.1, $T : C[0,1] \to C[0,1]$ is a completely continuous operator. From the hypothesis (F_3)-(i_3), T is an increasing operator. Further, by using the conditions (F_3), (F_4), Remark 3.1 and Lemma 3.2, for any $t \in [0,1]$, one obtains

$$\begin{aligned}
Tu_0(t) &= \int_0^1 G(t,qs)f(s,u_0(s))\,d_qs \\
&= \int_0^1 G(t,qs)f(s,cs^{\alpha-1})\,d_qs \\
&= \int_0^1 G(t,qs)\max\{f(s,cs^{\alpha-1}),0\}\,d_qs + \int_0^1 G(t,qs)\min\{f(s,cs^{\alpha-1}),0\}\,d_qs \\
&\geq \int_0^1 t^{\alpha-1}G(1,qs)\max\{f(s,cs^{\alpha-1}),0\}\,d_qs \\
&\quad + \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)}t^{\alpha-1}\min\{f(s,cs^{\alpha-1}),0\}\,d_qs \\
&= t^{\alpha-1}\bigg[\int_0^1 G(1,qs)\max\{f(s,cs^{\alpha-1}),0\}\,d_qs \\
&\quad + \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)}\min\{f(s,cs^{\alpha-1}),0\}\,d_qs\bigg] \\
&\geq t^{\alpha-1}c = u_0(t)
\end{aligned}$$

and

$$\begin{aligned}
Tv_0(t) &= \int_0^1 G(t,qs)f\bigl(s,v_0(s)\bigr)\,d_qs \\
&= \int_0^1 G(t,qs)f\bigl(s,ds^{\alpha-1}\bigr)\,d_qs \\
&= \int_0^1 G(t,qs)\max\bigl\{f\bigl(s,ds^{\alpha-1}\bigr),0\bigr\}\,d_qs + \int_0^1 G(t,qs)\min\bigl\{f\bigl(s,ds^{\alpha-1}\bigr),0\bigr\}\,d_qs \\
&\le \frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)}t^{\alpha-1}\max\bigl\{f\bigl(s,ds^{\alpha-1}\bigr),0\bigr\}\,d_qs \\
&\quad + t^{\alpha-1}\int_0^1 G(1,qs)\min\bigl\{f\bigl(s,ds^{\alpha-1}\bigr),0\bigr\}\,d_qs \\
&= t^{\alpha-1}\biggl[\frac{1}{\Gamma_q(\alpha)}\int_0^1 (1-qs)^{(\alpha-2)}\max\bigl\{f\bigl(s,ds^{\alpha-1}\bigr),0\bigr\}\,d_qs \\
&\quad + \int_0^1 G(1,qs)\min\bigl\{f\bigl(s,ds^{\alpha-1}\bigr),0\bigr\}\,d_qs\biggr] \\
&\le t^{\alpha-1}d = v_0(t).
\end{aligned}$$

Hence, $Tu_0 \ge u_0$, $Tv_0 \le v_0$. We construct the following sequences:

$$u_{n+1} = \int_0^1 G(t,qs)f\bigl(s,u_n(s)\bigr)\,d_qs, \qquad v_{n+1} = \int_0^1 G(t,qs)f\bigl(s,v_n(s)\bigr)\,d_qs,$$

$n = 0,1,2,\ldots$. According to the monotonicity of T, we get $u_{n+1} \ge u_n$, $v_{n+1} \le v_n$, $n = 1,2,\ldots$. From Lemma 2.5, we know that the operator T has two positive solutions $u^*, v^* \in C[0,1]$ with $u_0 \le u^* \le v^* \le v_0$, that is, $0 < ct^{\alpha-1} \le u^*(t) \le v^*(t) \le dt^{\alpha-1} \le d$, $0 < t \le 1$. In addition, $\lim_{n\to\infty} u_n = u^*$, $\lim_{n\to\infty} v_n = v^*$. □

By using the same proof as Theorem 3.2, we can easily obtain the following conclusions.

Theorem 3.3 *Assume that*

(F_5) *there exist a real number* $c < 0$ *and* $g \in L^1[0,1]$, *such that*

- (i_5) $f : [0,1] \times [c,0] \to \mathbf{R}$ *is continuous*, $|f(t,u)| \le g(t)$ *for* $(t,u) \in [0,1] \times [c,0]$ *and* $f(t,u) \le f(t,v)$ *for* $0 \le t \le 1$, $c \le u \le v \le 0$.

In addition, there exists $d \in (c,0)$ *such that* (F_3)-(i_4) *and* (F_4) *in Theorem* 3.2 *are also satisfied. Then the problem* (1.1) *has two negative solutions* $u^*, v^* \in D$, *where* $D = \{u \in C[0,1] \mid ct^{\alpha-1} \le u(t) \le dt^{\alpha-1}, t \in [0,1]\}$. *Let* $u_0(t) = ct^{\alpha-1}$, $v_0(t) = dt^{\alpha-1}$ *and we construct the following sequences*:

$$u_{n+1} = \int_0^1 G(t,qs)f\bigl(s,u_n(s)\bigr)\,d_qs, \qquad v_{n+1} = \int_0^1 G(t,qs)f\bigl(s,v_n(s)\bigr)\,d_qs,$$

$n = 0,1,2,\ldots$, *we can obtain* $\lim_{n\to\infty} u_n = u^*$, $\lim_{n\to\infty} v_n = v^*$.

Theorem 3.4 *Assume that*

(F_6) *there exist two real numbers* $c < 0 < d$ *and* $g \in L^1[0,1]$, *such that*

(i_6) $f : [0,1] \times [c,d] \to \mathbf{R}$ *is continuous,* $|f(t,u)| \leq g(t)$ *for* $(t,u) \in [0,1] \times [c,d]$ *and* $f(t,u) \leq f(t,v)$ *for* $0 \leq t \leq 1$, $c \leq u \leq v \leq d$.

In addition, (F_3)-(i_4) *and* (F_4) *are also satisfied. Then the problem* (1.1) *has two solutions* $u^*, v^* \in D$, *where* $D = \{u \in C[0,1] \mid ct^{\alpha-1} \leq u(t) \leq dt^{\alpha-1}, t \in [0,1]\}$. *Let* $u_0(t) = ct^{\alpha-1}$, $v_0(t) = dt^{\alpha-1}$ *and construct the following sequences*:

$$u_{n+1} = \int_0^1 G(t,qs) f\big(s,u_n(s)\big)\,d_qs, \qquad v_{n+1} = \int_0^1 G(t,qs) f\big(s,v_n(s)\big)\,d_qs,$$

$n = 0,1,2,\ldots,$ *we can get* $\lim_{n\to\infty} u_n = u^*$, $\lim_{n\to\infty} v_n = v^*$.

Remark 3.2 In Theorem 3.4, the two solutions may be sign-changing solutions.

4 An example

In this section, we give an example to illustrate our main results.

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} D_q^{\frac{5}{2}} u(t) + (3u^{\frac{2}{3}} + 4t) = 0, & t \in (0,1), \\ u(0) = D_q u(0) = D_q u(1) = 0, \end{cases} \tag{4.1}$$

here $\alpha = \frac{5}{2}$, $q = \frac{1}{2}$, $f(t,u) = 3u^{\frac{2}{3}} + 4t$, and $g(t) = 4t + 9$, let $c = \frac{\sqrt{2}}{4}$, $d = 3\sqrt{3}$, one can see that $f : [0,1] \times [0,3\sqrt{3}] \to [0,+\infty)$ is continuous, $f(t,u) \leq g(t)$ for $(t,u) \in [0,1] \times [0,3\sqrt{3}]$ and $f(t,u)$ is increasing in $u \in [0,3\sqrt{3}]$ for fixed $t \in [0,1]$. Since $f(s,cs^{\frac{3}{2}}) = \frac{11}{2}s$, $f(s,ds^{\frac{3}{2}}) = 13s$, by simple computation, we have

$$\begin{aligned} \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-2)} f\big(s,ds^{\alpha-1}\big)\,d_qs &= \frac{13}{\Gamma_{\frac{1}{2}}(\frac{5}{2})} \int_0^1 \left(1-\frac{s}{2}\right)^{(\frac{1}{2})} s\,d_qs \\ &= \frac{26}{(4-\sqrt{2})\Gamma_{\frac{1}{2}}(\frac{7}{2})} \approx 4.3184 \leq d, \\ \int_0^1 G(1,qs) f\big(s,cs^{\alpha-1}\big)\,d_qs &= \frac{11}{2} \int_0^1 \left[\left(1-\frac{s}{2}\right)^{(\frac{1}{2})} - \left(1-\frac{s}{2}\right)^{(\frac{3}{2})}\right] s\,d_qs \\ &= \frac{11}{(4-\sqrt{2})\Gamma_{\frac{1}{2}}(\frac{7}{2})} - \frac{11}{2\Gamma_{\frac{1}{2}}(\frac{9}{2})} \approx 0.6247 \geq c. \end{aligned}$$

So conditions (F_1) and (F_2) are satisfied. By Theorem 3.1, the problem (4.1) has two positive solutions $u^*, v^* \in C[0,1]$ with $\frac{\sqrt{2}}{4}t^{\frac{3}{2}} \leq u^* \leq v^* \leq 3\sqrt{3}t^{\frac{3}{2}}$. Moreover, let $u_0(t) = \frac{\sqrt{2}}{4}t^{\frac{3}{2}}$, $v_0(t) = 3\sqrt{3}t^{\frac{3}{2}}$ and we construct two sequences

$$u_{n+1} = \int_0^1 G(t,qs)\big(3u_n^{\frac{2}{3}} + 4s\big)\,d_qs, \qquad v_{n+1} = \int_0^1 G(t,qs)\big(3v_n^{\frac{2}{3}} + 4s\big)\,d_qs,$$

$n = 0,1,2,\ldots,$ where $G(t,qs)$ are given as in Lemma 3.1, we have $\lim_{n\to\infty} u_n = u^*$, $\lim_{n\to\infty} v_n = v^*$.

Competing interests
The authors declare to have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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